The elastic-plastic limit pressure of cylinder under internal pressure using a new yield criterion

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Abstract Elastic-plastic limit pressure(EPLP) of closed-end cylinders under internal pressure is derived, using a new yield criterion that depends on stress triaxiality and Lode angle. The von Mises, Tresca, twin-shear and Drucker-Prager yield criterions are encompassed in the new yield criterion for the EPLP. The results reveal that, with the increasing of the parameters of stress triaxiality and Lode angle, the EPLP decreases for cylinder with fixed outer-to-inner radius ratio(k). As k increases, the EPLP increases when the parameters of stress triaxiality and Lode angle are certain value. The parameter of Lode angle is much more important than the parameter of stress triaxiality affecting the EPLP of cylinders. The new yield criterion with smaller parameters of stress triaxiality and Lode angle is used instead of the conventional von Mises, Tresca, twin-shear and Drucker-Prager yield criterions in design, it can lead to substantial saving the material required. While the new yield criterion with larger parameters of stress triaxiality and Lode angle is used in design, it can make the design more safe and reliable.

Keywords Closed-end cylinder, New yield criterion, Stress triaxiality, Lode angle

1. Introduction

Closed-end cylinders are used in engineering, as pressure vessels, nuclear reactors and capsules. It is important to know the EPLP of cylinders under internal pressure. A considerable amount of work has been done on the problem of elastic-plastic analysis in a closed-end cylinder under internal pressure[1-4]. However, when the conventional yield criterions are adopted to derive the EPLP, the Tresca[5] and von Mises[6] yield criterions ignore the effect of the intermediate principal stress and octahedral normal stress on yield, respectively. They lead to conservative predictions of limit pressures. Although the twin-shear[7] and Drucker-Prager[8] yield criterions overcome above deficiencies, they still ignore the effect of stress triaxiality and Lode angle on yield. By the view of equivalent stress, stress triaxiality and Lode angle, the von Mises yield criterion ignore the effect of stress triaxiality and Lode angle on yield, and the Tresca yield criterion also ignore the effect of stress triaxiality on yield. But the effect of stress triaxiality and Lode angle on yield have been confirmed[9-13]. Based on the von Mises, Tresca and Drucker-Prager yield criterions, a new yield criterion, which considers the effect of stress triaxiality and Lode angle on yield, is proposed and confirmed by experiments[9]. In this paper, the EPLP of closed-end cylinders under internal pressure, which are two important parameters in the design of closed-end cylinders, is determined using the new yield criterion. Important results concerning the influence of the EPLP determined with the new yield criterion on the design of closed-end cylinders under internal pressure are presented.

2. Determined the EPLP of closed-end cylinders under internal pressure using the new yield criterion

The new yield criterion, which considers the effect of stress triaxiality and Lode angle on yield, as follows,

$$\sigma_{\mathrm{yld}} = \sigma_{\mathrm{eq}}\left[1 - c_{\eta}\left(\eta - \eta_0\right)\right] \cdot \left[c_{\theta} + \left(c_{\mathrm{ax}} - c_{\theta}\right)\gamma\right]. \tag{1}$$

Where γ and c_{ax} are two parameters defined by

$$\gamma = \frac{\sqrt{3}}{2\text{-}\sqrt{3}}\left[\sec\left(\theta\text{-}\frac{\pi}{6}\right)\text{-}1\right],\quad c_{\mathrm{ax}} = \begin{cases} c_{\mathrm{t}} & 0 \le \theta \le \frac{\pi}{6} \\ c_{\mathrm{c}} & \frac{\pi}{6} < \theta \le \frac{\pi}{3} \end{cases}. \tag{2}$$

Where σ_{eq}, η and θ are equivalent stress, stress triaxiality and Lode angle, respectively. σ_{yld} is the yield stress. η_0, c_η, c_θ, c_{t} and c_{c} are material constants, this depends on which type of reference test is used to calibrate the relationship of stress-strain.

