CRACK DELOPMENT AND DYNAMIC NATURE OF CRACK PROPAGATION IN STRUCTURES

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Abstract: This paper aims at finding the sources of crack development in modern structures at an elevated temperature and under severe vibrations due to excessive load magnitudes and also of ceismic vibrations. In the present paper attempts have been made for solving the problems relating to large amplitude vibrations of uniform elasto plastic plates and shells under both static and dynamic loads using the method of constant deflection contour lines[8-15]. Also, the effect of crack development and its propagation through the structures have been studied rigorously. The works in this field by numerous researchers are discussed shortly in the introduction and considering all the assumptions made by them after proper correction compatible with the boundary conditions, the present author tries to develop the equations with consideration of multi aspect crack generating factors and their impact on the time periods of nonlinear vibration of the structures and the results obtained for a crack structure are presented along with that of the non cracked structure.

I N T R O D U C T I O N

Modern structures are often subjected to severe vibrations and high temperatures. Sometimes the magnitudes of the applied forces or loads become very large, exceeding the elastic limit and brittle strength of the material used. Ceismic vibrations during earth quake, crashes and blasting of bombs etc. sometimes initiate the growth of fractures within the structures Also, if there exist any source of fracture like crystal defects, brittle disorder, imperfections etc within the structures (may or may not be visible from outsides), fracture will be developed and during the load variations in static cases or during vibrations in all dynamic cases and also during temperature elevation , the fractures have got dynamic character; i.e., crack begins to propagate through the structures causing a notable variation of the magnitude of the flexural rigidity of the material. Gradually, the crack developed within the structures causes a damage of the system due it its growth and dynamic nature. Griffith [1], Irwin[2], E. Erdogan [3], Orowan [4], G.I. Barenblatt [5], D.S, Dugdale [6], J. R. Wills [7] had made extensive researches on fracture mechanics. Some had attempted to explain the fracture character in structures assuming some sort of flaws within it. But all the works are based on some highly simplified hypothesis. To verify the flaw hypothesis, Griffith introduced an artificial flaw in his experimental specimens. The artificial flaw was in the form of a surface crack which was much larger than other flaws in a specimen. Irwin strategically partitioned the energy in two parts; the stored elastic strain energy which is released as a crack grows. This is the thermodynamic driving force for fracture and the dissipated energy which includes plastic dissipation and the surface energy also any other dissipative forces that may be at work.

The dissipated energy provides the thermodynamic resistance to fracture. The Stress Intensity Factor is given by the following equations.

$$G = G_I = \begin{cases} \dfrac{K_I^2}{E} & \text{plane stress} \\ \dfrac{(1-\nu^2)K_I^2}{E} & \text{plane strain} \end{cases}$$

where E is the Young's modulus, v is Poisson's ratio, and K_I is the stress intensity factor in mode I. Irwin also showed that the strain energy release rate of a planar crack in a linear elastic body can be expressed in terms of the mode I, mode II (sliding mode), and mode III (tearing mode) stress intensity factors for the most general loading conditions. Irwin adopted the additional assumption that the size and shape of the energy dissipation zone remains approximately constant during brittle fracture. This assumption suggests that the energy needed to create a unit fracture surface is a constant that depends only on the material. This new material property was given the name *fracture toughness* and designated G_{Ic}. Today, it is the critical stress intensity factor K_{Ic}, found in the plain strain condition, which is accepted as the defining property in linear elastic fracture mechanics.

By applying the theory of fracture mechanics one can study the propagation of cracks in materials. Fracture Mechanics uses methods of analytical solid mechanics to calculate the driving force on a crack and those of experimental solid mechanics to characterize the material's resistance to fracture. It applies the Physics of Stress and Strain, in particular the theories of elasticity and Plasticity, to the microscopic Crystallographic defects found in real materials in order to predict the macroscopic mechanical failure of bodies.A comparative study of the static and dynamic characteristics of the structure will provide a notice for of such a failure before hand . The prediction of crack growth is at the heart of the damage tolerance discipline. Still, by studying the static and dynamic characteristics of the structures and comparing the results with the standard results of the same structures without any crack, an estimation of the safety factor may be obtained with ease.

