MODE III PROBLEMS OF DOUBLY-PERIODIC RHOMBIC AND RECTANGULAR ARRAY OF EQUAL CRACKS IN AN INFINITE PLATE

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ABSTRACT

By using the Weierstrass elliptic function the authors obtain the exact solutions of K_{III} for the various doubly-periodic rhombic and rectangular array of equal cracks in various Mode III problems.

I. INTRODUCTION

Since Koiter[1] solved the problems of the doubly-periodic array of holes in 1960, many papers[2-4] on the doubly-periodic array of cracks have been published. However, most of the previous works reduced the problem to integral equations and solved numerically. In reference [5] Kuang and Zhan obtained the exact solution for Mode III problems of the doubly-periodic rectangular array of cracks by Jacobi elliptic functions. In the present paper we solve the Mode III problems of rhombic and rectangular array of cracks by Weierstrass elliptic functions.

The Weierstrass elliptic function P(z) with basic periods $2\omega_1$, $2\omega_3$ is defined as reference [7].

$$P(z) = \frac{1}{z^2} + {\sum}'\left[\frac{1}{(z-\Omega)^2} - \frac{1}{\Omega^2}\right] \quad (1)$$

where $\sum'$ means excluding the terms m=n=0 in summation, and $\Omega = 2m\omega_1+2n\omega_3$, $m,n = 0,\pm1,\pm2,\ldots$. Moreover the following relations hold

$$[P'(z)]^2 = 4P^3(z)-g_2P(z)-g_3 = 4[P(z)-e_1][P(z)-e_2][P(z)-e_3] \quad (2)$$

where

$$e_1 = P(\omega_1),\quad e_2 = P(\omega_2),\quad e_3 = P(\omega_3)$$
$$e_1+e_2+e_3 = 0,\quad e_1e_2+e_2e_3+e_3e_1 = -g_2/4,\quad e_1e_2e_3 = g_3/4 \quad (3)$$

Let a and b be real numbers. Take two special cases for Weierstrass elliptic function:

(1) $\omega_1 = 2a$, $\omega_3 = 2bi$ (Fig. 1) i.e. the basic periodic quadrilateral is a rectangle, then the values of P(z) on the sides and middle lines of the periodic quadrilateral are real and $e_1>e_2>e_3$.

(2) $\omega_1 = a-ib$, $\omega_3 = a+ib$ (Fig. 1) i.e. the basic quadrilateral is a rhombus, then the values of P(z) on the diagonals of the periodic quadrilateral are real. And e_2 is real, e_1, e_3 are conjugate complex numbers.

In both cases, g_2 and g_3 are real. In what follows we shall study the above mentioned cases.

II. RHOMBIC ARRAY OF EQUAL CRACKS

1. Uniform Stress on Crack Surfaces

Fig.2 gives a diagrammatic sketch of doubly-periodic rhombic array of cracks. Let the length of a crack be 2, and the shear stress be τ.

The problem of the doubly-periodic cracks can be simply reduced to finding a quasi-periodic analytic function F(z) in the infinite sheet such that:

$$W = \mathrm{Re}F(z),\qquad \tau_{XZ}-i\tau_{YZ} = GF'(z) \quad (4)$$

$$-G\,\mathrm{Im}F'(z) = \tau \quad \text{(on the crack faces)} \quad (5)$$

$$\left.\begin{aligned} 4b\tau^{\infty}_{XZ} &= G\int_{-2bi}^{2bi} ReF'(z)dz \\ 4a\tau^{\infty}_{YZ} &= -G\int_{-2a}^{2a} ImF'(z)dz \end{aligned}\right\} \qquad (6)$$

where W is the displacement, τ_{XZ}, τ_{YZ} are shear stresses. Eq.(5) is the boundary condition and Eq.(6) the mean stress condition.

For simplicity, we let $\tau^{\infty}_{XZ} = \tau^{\infty}_{YZ} = 0$ and take

$$F'(z) = -\frac{i\tau}{G} + \frac{iD}{\sqrt{P(1)-P(z)}} \qquad (7)$$

where the radical takes the positive value when $1<x<2a-1$. At the crack face $x<1$, $P(1)-P(x)$ is less than zero so the term, $iD/\sqrt{P(1)-P(x)}$ is real, so the Eq.(5) is satisfied. At $z = iy$, $-2b<y<2b$, $P(iy)$ is real and less than $P(1)$, $ReF'(iy) = 0$, so the first equation of Eq.(6) holds. And the second can be written as

$$2a\tau = G\int_{1}^{2a} \frac{Ddx}{\sqrt{P(1)-P}} \qquad (8)$$

where $P \equiv P(x)$. With aid of Eq.(2) the integral in the Eq.(8) can be reduced to[7]