Let us consider a closed-end cylinder under an internal pressure p. The inner and outer radius of the cylinder are a and b, respectively. In elastic stage, the σ_{eq}, η and θ are determined with Lame solutions of the elastic stress distribution in cylindrical coordinates system[14] as follows,

$$\sigma_{\mathrm{eq}} = \sqrt{3}\frac{\rho^2}{k^2-1}p, \eta = \frac{\sqrt{3}}{3}\frac{1}{\rho^2}, \theta = \frac{\pi}{6}. \tag{3}$$

Where $k=b/a$ and $\rho=b/r$ ($a\le r\le b$).

Yielding will appear at the inner surface of the closed-end cylinder at the elastic limit pressure(ELP) p_{e}. Substituting Eq. (3) into Eq. (1), the yield condition is satisfied at the inner surface $(\sigma_{\mathrm{r}})_{\mathrm{r=a}}=-p_{\mathrm{e}}$, the ELP obtained by the new yield criterion is

$$p_{\mathrm{e}} = \sigma_{\mathrm{yld}}\frac{\left(k^2-1\right)}{\sqrt{3}c_\theta k^2}\left[1-c_\eta\left(\frac{\sqrt{3}}{3k^2}-\eta_0\right)\right]^{-1}. \tag{4}$$

Because θ is a constant ($\theta=\pi/6$), the effect of Lode angle on the ELP is equal to c_θ. The relationship of stress-strain is obtained from smooth round bar tensile test, therefore, η_0 is equal to 1/3. c_η represents the effect of stress triaxiality on material plasticity.

When the internal pressure exceeds p_{e}, a plastic zone will appear at the inner surface and spread toward the outer surface. The elastic-plastic boundary at any stage has radius r_{y} ($a\le r_{\mathrm{y}}\le b$), in the elastic region ($r_{\mathrm{y}}\le r\le b$), the radial stress is obtained from Lame's equations using the boundary condition $\sigma_{\mathrm{r}}=0$ at $r=r_{\mathrm{b}}$ as follows,

$$\sigma_{\mathrm{r}} = \frac{1}{k_{\mathrm{y}}^2-1}\left(1-\rho^2\right)p_{\mathrm{y}}. \tag{5}$$

Where $p_{\mathrm{y}} = \frac{\sigma_{\mathrm{yld}}\cdot\left(k_{\mathrm{y}}^2-1\right)}{\sqrt{3}c_\theta\cdot k_{\mathrm{y}}^2}\left[1-c_\eta\left(\frac{\sqrt{3}}{3}\frac{1}{k_{\mathrm{y}}^2}-\frac{1}{3}\right)\right]^{-1}$ and $k_{\mathrm{y}} = \frac{b}{r_{\mathrm{y}}}$.

In plastic region, the material is assumed perfectly elastic-plastic, the equation of equilibrium is

$$\frac{d\sigma_{\mathrm{r}}}{dr} + \frac{\sigma_{\mathrm{r}}-\sigma_\theta}{r} = 0. \tag{6}$$

The cylinder is assumed at the plane strain condition, therefore, the longitudinal stress in the plastic region is

$$\sigma_{\mathrm{z}} = \frac{1}{2}\left(\sigma_{\mathrm{r}}+\sigma_\theta\right). \tag{7}$$

The σ_{eq}, η and θ are determined with Eq. (7), σ_{r} and σ_θ as follows,

$$\sigma_{\mathrm{eq}}=\frac{\sqrt{3}}{2}\left(\sigma_{\theta}-\sigma_{\mathrm{r}}\right),\ \eta=\frac{\sqrt{3}}{3}\frac{\sigma_{\theta}+\sigma_{\mathrm{r}}}{\sigma_{\theta}-\sigma_{\mathrm{r}}},\theta=\frac{\pi}{6}. \tag{8}$$

Substituting Eq. (8) into Eq. (1), we have

$$A=\sigma_{\theta}-B\sigma_{\mathrm{r}}. \tag{9}$$

Where $A=\frac{2\sqrt{3}}{c_{\theta}}\frac{\sigma_{\mathrm{yld}}}{3+\left(1-\sqrt{3}\right)c_{\eta}}$ and $B=\frac{3+\left(1+\sqrt{3}\right)c_{\eta}}{3+\left(1-\sqrt{3}\right)c_{\eta}}$.