Derivation Of The Governing Differential Equations for A Non Cracked Structure

A shallow shell of uniform thickness `h ` is considered. Let the equation of the middle surface of the shell, referred to an orthogonal coordinate system (x,y,z), be given by

$$Z = (x^2 / 2 R_x) + (xy / R_{xy}) + (y^2 / 2 R_y) \qquad \textbf{(1)}$$

For a shallow shell $r = \sqrt{(x^2 + y^2)}$ considered small in comparison to the least of the radii of curvature, R_x , R_{xy} and R_y which are taken to constants.When the shell experiences axisymmetric free vibration the intersections of the deflected surface and the parallels z = constant yield contour lines of constant deflection. Application of D` Alembert`s principle to an element of the shell bounded by such a contour at any time τ and subsequent summation of the forces in the direction normal to the surface yields the following dynamical equations [1] :

$\int V_n \, ds + \int\int[\rho h (\partial^2 w) / \partial \tau^2 + (N_x) / R_x + (N_y) / R_y + 2 (N_{xy})/R_{xy}] \, dx \, dy = 0$ (2)

where the transverse reaction forces $V_n = Q_n - \partial/\partial s(M_{nt})$ in absence of fractures, (2,a)

$V_n = Q_n - \partial/\partial s(M_{nt}) – f(a, G, K)$, the last term is due to fractures, (2,b) represents the effect of the shearing force Q_n and the edge-rate of change of twisting moment M_{nt} along the contour C_u. According to Ilyushin`s theory of the elastic plastic deformation (1948), the bending moments M_x, M_y, M_{xy} and their shear forces Q_x, Q_y are given by the following relations:

$$M_x = -D(1-\nu)\{(\partial^2 w/\partial x^2) + \nu(\partial^2 w/\partial y^2)\} \quad (3)$$
$$M_y = -D(1-\nu)\{(\partial^2 w/\partial y^2) + \nu(\partial^2 w/\partial x^2)\}$$
$$M_{xy} = D(1-\nu)(1-\Omega)(\partial^2 w/\partial x\partial y)$$
$$Q_x = (\partial/\partial x)\{M_y\} - (\partial/\partial y)\{M_{xy}\}$$
$$Q_y = (\partial/\partial y)\{M_x\} - (\partial/\partial x)\{M_{xy})\}$$
$$Q_n = Q_x \cos\alpha + Q_y \sin\alpha$$

$$M_{nt} = M_{xy}(\cos^2\alpha - \sin^2\alpha) + (M_x - M_y)\sin\alpha\cos\alpha \quad (4)$$

Where, $\cos\alpha = (dy/ds)$ and $\sin\alpha = -(dx/ds)$.

Here, ρ, h and w are, respectivly, the mass density, the shell thickness and the deflection. Using the well known expressions for the moments and shearing forces and assuming that the membrane forces N_{x}, N_y and N_{xy} are given by

$$N_x = (\partial^2\Phi/\partial y^2),\quad N_y = (\partial^2\Phi/\partial x^2),\quad N_{xy} = -(\partial^2\Phi/\partial x\partial y) \quad (5)$$

Equation (2) finally reduces to:

$$(\partial^3 w/\partial u^3)\int(1-\Omega)R\,ds + (\partial^2 w/\partial u^2)\int(1-\Omega)F\,ds + (\partial w/\partial u)\int(1-\Omega\ G\,ds$$
$$+(\partial^2 w/\partial u^2)\int D[(\partial\Omega/\partial x)(\partial u/\partial x) + (\partial\Omega/\partial y)(\partial u/\partial y)]\sqrt{t}\,ds + (\partial w/\partial u)\int(D/\sqrt{t})[K(\partial\Omega/\partial x) + L(\partial\Omega/\partial y)]\,ds + \iint[\rho h\ (\partial^2 w/\partial\tau^2\ (1/R_x)(\partial^2\Phi/\partial y^2) + (1/R_y)(\partial^2\Phi/\partial x^2)$$
$$-2/(R_{xy}\partial^2\Phi/\partial x\partial y]\,dx\,dy = 0 \quad (6)$$