$$\begin{aligned} \int_{1}^{2a} \frac{dx}{\sqrt{P(1)-P}} &= \frac{P'(x)dx}{2\sqrt{[P(1)-P](P-e_2)(P-e_1)(P-e_3)}} \\ &= \frac{1}{2}\int_{e_2}^{P(1)} \frac{dp}{\sqrt{[P(1)-P](P-e_2)(P-e_1)(P-e_3)}} \\ &= \tfrac{1}{2}gF(\pi, k) = gK(k) \end{aligned} \qquad (9)$$

thus we obtain

$$D = 2a\tau/GK(k)g \qquad (10)$$

where

$$\begin{aligned} &g = 1/\sqrt{AB}, \quad k^2 = \{[P(1)-e_2]^2-(A-B)^2\}/(4AB) \\ &b_1 = (e_1+e_3)/2 = -e_2/2, \quad a_1^2 = -(e_1-e_3)^2/4 \\ &A^2 = [P(1)-b_1]^2+a_1^2, \qquad B^2 = (e_2-b_1)^2+a_1^2 \\ &K(k) = \int_0^{\pi/2} d\theta/\sqrt{1-k^2\sin^2\theta} \end{aligned} \qquad (11)$$

Here symbol $K(k)$ is the complete elliptic integral of the first kind with modulus k,

The stress intensity factor

$$K_{III} = \lim_{z\to 1}\sqrt{2\pi(z-1)}\,\tau_{YZ} = -G\lim_{z\to 1}\sqrt{2\pi(z-1)}\,ImF'(z)$$

So we have

$$\frac{K_{III}}{\tau\sqrt{\pi l}} = \frac{2\sqrt{2}a}{gK\sqrt{-P'(1)l}} \qquad (12)$$

Because there is only one crack in the basic periodic rhombus the single-valuedness condition of the displacement is automatically satisfied[5]. Some special cases may be derived:

(1) As $b\to\infty$,

$$K_{III} = \tau\sqrt{4a\tan(\pi l/4a)}$$

which is the case of the colinear cracks.

(2) As $a\to\infty$,

$$K_{III} = \tau\sqrt{4b\tanh(\pi l/4b)}$$

which is the case of the parallel cracks.

(3) If $\tan(b/a) = \sqrt{3}$ then

$$K_{III} = 1.0045\tau\sqrt{\pi l}$$

This result coincides with that obtained in reference [4].

Fig. 3 gives the relationship of $K_{III}/\tau\sqrt{\pi l}$ vs. L/a at various b/a. It is found that $k_{III}/\tau\sqrt{\pi l}$ is almost independent of b/a as b/a>1.

2. Two Equal and Opposite Concentrated Forces on Crack Surfaces

Let the symmetric concentrated forces T be applied on the crack faces at $z = \pm c \pm 4ma \pm 4nbi$. At the point under the concentrated force there is a pole of first order while at the crack tip there is a pole of half order. On the faces of cracks we have $ImF'(z) = 0$ except at the points under the concentrated forces. The boundary condition at these points can be wirtten as

$$2T = G\int_{-1}^{1} \tau_{YZ} dz = -2G\int_{0}^{1} \mathrm{Im}F'(z)dz \tag{13}$$

Thus we finally obtain:

$$F'(z) = \frac{Ti}{\pi G\sqrt{P(1)-P(z)}}\left\{\frac{\pi}{gK} + \frac{(A+B)\sqrt{P(c)-P(1)}P'(c)}{[P(c)-e_2]A-[P(c)-P(1)]B}\right.$$

$$\left.\times\left[\alpha_1 + \frac{\alpha-\alpha_1}{1-\alpha^2}\frac{\Pi(\pi/2,\alpha^2/(\alpha^2-1),k)}{K(k)}\right] - \frac{\sqrt{P(c)-P(1)}}{P(c)-P(z)}P'(c)\right\} \tag{14}$$

$$K_{III} = \frac{\sqrt{2}T}{\sqrt{-\pi P'(1)}}\left\{\frac{\pi}{gK} + \frac{(A+B)\sqrt{P(c)-P(1)}P'(c)}{[P(c)-e_2]A-[P(c)-P(1)]B}\right.$$

$$\left.\times\left[\alpha_1 + \frac{\alpha-\alpha_1}{1-\alpha^2}\frac{\Pi(\pi/2,\alpha^2/(\alpha^2-1),k)}{K}\right] - \frac{P'(c)}{\sqrt{P(c)-P(1)}}\right\} \tag{15}$$

where $x=\pm c$ are the points where the concentrated loads are applied. $\Pi(\pi/2, \alpha^2/(\alpha^2-1),k)$ is the complete elliptic integral of the third kind.

$$\alpha_1 = \frac{A-B}{A+B}, \quad \alpha = \frac{[P(c)-e_2]A-[P(c)-P(1)]B}{[P(c)-e_2]A+[P(c)+P(1)]B} \tag{16}$$

And other symbols are given by Eq.(11).

