Size effects on compressive strength from a statistical physics perspective

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Abstract Compression is a loading mode that stabilizes microcrack propagation. Consequently, the weakest-link approach becomes inappropriate to account for size effects on compressive strength of brittle materials such as rocks, ice, or concrete. Instead, compressive failure is characterized by an apparent power law decay of the mean strength at small sizes but a non-vanishing strength towards large sizes, associated to an increasing variability towards small sizes. Here we show from a progressive damage model that compressive failure can be considered as a critical phase transition, with a correlation length diverging at failure. Specific scaling laws for the mean as well as the standard deviation of the strength ensue, which are in full agreement with the experimental observations.

Keywords compressive strength, size effect, critical transition, rocks

1. Introduction

The size effect on strength of materials is an old problem, already discussed by Leonardo da Vinci [1] and Edmé Mariotte [2] several centuries ago. The experimental tensile strength is generally orders of magnitude lower than expected from atomic scale calculations, decreases with increasing scale and is associated to a large scatter that also decreases with increasing scale. A statistical approach based on the presence of internal defects and on the weakest-link concept has been developed for structural materials a long time ago [3,4]. This weakest-link approach is based on the following assumptions: (i) defects do not interact with one another, (ii) failure of the whole system is dictated by the activation of the largest pre-existing flaw, and (iii) the material strength can be linked directly to the critical defect size. Assuming a power law distribution of defect size *s*, $P(s) \sim s^{-\alpha}$ (with generally α>>3), and following linear elastic fracture mechanics (LEFM) principles for which the activation of a flaw of size *s* occurs at a stress $\sigma_c \sim s^{-1/2}$, one gets extremal Weibull statistics for the strength σ_f, and the following scalings for the mean strength $\langle \sigma_f \rangle$ and the associated standard deviation $std(\sigma_f)$:

$$\langle \sigma_f \rangle(L) \sim std(\sigma_f)(L) \sim L^{-d/m} \tag{1}$$

where m=2(α-1) is Weibull's modulus and d the topological dimension [5]. Although based on strong assumptions, this approach has been successfully applied to the statistics of failure strength of structural materials under tension (e.g.[4,6]), with m in the range 6 to 25. Relation (1) implies a continuously decreasing average strength towards large scales, i.e. a vanishing strength for $L \to +\infty$, although this decrease can be rather shallow, owing to the large values of m often reported.

These assumptions are reasonable for materials with relatively weak disorder loaded under tension, but do not hold for heterogeneous materials with a broad distribution of initial disorder, or for loading conditions that stabilizes crack propagation, such as compression. Nevertheless, it has been shown recently that the weakest link hypothesis remains essentially valid even in the presence of long-range elastic interactions in heterogeneous media [7]. We consider here a different problem, the case of compressive failure of brittle materials such as rocks, concrete, or ice; a loading mode that stabilizes microcrack propagation, making the assumptions of the weakest-link approach inappropriate.

2. Brittle compressive failure and associated size effects

Brittle compressive failure is a complex process, as the local tensile stresses at crack tips are counteracted by the far-field compressive stresses. Consequently, Griffith-like energy balance arguments, or related LEFM tools such as fracture toughness, cannot be developed in this case to describe the instability leading to terminal failure, thus making the weakest-link approach inoperative. Instead, compressive failure involves an initiation phase, elastic interactions and stress redistributions, as well as frictional sliding along rough surfaces. During the initiation phase, secondary cracks nucleate from the local tensile stresses generated by the frictional sliding along pre-existing defects such as grain boundaries, small joints, or microcracks [8,9]. The propagation of these mode I secondary cracks is however rapidly stopped by the far-field compression. Instead, such nucleation events locally soften the material and thus cause a redistribution of elastic stresses, which in turn can trigger other microcracking events. Then, in the course towards the failure, the linking of en echelon arrays of secondary cracks is considered to be at the onset of shear fault formation, from which the macroscopic instability is thought to result [9]. This process is characterized by a progressive localization of microfracturing and deformation along a fault [10]. From this qualitative description, one sees that all the assumptions of the weakest-link theory listed above are inappropriate.