Where R,F,G are given in Ref.13, $D = (E h^3/12(1-\nu^2)$,is the flexural rigidity.
Here, $\Omega = 0$ when $e \le 1$, the region is elastic ; when $e > 1$ the region is plastic. Also, $\Omega = \lambda[1-(3/2e) + (1/2e^3)$

and $e^2 = (h^2/3e_s^2)[(\partial^2 w/\partial x^2)+(\partial^2 w/\partial y^2)+(\partial^2 w/\partial x\partial y)+(\partial^2 w/\partial x^2)(\partial^2 w/\partial y^2)$

$$= (h^2/3e_s^2)[M\ (\partial w/\partial u)^2 + N(\partial w/\partial u)(\partial^2 w/\partial u^2)(\partial w/\partial u) + t\,2\,(\partial^2 w/\partial u^2) \quad (7)$$

in which e_s is the yield strain, ν is the poisson`s ratio, D is the flexural rigidity of the plate material, λ is a material constant .

Here , $M = [u_{,xx}^2 + u_{,yy}^2 + u_{,xx}u_{,yy} + u_{,xy}^2]$
$N = [2u_{,x}^2 u_{,xx} + 2u_{,y}^2 u_{,yy} + u_{,xx}u_{,y}^2 + u_{,x}u_{,yy} + 2u_{,x}u_{,y}u_{,xy}]$
$t^2 = (u_{,x}^2 + u_{,y}^2)$

Considering only the transverse vibration, we assume that

$$w = W(x,y)f(t) \quad (8)$$
$$\Phi = \Phi(x,y)f(t) \quad (9)$$

Equation (6) will now reduce to

$$[(\partial^3 W/\partial u^3)\int(1-\Omega)R\,ds + (\partial^2 W/\partial u^2)\int(1-\Omega)F\,ds + (\partial W/\partial u)\int(1-\Omega)\ G\,ds$$
$$+(\partial^2 W/\partial u^2)\int D[(\partial\Omega/\partial x)(\partial u/\partial x) + (\partial\Omega/\partial y)(\partial u/\partial y)]\sqrt{t}\,ds + (\partial W/\partial u)\int(D/\sqrt{t})[K(\partial$$

$\Omega / \partial x) + L (\partial \Omega / \partial y)] ds] f(t) + \iint [\rho h W f``(t) + \{ (1/R_x)(\partial^2 \Phi / \partial y^2) + (1/R_y)(\partial^2 \Phi / \partial x^2) - 2/(R_{xy} \partial^2 \Phi / \partial x \partial y \} f(t)] dx\, dy = 0 \qquad (10)$

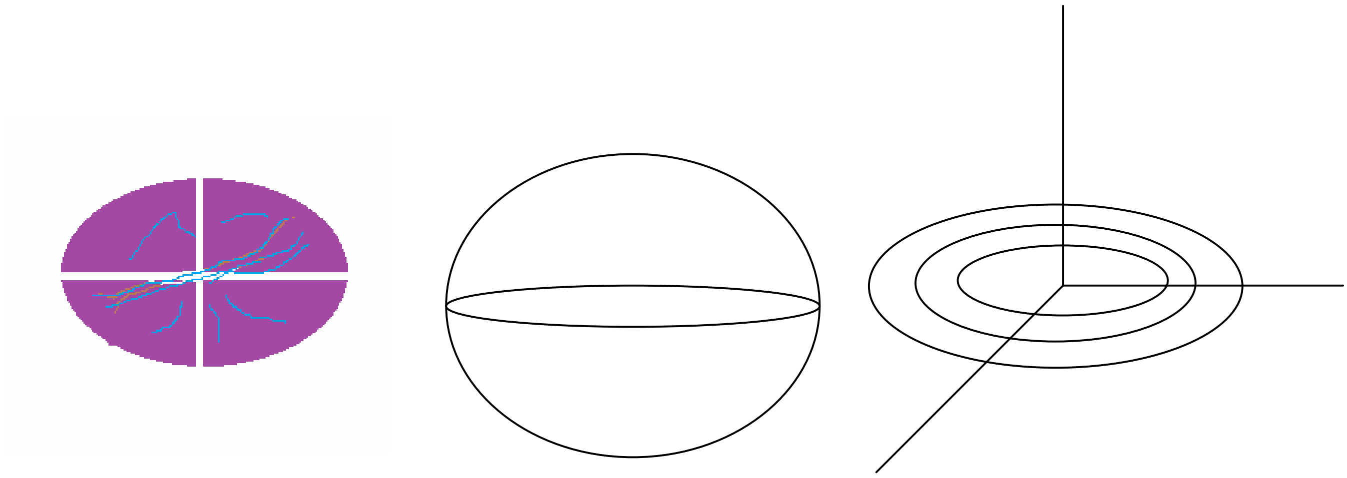
Consequently, the condition for continuity of deformation reduces to