When $c_{\eta}\neq0$, Substituting Eq. (9) into Eq. (6), then the boundary condition $(\sigma_{\mathrm{r}})_{\mathrm{r=a}}=-p$ is satisfied, the stress distribution in the plastic region ($a\leq r\leq r_{\mathrm{y}}$) is

$$\sigma_{\mathrm{r}}=C\cdot r^{B-1}-\frac{A}{B-1},\sigma_{\theta}=B\cdot C\cdot r^{B-1}-\frac{A}{B-1}. \tag{10}$$

Where $C=\left(\frac{A}{B-1}-P\right)a^{-(B-1)}$.

According to the stress continuity of radial stress σ_r across $r=r_{\mathrm{y}}$, it requires that

$$\left(\sigma_{\mathrm{r}}\right)_{\mathrm{r=r_y}}\text{ (elastic zone)}=\left(\sigma_{\mathrm{r}}\right)_{\mathrm{r=r_y}}\text{ (plastic zone)}. \tag{11}$$

Substituting the radial stress Eq. (5) for the elastic zone and the radial stress Eq. (10) for the plastic zone into Eq. (11), the relation of pressure p with plastic zone radius is obtained as follows

$$p=\frac{A}{B-1}+\left\{\frac{\sigma_{\mathrm{yld}}\cdot\left(k_{\mathrm{y}}^{2}-1\right)}{\sqrt{3}c_{\theta}\cdot k_{\mathrm{y}}^{2}}\left[1-c_{\eta}\left(\frac{\sqrt{3}}{3}\frac{1}{k_{\mathrm{y}}^{2}}-\frac{1}{3}\right)\right]^{-1}-\frac{A}{B-1}\right\}\left(\frac{r_{\mathrm{y}}}{a}\right)^{-(B-1)}. \tag{12}$$

When r_{y} becomes equal to b, the closed-end cylinder is completely plastic, the plastic limit pressure(PLP) for closed-end cylinder is, therefore, obtained as

$$p_{\mathrm{p}}=\frac{\sigma_{\mathrm{yld}}}{c_{\theta}c_{\eta}}\left[1-k^{\frac{-2\sqrt{3}c_{\eta}}{3+\left(1-\sqrt{3}\right)c_{\eta}}}\right]. \tag{13}$$

When $c_{\eta}=0$, the PLP for closed-end cylinder also is obtained as follows,

$$p_{\mathrm{p}}=\frac{2\sigma_{\mathrm{yld}}}{\sqrt{3}c_{\theta}}\ln k. \tag{14}$$

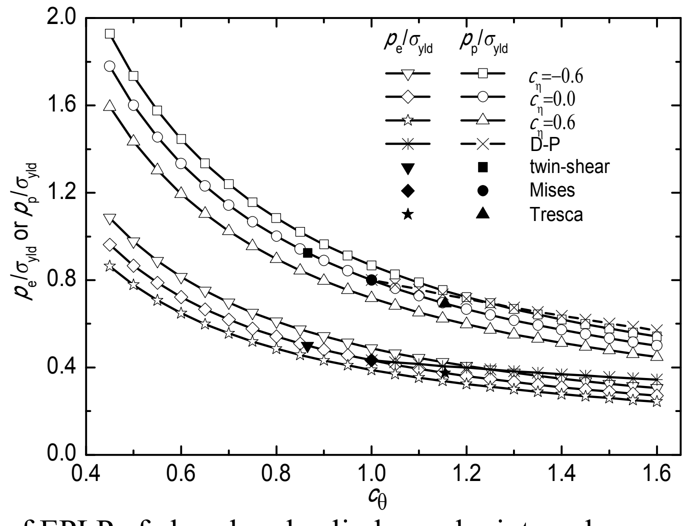

Fig.1 Relation of EPLP of closed-end cylinder under internal pressure with c_{θ} (k=2).

