Structural Hierarchy and Dynamic Size Effect of Rock Strength

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Abstract The mechanism of dynamic size effect of strength of rock is studied from the viewpoint of structural hierarchy. Relaxation model of Maxwell type for rock is used to obtain the relationship between strength, sample size and strain rate. It is shown that because of the finiteness of crack propagation velocity, when the strain rate is well above certain characteristic strain rate, dynamic loading process takes predominant role, the stresses in sample have not enough time to relax, the larger the sample size is, the more time is needed for cracks in sample to go through the sample, the ultimate applied stresses before macrofracture are greater and the strength is higher. From other hand because of the size effect the higher dynamically applied stresses will activate the cracking at smaller scale levels of rock sample, and the fragment size is smaller. Based on the presented model the characteristic strain rate separating the predominant diapasons for static and dynamic size effect is determined, and the fragment size is predicted satisfactorily.

Keywords rock, structural hierarchy, dynamic size effect, static size effect

1. Introduction

The strength of rock-like materials has size effect, i.e. it depends on the size of the samples. The general law of the size dependence of strength is that it decreases with the increase of the sample size.

Static size effect of strength has been studied by many scientists. According to Bazant size effect of strength can be separated into two kinds [1]：(1) statistical, described by the Weibull [2, 3, 4] theory of random local material strength, and (2) energetic (deterministic). The latter includes type I size effect[5-10], occurring in materials that fail at crack initiation from a smooth sample surface, and type II size effect[5, 11-14], occurring in materials with a deep notch or deep stress-free crack formed stably before reaching the maximum load. Another approach to size effect is based on the concept of fractality. The self-similarity (fractality) in concrete deformation and fracture has been studied by Z. Bazant Z.[7], Carpinteri A., Pussi. S., Pugno N.M. et al [15, 16]. And the mechanics of hierarchical materials has been also developed [17].

As to size effect of dynamic strength of rocks, the research is relatively rare and the reached conclusions are controversial [18, 19, 20, 21]. For clarification of the size effect of rock strength under dynamic loading condition, recently Hong Liang, Li Xibing et al. have performed refined research on size effect of rock dynamic strength and strain rate sensitivity[22]. The reached conclusions are interesting. The test results show that the rock dynamic strength increases with strain rate in power law which agrees with the research results of many other scientists. The interesting result is that the larger the specimen size is, the more notable the strain rate sensitivity of dynamic strength of rock is; the rock dynamic strength increases with the increase of specimen size under the same strain rate which is opposite to the size effect under static loading condition (see Fig.1 and 2 for granite). The size effect of dynamic strength becomes weaker with the decrease of strain rate and there exist a critical strain rate below which static size effect takes dominant position. In addition, experiments show that the fragment size decreases significantly with the increase of sample size (see Fig.3). The less the sample size is, the more is the dispersion of the experimental results.

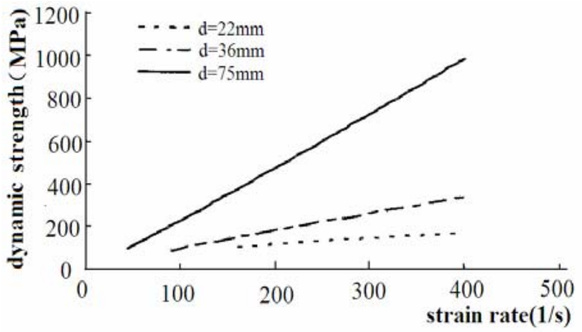

Fig. 1 Strain rate effect of strength of granite under different strain specimens diameters

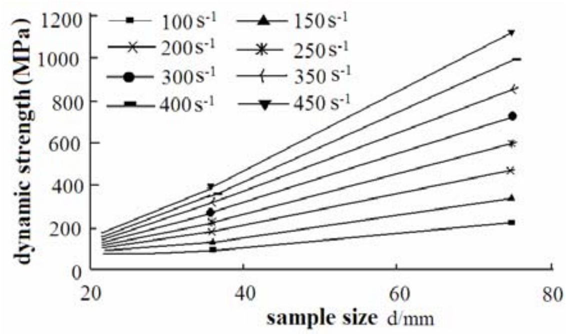

Fig. 2 Size effect of strength of granite under different strain rates

(a) d=75mm, $\dot{\varepsilon}= 98.23s^{-1}$ (b) d=36mm, $\dot{\varepsilon}= 99.75s^{-1}$ (c) d=22mm, $\dot{\varepsilon}= 164.38s^{-1}$

Fig.3 Fragmentation of granite specimens with different sizes

Until now the underlying mechanism of dynamic size effete of rocks has not been intepretated. Therefore in the present paper we will make a trial to perform some theoretical study aiming at clarifying the underlying mechanism of dynamic size effect of rock.