When the compressive strength of brittle materials is measured from laboratory tests over a limited scale range (generally between ~ 10^{-2} m and ~ 10^{-1} m), either non-significant [11,12] or limited (e.g. [13]) size effects are reported on $\langle\sigma_f\rangle$, whereas, when reported, the associated variance is relatively large and increases towards small scales [11]. Consequently, empirical or theoretical size effect formulations are hardly constrained by these results. Some studies were performed several decades ago over a much larger scale range (~ 10^{-2} m to few m), combining laboratory experiments and in-situ tests [14,15,16]. All of them reported a significant decrease of $\langle\sigma_f\rangle$ at small scales, which can be tentatively and empirically fitted as a power law decrease $\langle\sigma_f\rangle(L)\sim L^{-\beta}$ [16], but also a saturation of this decay towards large scales which is not explained by the weakest link approach. Note that these results were obtained for natural rock samples that did not contain a pre-existing fault or joint coming through the entire sample. Indeed, in this latter case, one may expect that the flaw size dependency on sample size and the reactivation of this flaw would lead to a power law decrease of $\langle\sigma_f\rangle$ with L, with $\beta\approx1/2$ and without saturation at large sizes [17].

So far, there is no clear explanation for this non-vanishing compressive strength at large sizes in the literature. Instead, empirical [16,18] or more theoretical formulations (based on stored strain energy caused by buckling [5]) of size effects on compressive strength of brittle materials generally ignore such asymptotic behaviour. Following observations at small sizes, all these formulations share a common power law scaling $\langle\sigma_f\rangle(L)\sim L^{-\beta}$, with β varying from very small values [13] (i.e. almost no size effect), to values between 1/2 and 1. The weakest-link concept has been sometimes put forth to explain this scaling for small β values [13], although it is clear from above that this approach is irrelevant in case of compressive failure.

In what follows, we propose an entirely different approach, based on the mapping of brittle compressive failure on a critical phase transition. Using a numerical progressive damage model, we demonstrate the relevance of this mapping, and show that it implies a formulation of size effects on strength which explains (i) the power law-like decay of the mean strength at small sizes, (ii) a non-vanishing strength for $L\rightarrow+\infty$, and (iii) an increasing variability towards small sizes.

3. A model of progressive damage

The model, described in more details elsewhere [19,20], considers a continuous 2D elastic material (Hooke's law) under plane stress, with progressive local damage. Damage is represented by a reduction of the isotropic elastic modulus Y_i of the element i, $Y_i(n+1)=Y_i(n)d_0$, with $d_0 = 0.9$, each time the stress state on that element exceeds a given threshold. This elastic softening simulates an increase in microcrack density at the element scale [21] as supported by experiments [22]. The stress field is recalculated each time a damage event occurs by solving the equation of static equilibrium using a finite element scheme. As the result of elastic interactions, the stress redistribution following a damage event can trigger an avalanche of damage, which stops when the damage threshold is no longer fulfilled by any element.

The Coulomb criterion, $\tau=\mu\sigma_N+C$, of wide applicability for brittle materials under compressive stress states [23], defines the damage threshold. τ and σ_N are respectively the shear and normal stress on the element (sign convention positive in compression), μ is an internal friction coefficient identical for all elements, whereas quenched disorder is introduced through the cohesion C randomly drawn from a uniform distribution ($0.2\times10^{-3}Y_0\leq C\leq 10^{-3}Y_0$). We use $\mu = 0.7$, a common value for most geomaterials [23]. This envelope is completed by a truncation in tension in the Mohr's plane, i.e. the element is damaged if $\sigma_N = -2.10^{-3}\times Y_0$. The simulations start with undamaged material ($Y_i = Y_0$ = const) and are performed on rectangular meshes of randomly oriented triangular elements. Uniaxial compression is applied by increasing the vertical displacement of the upper boundary (strain-driven loading), while left and right boundaries can deform freely.

Series of simulations with meshes of linear size L varying from 8 to 128, composed of $N=4L(L-1)$ triangular elements, were performed with the following number of independent simulations: 5×10^4 for L=8, 3×10^4 for L=16, 5×10^3 for L=32, 10^3 for L=64, and 100 for L=128.

It was shown previously that this model remarkably well reproduces both the macroscopic (strain softening before failure, large stress drop at failure) and microscopic (progressive localization of damage towards the failure along an inclined shear fault, increasing rate of damage avalanches,..) features of compressive failure [19,20]. In a recent work [20,24], we have shown from this modeling framework that brittle compressive failure can be considered as a critical phase transition: (i) the size of the largest damage cluster as well as of the largest damage avalanche diverge at peak load, which just precedes failure, and (ii) the divergence of a correlation length ξ at failure can be identified either from a spatial correlation analysis of damage events, or from a coarse-graining analysis of the strain-rate field. This divergence takes the form $\xi \sim \Delta^{-1/\nu}$, where $\Delta = \frac{\epsilon_{mf}-\epsilon_m}{\epsilon_{mf}}$ is the control parameter, ϵ_m the macroscopic (applied) strain, ϵ_{mf} the corresponding value at peak load (failure), and ν=1.0±0.1 a critical exponent.