If the ratio of outer radius to the inner radius is 2(k=2), the relation of EPLP of closed-end cylinders under internal pressure with c_{θ} is illustrated in Fig.1. It can be seen that, the EPLP calculated by von Mises, Tresca and twin-shear yield criterions equal these obtained by the new yield criterion when $c_\eta=0$ and $c_\theta=1$, $2/\sqrt{3}$ and $\sqrt{3}/2$, respectively. While $c_\eta\neq0$, $c_\eta=3(1-c_\theta)/c_\theta$ and $c_\theta>1$, one will get the EPLP determined with Drucker-Prager. Therefore, it can be concluded that the von Mises, Tresca, twin-shear and Drucker-Prager yield criterions are encompassed in the new yield criterion for the EPLP. It is also found that, when $c_\eta=0.0$ and $c_\theta>\sqrt{3}/2$, the EPLP is higher than those obtained by the von Mises, Tresca, twin-shear and Drucker-Prager yield criterions. When $c_\eta=0.0$ and $c_\theta<2/\sqrt{3}$, the EPLP is lower than those obtained by conventional yield criterions. When c_θ is certain value, the EPLP in range of $c_\eta<0.0$ is larger than that at $c_\eta=0.0$, the EPLP in range of $c_\eta>0.0$ is smaller than that at $c_\eta=0.0$. c_θ is much more important than c_η affecting the EPLP of cylinder, similar conclusions are also found in earlier report[9, 15]. The EPLP decreases with the increasing of c_θ and c_η.

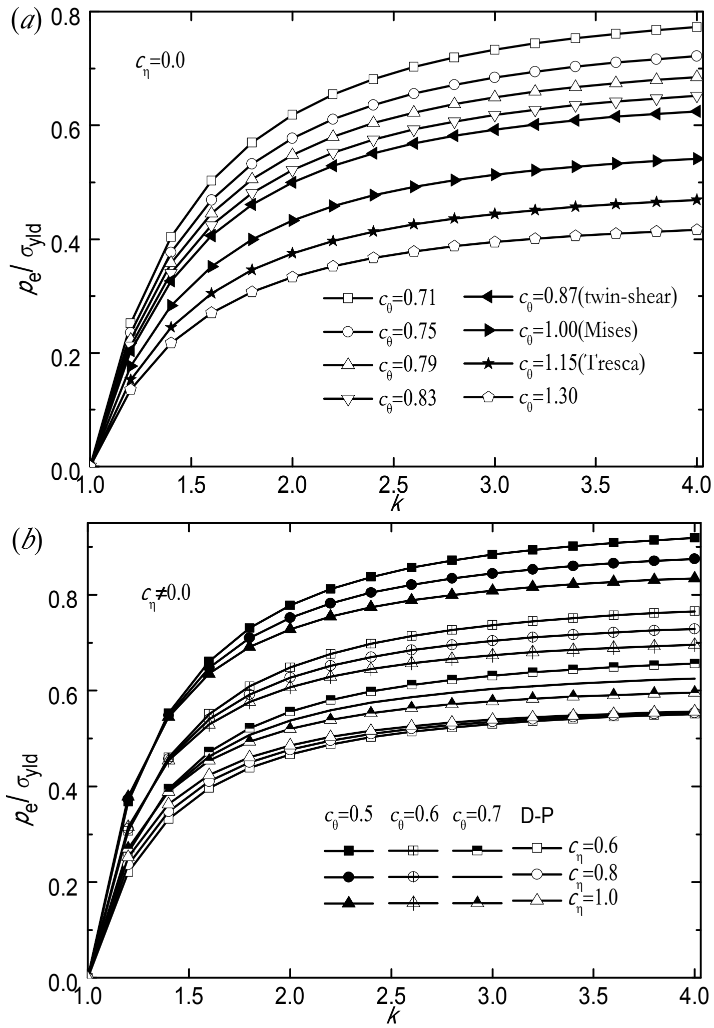

Fig.2 Relation of ELP of closed-end cylinders under internal pressure with *k* at $c_\eta=0$(*a*) and $c_\eta\neq0$(*b*).