2. Mechanism underlying dynamic size effect of rock strength

As natural materials rock-like materials have complex internal structure, the scales of which span a huge scale range. For rock mass an important peculiarity is the similarity of the internal structure in a wide range of sizes. Investigations [23]showed that, a fundamental canonical series for the sizes Δ_i of geo-blocks exists：

$$\Delta_i = \left(\sqrt{2}\right)^{-i} \Delta_0 \tag{1}$$

where $\Delta_0 = 2.5 \times 10^6$ m is the radius of Earth's core; i is positive integers. As demonstrated in [24], the atomic-ionic radii of different valent orbits of 98 elements in the table of Medeleev also obey canonical series of Eq.(1). Therefore canonical series of Eq.(1) is valid for huge range of scale sizes from continental level to atomic-ionic level.

Some relation between the thickness of the weakened structural surfaces (or the opening of cracks) separating structural elements and the characteristic size of the elements at the given scale level exists. According to investigation in [25], the ratio of the openings of cracks δ_i to the characteristic linear size of the blocks Δ_i separated by the weakened structural surfaces (or cracks) at i-th scale level is stable, and can be described by the following relation which has a normal statistical distribution:

$$\mu_\Delta(\delta) = \frac{\delta_i}{\Delta_i} = \Theta \cdot 10^{-2} \tag{2}$$

where Θ is a coefficient changing in the interval 1/2—2, and parameter μ_Δ is termed as "geo-mechanical invariant" in [25].

In-situ observations of destruction of earth's crust[26] and theoretical and experimental studies on smaller scale rock samples [27] revealed that the deformation and fracture of rock-like materials are governed by laws of Maxwell bodies. This conclusion allows us to describe the deformation and fracture of rock-like materials by Maxwell model.

The internal structure of rocks has decisive impact on mechanical behavior of rocks. If the strength of crystals with ideal regular lattices is their theoretical strength, then the strength of real materials is about 2-3 orders lower than the theoretical strength for ideal crystals. Obviously, the complex hierarchic internal structure of real materials will causes the stress concentration and strain localization which are responsible for lowering of real material strength.

As a reference medium we take ideal crystal with ideal regular lattices. Image that such ideal crystal is subjected to intensive external loading, the intensity of which is high enough, but the induced stress state is well below the strength limit. In this case in crystal no damage and fracture occur and no stress relaxation takes place. But if the ideal crystal is replaced by real rock mass with complex internal structure, then under the action of such intensive external loading stress concentration and successively damage and fracture will occur. Consequently part of the stresses in rock mass will relax. Therefore we can think that in such solid stresses are consisted of two components: elastic stresses caused by the reversible volume and shear deformations, and the local inelastic stresses in heterogeneities which are responsible for the irreversible deformations. The elastic stresses are related to the reversible deformations linearly. As to the residual stresses (inelastic stresses), they arise at definite strain rate, and relax with time. The evolution equation for the residual stress deviator Δs_{ij}^l in heterogeneities may be described by Maxwell model

$$\frac{d\Delta s_{ij}^l}{dt} = 2\rho c_s^2 \dot{e}_{ij} - v\frac{\Delta s_{ij}^l}{l} \qquad (3)$$

where Δs_{ij}^l is the residual stress deviator components in heterogeneities with characteristic scale l; $\dot{e}_{ij}$ is the residual strain rate deviator components; ρ is the density of the medium; v is the relaxation velocity, which may be looked at as the effective propagation velocity of single or multiple cracks depending on the loading conditions; c_s is the propagation velocity of the elastic shear wave. Here we suppose that all residual stress components relax with the same relaxation time. Essentially l/v may be considered as relaxation time $\tau = l/v$.

The main feature of this model is that, the relaxation rate of the residual stresses in heterogeneities is proportional to the magnitude of the residual stresses, and inversely proportional to the size of the heterogeneities. The growth of residual stresses is controlled by two contradicting factors in the right hand side of Eq.(1): the residual stress growth rate $2\rho c_s^2 \dot{e}_{ij}$ and the relaxation rate of residual stresses $v\Delta s_{ij}^l / l$. It is necessary to note that this model is applicable not only to different rock-like materials with great variation range of relaxation times, but also applicable to highly viscous fluids for which the relaxation time is relatively short [28].