If this interpretation is correct, ν is a finite-size exponent and one may expect and the following size effect on strength, from a mapping of this critical transition to the depinning transition of an elastic manifold [25,26]:

$$std(\sigma_f)(L) \sim L^{-\frac{1}{\nu}} \qquad (2a)$$

$$\langle\sigma_f\rangle(L) = A\, L^{-\frac{1}{\upsilon}} + \sigma_\infty \qquad (2b)$$

where A is a constant and σ_∞ a non-vanishing asymptotic value of the strength for $L \rightarrow +\infty$.

4. Results and conclusion

Figure 1 shows the standard deviation of the compressive strength in the model, defined as the maximal macroscopic stress in the direction of loading, as a function of system size. The power law scaling is in full agreement with equation (2a), however with a corresponding finite-size scaling exponent ν=1.47 larger than the value expected from the divergence of the correlation length at failure.

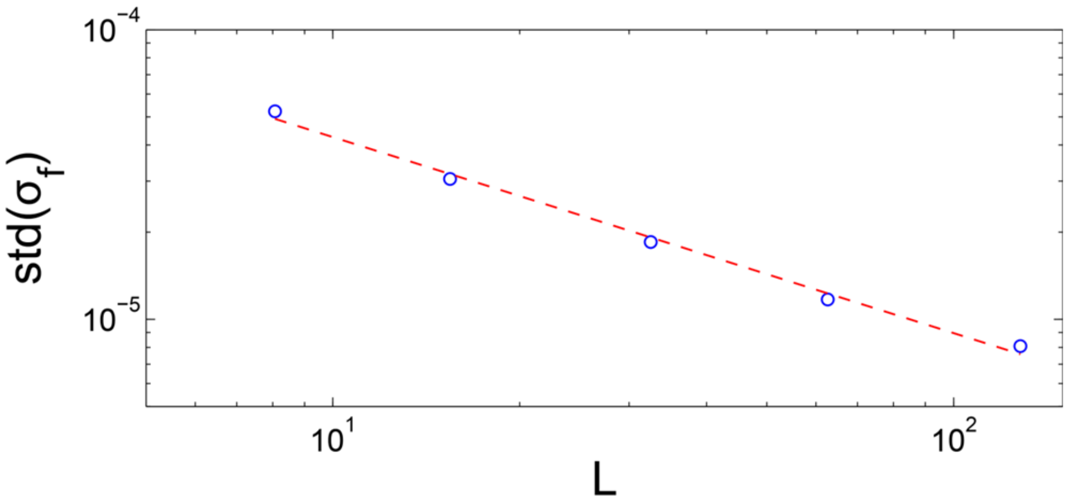

Figure 1. Progressive damage model: Evolution of the standard deviation of the compressive strength, $std(\sigma_f)$, in units of Y_0, as a function of system size L. The red dashed line shows a power law scaling with a exponent of -0.68, corresponding to a finite-size exponent of ν=1.47, in agreement with relation (2a).

This value of ν can be used to test the relevance of equation (2b) to describe the scale dependence of the mean strength. Figure 2 demonstrates the validity of this scaling and the existence of a non-vanishing strength for $L \rightarrow +\infty$.