A number of the relations of EPLP with the ratio of outer radius to the inner radius, are shown in Fig. 2 and Fig. 3. In Fig.2 and Fig.3, the EPLP increases with the increasing of *k*, when c_θ and c_η are certain value. In Fig.2(*a*), when the effect of stress triaxiality on yield is ignored ($c_\eta=0$), it is worth noting that, if *k* approaches the value 3.0, the increase in the ELP is very small with the increasing of *k*. While the effect of stress triaxiality on yield is considered ($c_\eta\neq0$), the increase in the ELP is also very small when $k\geq2.5$, shown in Fig.2(*b*). From what I have mentioned above, we can see the new yield criterion with smaller c_θ and c_η is used instead of the conventional von Mises, Tresca, twin-shear and Drucker-Prager yield criterions in design, it can lead to substantial saving the material required. While the new yield criterion with larger c_θ and c_η is used in design, it can make the design more safe and reliable.

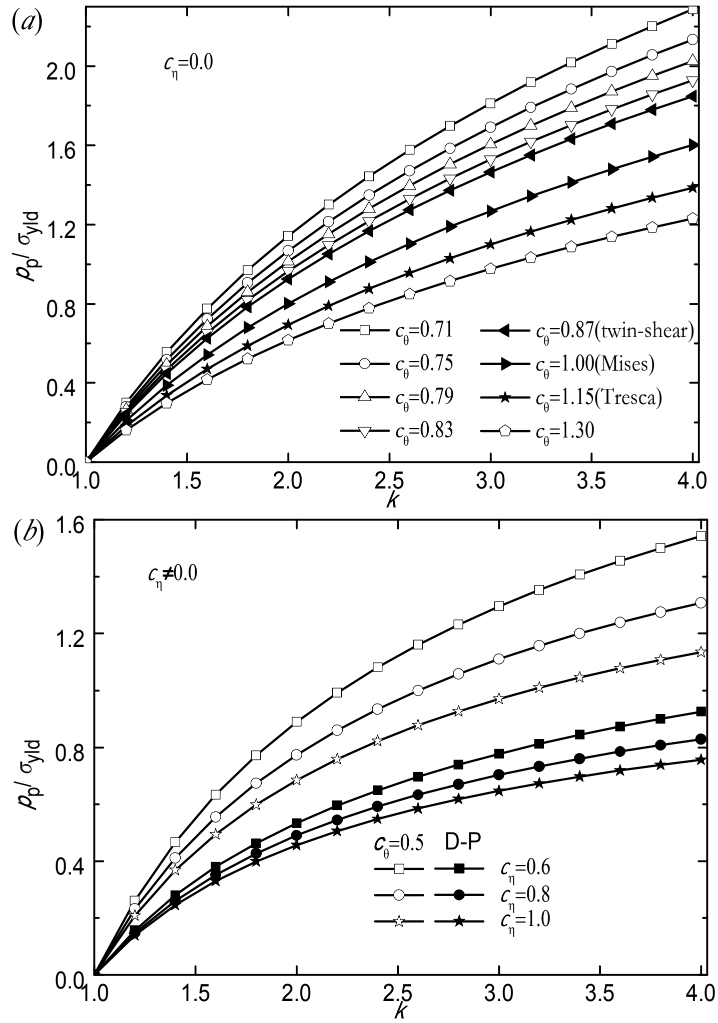

Fig.3 Relation of PLP of closed-end cylinders under internal pressure with k at $c_\eta=0(a)$ and $c_\eta\neq0(b)$.

3. Summary

The EPLP of closed-end cylinders under internal pressure is derived using the new yield criterion that depends on stress triaxiality and Lode angle. The von Mises, Tresca, twin-shear and Drucker-Prager yield criterions are encompassed in the new yield criterion for the EPLP. The results reveal that, with the increasing of c_θ and c_η, the EPLP decreases for cylinder with fixed k. As k increases, the EPLP increases when c_θ and c_η are certain value. c_θ is much more important than c_η affecting the EPLP of cylinder, The new yield criterion with smaller c_θ and c_η is used instead of the conventional von Mises, Tresca, twin-shear and Drucker-Prager yield criterions in design, it can lead to substantial saving the material required. While the new yield criterion with larger c_θ and c_η is used in design, it can make the design more safe and reliable.

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