$$\nabla^4 \Phi = \{ 12D (1 - v^2) \} / h^2 (1 - \Omega) [(1/R_x)(\partial^2 \Phi / \partial y^2) + (1/R_y)(\partial^2 \Phi / \partial x^2) - 2/(R_{xy}) \qquad (11)$$

This equation must hold over all points in the interior of the shell. After integration over the area and application of Greens theorem one obtains :

$$(d^3\Phi/du^3) \int Rds + (d^2 \Phi/du^2) \int Fds + (d\Phi/du) \int Gds * 12D^2 (1 - v^2) / h^2 (1 - \Omega)(dW/du) \int K_x (\partial u / \partial y)^2 + Ky (\partial u / \partial x)^2 / t^{1/2} ds = 0 \qquad (12)$$

where K_x and K y denote curvatures at a point and K xy has been assumed to be zero in accordance with the shallowshell theory. Equations (11) and (13) are now the two basic equations for large amplitude vibration of shallow shell.



Internally Cracked shallow domes and isodeflection contour lines

Figure-I

ILLUSTRATION:

Let us now consider a clamped dome of non-zero curvature upon an elliptic base. For the first approximation under symmetry consideration we may write

$$u = 1 - x^2 / a^2 - y^2 / b^2 \qquad (13)$$

Performing the contour integrations taken around the closed contour

$$u = 1 - x^2 / a^2 - y^2 / b^2$$

And the double integration extending over the ellipse

$$x^2 / a^2 + y^2 / b^2 = 1 - u \qquad (14)$$

Equation (11) in non dimensional form becomes

$$(1 - \Omega)(1 - u)(d^3W/du^3) - 2(1 - \Omega)(d^2W/du^2) - (d\Omega/du)[(1 - u)(d^2W/du^2) - 2P\{(1/a^4) + (1/b^4) + 2\nu/a^2b^2)(dW/du)] + (\rho h^2 \omega^2 P)/(2 D e_s a^2) - \{(E h \gamma)/D\}(d\Phi/du) = 0 \qquad (15)$$

where $P = (a^4 b^4)/(3a^4 + 2a^2b^2 + 3b^4)$, while equation (12) in non dimensional form will reduce to

$$(1 - u)(d^3\Phi/du^3) - 2(d^2\Phi/d^2u) + (1 - \Omega)\gamma(dW/du) = 0 \qquad (16)$$

with $\gamma = p(k_x/b^2 + k_y/a^2)$; $W = wh/ e_s a^2$; $\Phi = \varphi/Ee_s a^2$ (17)

(18)

METHOD OF SOLUTION

On substitution of the value of Ω into equations (16) & (17), one obtains

$$[(1 - u)(d^3W/du^3) - 2(d^2W/du^2)] Q_1 f(t) - [2M(d^2W/du^2)(dW/du) + N(d^2W/du^2)^2 N(d^3W/du^3)(dW/du) + 2t^2(d^3W/du^3)(dW/du)] [(1 - u)(d^2W/du^2) - 2P_1(dW/du)Q_2 f^3(t) - (Eh\gamma/D) f(t)(d\Phi/d + (\rho h / (2 D e_s a^2) f``(t) = 0 \qquad (18)$$

$$\text{and} \quad (1 - u)(d^3\Phi/du^3) - 2(d^2\Phi/d^2u) + Q_1\gamma(dW/du) = 0 \qquad (19)$$

Where , $Q_1 = [2e^3(1 - \lambda) + \lambda(3e^2 - 1)]/2e^3$; $Q_2 = (\lambda/4e^5)(e^2 - 1)$

$P_1 = P(1/a^4 + 1/b^4 + 2\nu/a^2b^2)$

Also, e^2 is given by

$$e^2 = ⅓[M(dW/du)2 + N(dW/du)(d^2W/du^2) + t2(d^2W/du)^2] \qquad (20)$$

Suppose the shell is completely clamped along the boundary.