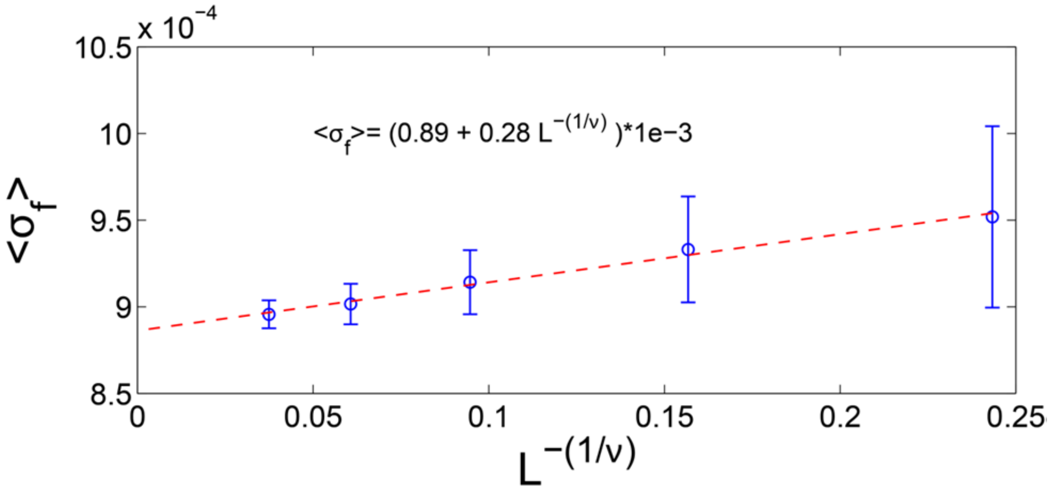

Figure 2. Progressive damage model: Mean compressive strength $\langle \sigma_f \rangle$ as a function of $L^{-\frac{1}{\nu}}$, where L is the system size and ν=1.47 is obtained from Figure 1. The red dashed line corresponds to equation (2b) with an asymptotic strength σ_∞=8.9 $10^{-4} \times Y_0$.

Figure 3 shows the distributions of compressive strength at various scales in a normal probability plot. The collapse along a straight line (i) demonstrates that compressive strengths are distributed according to a Gaussian PDF, and (ii) confirms the power law scaling of the standard deviation (equation (2a)). Compressive strengths are therefore clearly not distributed according to a Weibull distribution (the same data are not aligned and do not collapse in a Weibull probability plot). This is a further confirmation of the irrelevance of the weakest-link concept in compressive failure.

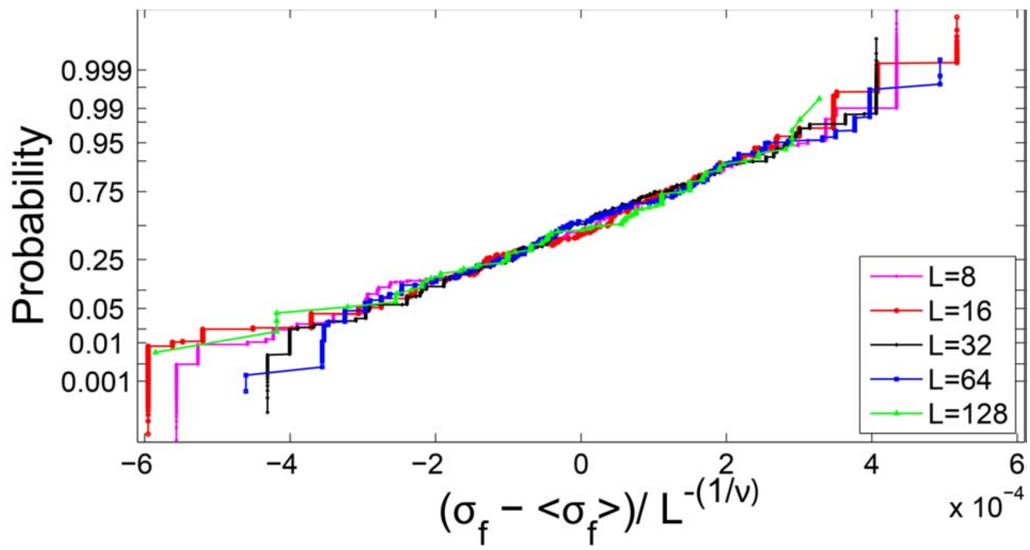

Figure 2. Normal probability plot of the distributions of strength at various scales. Albeit minute deviations, the plot shows a collapsed straight line, as expected for a Gaussian distribution. The collapse also confirms the power law scaling of the standard deviation (equation (2a)).

In conclusion, we have shown that the weakest-link concept is irrelevant for brittle failure under compressive stress states. Instead, compressive failure is a complex process characterized by the divergence of a correlation length at peak (failure) load. This argues for an interpretation of

compressive failure as a critical phase transition. From this interpretation, and mapping this transition to the well-described depinning transition, we propose specific scaling laws for the mean compressive strength as well as the associated variability (equation (2)). This scaling implies (i) an apparent power law decay of the mean strength at small sizes, (ii) a non-vanishing strength for $L \to +\infty$, and (iii) an increasing variability towards small sizes; these three aspects being in full agreement with experimental data (see section 2). In addition, modelling results show that compressive strengths are normally distributed.
This has important consequences including the fact that brittle compressive strengths are expected to be less scattered (normally distributed) than tensile strengths (extremal Weibull statistics), and a reasonable estimate of large scale compressive strength can be obtained from laboratory tests if the obtained experimental size effect is limited.

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