3. More Complicated Cases

The present method may also be used to treat more complicated cases. For example, the solution for the problem of two cracks symmetrically located with respect to the coordinate axes in a basic periodic rhombus subjected to uniform stress can be expressed as

$$F'(z) = i\tau/G\{-1+(DP(z)+C)/\sqrt{[P(l_1)-P(z)][P(l_2)-P(z)]} \tag{17}$$

where l_1, l_2 refer to the distances of the crack tips from the origin. Obviously Eq.(17) satisfies the uniform stress condition on the crack faces. The constants C and D can be determined by the mean stress condition and the single-valuedness condition of the displacement.

III. RECTANGULAR ARRAY OF EQUAL CRACKS

Fig. 4 shows the doubly-periodic rectangular array of cracks subjected to uniform stress. The solution is

$$F'(z) = \frac{\tau i}{G}\{-1+ \frac{2}{GgK(k)\sqrt{P(1)-P(z)}}\} \tag{20}$$

$$\frac{K_{III}}{\tau\sqrt{\pi l}} = \frac{2\sqrt{2}a}{gK(k)\sqrt{-lP'(l)}} \tag{21}$$

where

$$k^2 = \frac{[P(1)-e_1](e_2-e_3)}{[P(1)-e_2](e_1-e_3)} \qquad g = \frac{1}{[P(1)-e_2](e_1-e_3)}$$

Fig. 5 shows the relationship of $K_{III}/(\tau\sqrt{\pi l})$ with b/a and l/a. We find that $K_{III}/(\tau\sqrt{\pi l})$ is almost independent of b/a as b/a>1.

For other cases they can be solved in the similar manner.

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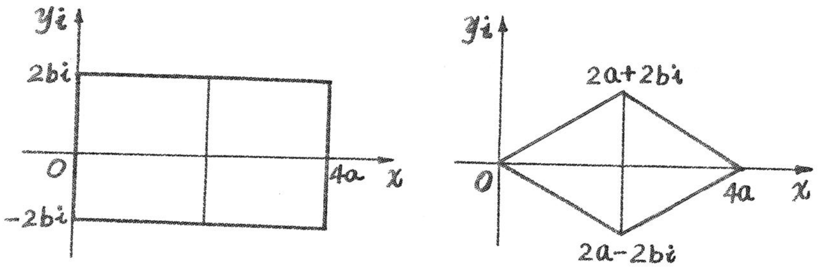

Fig. 1

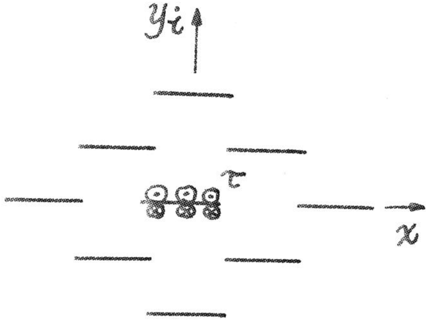

Fig. 2

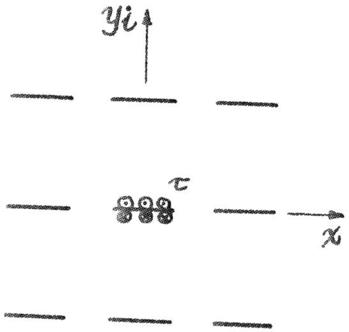

Fig. 4

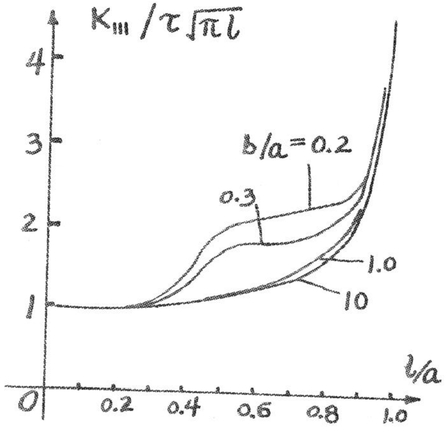

Fig. 3

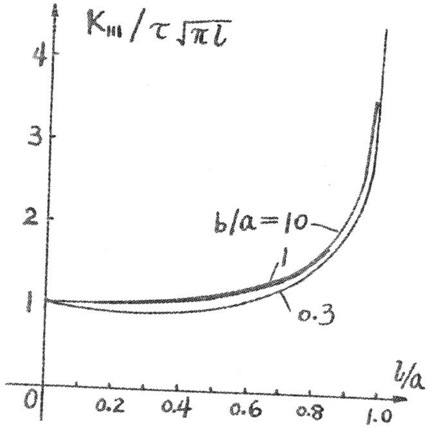

Fig. 5