The solution of Eq. (4) has the following form:

$$\Delta s_{ij}^l = 2\rho c_s^2 \dot{e}_{ij} \frac{l}{v}\left[1 - e^{-vt/l}\right] = 2\rho c_s^2 \dot{e}_{ij}\tau\left[1 - e^{-t/\tau}\right] \qquad (4)$$

For short loading time $t << \tau$, relaxation process has not enough time to develop, and the loading process is the predominant factor, in this case Eq.(4) gives

$$\Delta s_{ij}^l \approx 2\rho c_s^2 \dot{e}_{ij} t = 2\rho c_s^2 e_{ij} \qquad (5)$$

i.e. the residual stresses will increase almost linearly.

For long loading time $t >> \tau$, relaxation process has enough time to develop, loading process is limited by the relaxation time, in this case Eq.(4) gives

$$\Delta s_{ij}^l \approx 2\rho c_s^2 \dot{e}_{ij}\tau = 2\rho c_s^2 \dot{e}_{ij} \frac{l}{v} \qquad (6)$$

For the occurrence of macrofracture, it is necessary that the loading time is greater than relaxation time $t > \tau$, therefore Eq.(6) is appropriate for study of macrofracture of the samples.

Substituting Eq.(6) into the expression $\Delta\sigma_I = \sqrt{3\Delta s_{ij}^l \Delta s_{ij}^l / 2}$ for intensity of residual stress deviator, we obtain

$$\Delta\sigma_I = 3\rho c_s^2 \dot{e}_I \frac{l}{v} \qquad (7)$$

where $\dot{e}_I = \sqrt{2\dot{e}_{ij}\dot{e}_{ij}/3}$ is the residual strain rate intensity.

It can be seen from Eq. (7) that if we fix the applied strain rate, then the greater the size of the heterogeneities is, the greater the residual stresses are. If the size of the body is infinite, then we can always find large enough heterogeneities that their residual stresses are large enough to cause the fracture of the body. In this way, at constant strain rate among the parameters of solid a parameter with dimension of length arises. Stress concentration causing residual stress in heterogeneous media is the main cause for material fracture, the limit residual stress causing fracture of the body σ^* may be looked at as the strength σ_Y of the sample with size D. Therefore Eq.(7) may be rewritten as

$$\sigma_Y = 3\rho c_s^2 \dot{\varepsilon}_Y \frac{D}{v} \tag{8}$$

Eq.(8) explicitly shows that dynamic strength is proportional to the size of sample, and inversely proportional to the the relaxation velocity. The physical mechanism is as follows. Experiments show that the maximum crack growth velocity is limited and is below the Rayleigh wave speed C_R [29] (Fineberg, Marder, 1999). Therefore the relaxation velocity is also limited. The larger the sample size is, the more is the time needed for the occurrence of macrofracture, and the higher is the reached ultimate amplitude of loading.

Now let us use the obtained fequation (8) to model the dynamic size effect.

For granite we take the Young's elastic modulus as $E = 5.5 \times 10^{10} Pa$, Poisson's ratio as $\mu = 0.29$, then the shear modulus is $G = 2.13 \times 10^{10} Pa$. Now let us with the help of Eq.(8) to fit the experimental data.

For sample with size $D = 22mm = 0.022m$, we obtain that the effective relaxation velocity is $v = 3515 m/s$.

For sample with size $D = 36mm = 0.036m$, we obtain that the effective relaxation velocity is $v = 2465 m/s$.

For sample with size $D = 75mm = 0.075m$, we obtain that the effective relaxation velocity is $v = 1867 m/s$.

It is clear that the effective relaxation velocity is dependent of the specimen size and decreases with the increase of sample size. The dependence of effective relaxation velocity on sample size is shown in Fig.4.

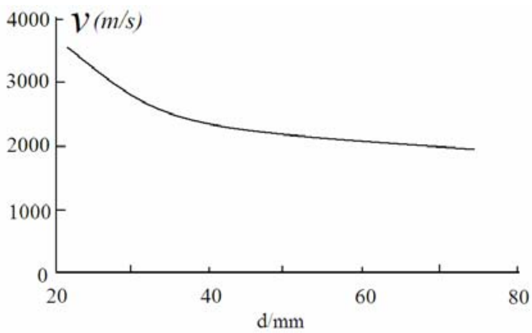

Fig.4 The dependence of velocity of crack propagagtion on specimen size

In the diapason of the sample sizes tested we will use the following equation to interpolate the dependence of crack propagation velocity on the sample size

$$v(D) = 112.59D^2 - 1403D + 6056.6 \tag{9}$$

where the unit of d is cm.