The boundary conditions are given by

$$W\big|_{u=0} = 0 = (dW/du)\big|_{u=0}$$

$$\Phi\big|_{u=0} = 0 = (d\Phi/du)\big|_{u=0}$$

To find an approximate solution, we assume the following trial solutions:

$$W = \Sigma a_j u^j ; \quad \Phi = \Sigma b_j u^j \qquad (21)$$

On substitution of these trial solutions in equations (16) & (18), we get the residuals R_1 & R_2 which, after the application of Galerkin`s procedure, yield the following results.

$$(\rho h^2 P)/(6 D e_s a^2) f``(t) = [(4/3) Q_1 a_2 + (Eh\gamma/2D) b_2] f(t) + [(4/5)M + (2/3)N - (32/5)MP_1 - 4NP_1] Q_2 a_2^3 f^3(t) \qquad (22)$$

$$\text{and} \quad b_2 = (3/8) Q_1 \gamma a_2 \qquad (23)$$

while the average value of `e` happens to be

$$e = a_2 \sqrt{[(1/a^4) + (1/b^4) + (2/a^2 b^2)] (40/9)} \qquad (24)$$

From equations (22) & (23), one obtains the following time differential equations in f(t):

$$f``(t) + \mu f(t) + \zeta f^3(t) = 0 \qquad (25)$$

where, $\mu = -(6 D e_s a_2 / \rho h^2 P) [(4/5) M + (2/3) N - (32/5) M P_1 - 4 N P_1] Q 2 a_2^3$

Since the series is rapidly converging hence considering the first few terms one may obtain the approximate value of the central deflection w* as

$$w^* = \sum_j a_j$$

The solution of equation (25) is given by

$$f(t) = a_0 \operatorname{Sin} [\mu t \{ 1 + (3/8) a_0^2 (\zeta/\mu) \} + \theta_0] \qquad (26)$$

The time periods of the non-linear and linear oscillations are

$T^* = 2\pi / [\mu \{ 1 + (3/8) a_0^2 (\zeta/\mu) \}]$ and $T = 2\pi / \mu$.

Thus $[T / T^*] = 1 / \{ 1 + (3/8) a_0^2 (\zeta/\mu)$ (27)

NUMERICAL RESULTS

Numerical results are computed both for circular and elliptic plates & shallow domes upon the circular & elliptic bases in the elastic and elastic plastic regions and these are presented in tables (1 – 4). The computations are made with different values of the shallowness parameter ($2\gamma / h$) and material constant $\nu = 0.3$. Dynamic responses of the elasto-plastic shells for moderately large amplitude are obtained from the same differential equations. Moreover, effect of crack / fractures are computed with the same equation only changing the term for Vn in equation (2,b) and making subsequent changes in other equations and the results are presented in the tables from (1-4, red coloured) .This is no doubt advantageous as such both static and dynamic behaviors are obtained simultaneously with least effort.

TABLE -1

Free vibrations of clamped plastic shallow shell with circular planform. First row results for without crack while the second row results[Red coloured] are with crack.
e >1 , a = b, ν = 0.3 ,λ =1

T*/T→

W^*a_0→	0	0.5	1.0	1.5	2.0
2γ/=0	1.0000 , 1.0276	1.0256 1.0453	1.1111 1.0689	1.2903 1.3561	1.6666 1.7860
2γ/h=1	1.0000 1.0045	1.0184 1.0235	1.0778 1.0976	1.1940 1.2357	1.4063 1.5872
2γ/h=2	1.0000 1.0011	1.0099 1.0212	1.0411 1.1676	1.0975 1.2879	1.1874 1.3654
2γ/h=3	1.0000 1.0078	1.0056 1.0487	1.0230 1.1432	1.0532 1.1769	1.0985 1.1980
2γ/h=4	1.0000 1.0061	1.0037 1.0764	1.0149 1.1267	1.0345 1.1478	1.0621 1.1645
2γ/h=5	1.0000 1.0032	1.0024 1.0989	1.0095 1.1087	1.1215 1.1112	1.0390 1.1313

TABLE -2

Free vibrations of clamped plastic shallow shell with elliptic planform . e >1 , a =2 b, ν = 0.3 ,λ = 1. First row results for without crack while the second row results[Red coloured] are with crack.

T*/T→

W^*a_0→	0	0.5	1.0	1.5	2.0
2γ/h=0	1.0000 1.0087	1.0866 1.1432	1.4682 1.8760	2.5398 2.7890	4.6241 5.0023
2γ/h=1	1.0000 1.1123	1.0611 1.1564	1.2995 1.8709	2.0768 2.9556	2.2857 3.0986
2γ/h=2	1.0000 1.2345	1.0325 1.4532	1.1439 1.6750	1.3947 1.9870	2.6125 3.1012
2γ/h=3	1.0000 1.5680	1.0182 1.7690	1.0771 1.9080	1.1919 2.3240	1.4009 3.4532
2γ/h=4	1.0000 1.6730	1.0133 1.8750	1.0468 2.0021	1.1117 2.0234	1.2176 3.6700
2γ/h=5	1.0000 1.8870	1.0075 2.0222	1.0310 2.3421	1.0726 2.4562	1.1367 3. 7800

TABLE - 3

Free vibrations of clamped elastic shallow shell with circular planform . e >1 , a = b, ν = 0.3 ,λ = 1. First row results for without crack while the second row results[Red coloured] are with crack. T*/T→

W^*a_0→	0	0.5	1.0	1.5	2.0
2γ/=0	1.0000 1.0500	1.9582 2.1865	0.8513 2.5087	0.7179 3.1254	0.5887 3.2754
2γ/h=1	1.0000 1.0836	1.9695 2.3905	0.8880 2.6540	0.7789 3.2300	0.6646 3.4322
2γ/h=2	1.0000 1.1021	1.9831 2.5214	0.9355 2.8800	0.8657 3.3500	0.7840 3.4804
2γ/h=3	1.0000 1.1208	1.9903 2.7806	0.9623 2.9231	0.9189 3.5602	0.8645 3.7120
2γ/h=4	1.0000 1.1457	1.9936 2.9807	0.9751 3.0061	0.9457 3.6615	0.9074 3.8654
2γ/h=5	1.0000 1.1674	1.9960 3.0085	0.9838 3.0120	0.9645 3.7211	0.9385 3.9800

TABLE - 4

Free vibrations of clamped elastic shallow shell with elliptic planform . e >1 , a = 2b, ν = 0.3 ,λ = 1 . First row results for without crack while the second row results[Red coloured] are with crack.

T*/T→

W^*a_0→	0	0.5	1.0	1.5	2.0
2γ/=0	1.0000 1.0764	0.8778 2.3421	0.6423 2.3675	0.4436 3.1586	0.3098 3.4210
2γ/h=1	1.0000 1.0887	0.9085 2.4531	0.7130 2.4722	0.5247 3.4326	0.3883 3.4789
2γ/h=2	1.0000 1.0912	0.9479 2.6487	0.8200 2.7120	0.6692 3.6870	0.5323 3.7123
2γ/h=3	1.0000 1.1105	0.9697 2.7896	0.8888 2.8760	0.7805 3.8902	0.6667 3.9431
2γ/h=4	1.0000 1.1236	0.9808 2.9807	0.9276 3.0023	0.8507 3.9034	0.7622 3.9890
2γ/h=5	1.0000 1.1432	0.9865 3.0231	0.9480 3.0765	0.8902 3.9986	0.8203 4.0064

OBSERVATIONS AND CONCLUSIONS

It is observed that the results obtained for elastic shallow shells are in excellent agreement with those obtained in [4, 5].Also, the results for plastic shells based on circular and elliptic domes are in good agreement with those obtained in [4]. It is found in both the cases that the ratio of nonlinear to linear time periods become larger in case of plastic shell in comparison to those obtained in case of elastic shells and which is expected. This actually supports the application of the isodeflection contour lines method to analyse such problems. Inspite of that the method has some limitations. It heavily relies on the accuraqcy of the choice of the iso-deflection contour function u(x,y). The main advantage of the method lies in the fact that once the form of the function u (x,y) is chosen suitably the remaining task can be tackled with ease and accuracy. Numerical results [red coloured] obtained for the plates and shells in presence of fractures reveal the fact that due to reduction of the values of elastic constants reaction falls and some irregularities are observed when the cracks propagating through the specimen increasing their dimensions which is expected.

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