Therefore we now should use the following equation for determination of the dynamic strength of material.

$$\sigma_Y = 3\rho c_s^2 \dot{\varepsilon} \frac{D}{v(D)} \tag{10}$$

Using Eq. (10) we can accurately fit the experimental data shown in Fig.(2) and (3).

Now let's analyze the size effect on dynamic fragmentation.

3. The determination of fragment size of rock

The static size effect and dynamic size effect are depicted in Fig.5.

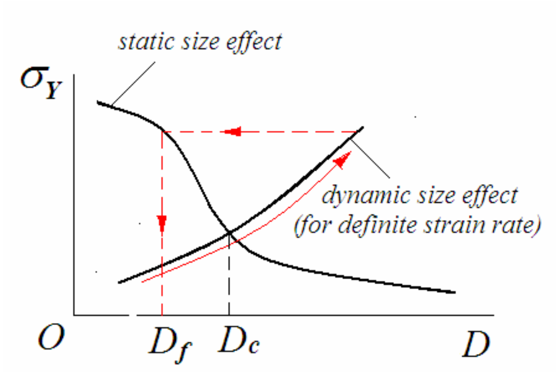

Fig.5 The mechanism of dynamic fragmentation

The static strength of rock mass depends on the sample size. Generally, the compressive strength of materials σ_D can be expressed as a function of the sample size *D* as follows [5]:

$$\sigma_D = \sigma_0 (1 + D/D_0)^{-1/2} \tag{11}$$

where σ_0 and D_0 are constants. Eq.(11) can be rewritten as

$$D = D_0 \left[(\sigma_0/\sigma)^2 - 1 \right] \tag{12}$$

where parameter σ_D is replaced by σ representing the applied loading.

For fast dynamic loading process, because of the finiteness of relaxation velocity, failure will be delayed, and overloading will take place. Therefore for large enough rock sample dynamic strength will be higher than static one. The higher loading will activate the deformation and fracture process in rock at smaller scale element levels, and the rock will fracture according to the static size effect law of rock as shown in Fig.5.

Replacing σ in Eq. (12) by σ_Y in Eq. (11), we obtain the following formula for determining the fragment size of fractured rock mass D_f :

$$D_f = D_0\left[\left(\frac{\sigma_0}{\sigma_Y}\right)^2 - 1\right] = D_0\left[\left(\frac{\sigma_0 v(D)}{3\rho c_s^2 \dot{\varepsilon} D}\right)^2 - 1\right] \tag{13}$$

For determing σ_0 and D_0, we use the data in Fig.3(a) and (b):

For $d = 75mm, \dot{\varepsilon} = 98.23s^{-1}$, the dynamic strength is $\sigma_Y = 220MPa$, and the fragment size is approximately $D_f = 0.2cm$

For $d = 36mm, \dot{\varepsilon} = 99.75s^{-1}$, the dynamic strength is $\sigma_Y = 105MPa$, and the fragment size is approximately $D_f = 1.47cm$.

From these data we obtain $D_0 = 1.75mm$, $\sigma_0 = 3.22 \times 10^8 Pa$.

Hence we have

$$D_f = D_0\left[(\sigma_0/\sigma)^2 - 1\right] = 1.75 \times 10^{-3}\left[\left(3.22 \times 10^8/\sigma_Y\right)^2 - 1\right] \tag{14}$$

Now let us use Eq.(14) to predict the experimental result shown in Fig.3(c). For the case shown in Fig.3(c), $d = 22mm$. We know from Fig.1 (a) that for $\dot{\varepsilon} = 164.38s^{-1}, \sigma_Y \approx 100MPa$. Substituting $\sigma_Y \approx 100MPa$ into Eq.(14) we have $D_f = 16.4mm$, which is very close to the fragment size in Fig.3 (c): $D_f \approx 22/\sqrt{2} = 15.6mm$. Therefore the model is sufficiently good for the description of the dynamic effects on strength and on dynamic fragmentation.

4. The determination of characteristic size and characteristic strain rate for rock

Now let us discuss the characteristic size and characteristic strain rate for rock samples.

For fixed strain rate $\dot{\varepsilon}$, from the following equation

$$\sigma_D = \sigma_0(1 + D_c/D_0)^{-1/2} = \sigma_Y = 3\rho c_s^2 \dot{\varepsilon} \frac{D_c}{v(D_c)}$$

we can determine the characteristic size D_C for rock, above which dynamic size effect of rock strength will predominate.

From other hand for fixed rock sample size D from equation

$$\sigma_D = \sigma_0(1 + D/D_0)^{-1/2} = \sigma_Y = 3\rho c_s^2 \dot{\varepsilon}_c \frac{D}{v(D)}$$

we obtain the characteristic strain rate

$$\dot{\varepsilon}_c = \frac{\sigma_0 v(D)}{3\rho c_s^2 D(1 + D/D_0)^{1/2}} = \frac{\sigma_0 v(D)}{3GD(1 + D/D_0)^{1/2}} \tag{15}$$

From Eq. (15) it is clear that under fixed rock sample size D, when strain rate $\dot{\varepsilon} > \dot{\varepsilon}_c$, dynamic size effect takes dominant position, and when $\dot{\varepsilon} < \dot{\varepsilon}_c$ static size effect predominates. From other hand, under fixed strain rate $\dot{\varepsilon}_c$, when strain rate $D > D_c$, dynamic size effect takes dominant position, and when $D < D_c$ static size effect predominates.

Now let's evaluate the order of the predicted characteristic strain rates.

For $d = 75mm$, $v = 1867m/s$, $G = 2.13\times10^{10}Pa$, $D_0 = 1.75mm$, $\sigma_0 = 3.22\times10^8 Pa$, the predicted characteristic strain rate is

$$\dot{\varepsilon}_c = \frac{\sigma_0 v(D)}{3GD(1+D/D_0)^{1/2}} = \frac{3.22\times10^8\times1.867\times10^3}{3\times2.13\times10^{10}\times7.5\times10^{-2}\times(1+75/1.75)^{1/2}} = 18.9/s$$

For $d = 36mm$, $v = 2645m/s$, $G = 2.13\times10^{10}Pa$, $D_0 = 1.75mm$, $\sigma_0 = 3.22\times10^8 Pa$, the predicted characteristic strain rate is

$$\dot{\varepsilon}_c = \frac{\sigma_0 v(D)}{3GD(1+D/D_0)^{1/2}} = \frac{3.22\times10^8\times2.465\times10^3}{3\times2.13\times10^{10}\times3.6\times10^{-2}\times(1+36/1.75)^{1/2}} = 74/s$$

For $d = 22mm$, $v = 3515m/s$, $G = 2.13\times10^{10}Pa$, $D_0 = 1.75mm$, $\sigma_0 = 3.22\times10^8 Pa$, the predicted characteristic strain rate is

$$\dot{\varepsilon}_c = \frac{\sigma_0 v(D)}{3GD(1+D/D_0)^{1/2}} = \frac{3.22\times10^8\times3.515\times10^3}{3\times2.13\times10^{10}\times2.2\times10^{-2}\times(1+22/1.75)^{1/2}} = 227/s$$

Comparing with Fig.1 we can see that if the static strength of granite is 70 Mpa, the predictions are good enough.

5. Conclusion

Rock-like materials have complex internal structure, the scales of such complex structure span a huge scale range. At the same time deformation and fracture of rocks proceed in real time, and their temporal scales are related to the internal structure and physical-mechanical properties of rocks. To understand the nature of dynamic size effect of strength of rocks it is necessary to consider the structural hierarchy and the temporal properties of deformation and fracture process of rocks. In the present paper relaxation model of Maxwell type for rock is used to obtain the relationship between strength, sample size and strain rate. It is shown that when the strain rate is well above certain characteristic strain rate dynamic loading process takes predominant role. Because of the finiteness of crack propagation velocity, the larger the sample size is, the more time is needed for cracks in sample to go through the sample, and the ultimate applied stresses before macrofracture are more and the strength is higher. Factually the dynamic strength is induced by overloading. From other hand because of the size effect of rock strength, the overloading will activate the cracking at smaller scale levels of rock sample, and the fragment size is smaller. From the present model the characteristic strain rate separating the predominant diapasons for static and dynamic size effect is determined, the dynamic fragmentation size is predicted well.

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