Nonlinear fracture mechanics metal foams

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Abstract Nonlinear fracture mechanics of metal foams is discussed. The equivalent continuum constitutive model is introduced first. Then we studied analytic solutions on plastic analysis based on generalized cohesive force model, approximate analytic solution of elasto-plastic analysis for central crack, single edge crack specimens in tension and bending, respectively. The finite element analysis is also conducted. In addition, the crack slowly growth and fast propagation are studied as well.Atthe end we proposed a fracture criterion on metal foam.

Keywords Cellular/foam material, Crack tip opening displacement, Analytic solution in closed foam, Approximate solutions, Finite element analysis

1. Introduction

Cellular/foam material, which is one of advanced materials, has become an important engineering material to date due to its lower density and higher specific strength and other excellent mechanical, thermal and acoustical behaviour. For engineering usage, the structural integrity requires materials to have sufficient strength and toughness. Hence the study of crack problems in the material is significant. It is well-known that the basis of deformation and fracture investigation is constitutive law for any engineering materials. Triantafilou and Gibson [1], Gibson et al [2], Deshpande and Fleck [3], Miller [4] and others carried on considerable studies in this respect. Based on the equivalent continuum model they put forward some new macroscopic parameters out of the conventional materials to describe the influence of the substructure, which is called cells. The existence of cells leads to unusual plasticity of metal foams, that is the plasticity dependent from volume deformation.Thus the effect of hydrostatic pressure or the average stress should be considered, and these quantities should be contained into the new constitutive equations in the so-called equivalent continuum model. With these constitutive equations, the elasto-plastic analysis for cellular/foam material can be done. The nonlinearity of the governing equations of the material makes it difficult to construct analytic solutions of boundary value problems. Fan and co-workers [5,6] paid attention to developing a generalized cohesive force model and making the problem being linearized for static crack and moving crack problems, and they developed an asymptotic analysis method by using singular perturbation procedure for slowly steady crack growth problem. Within the linearization regime, the complex analysis presents its particular role and in this paper we report one of the works. The application of conformal mapping helps us to construct some solutions of finite size crack specimens, and it may be useful to experimental investigation and engineering application.

2. Overview in brief of equivalent continuum constitutive laws

The artificial cellular materials include metal, polymer and ceramic foams respectively. The first two foams display plasticity of dependent of volume deformation, so that the hydrostatic pressure p or the average stress σ_m must be taken into account in the constitutive law of the material. The yield/loading surface can be expressed by

$$\Phi = \hat{\sigma} - Y = 0 \tag{1}$$

where $\hat{\sigma}$ represents the generalized effective stress which will be discussed in the following, and if

$$Y = \sigma_{\mathrm{Y}} = \mathrm{const} \tag{2}$$

in which σ_{Y} denotes the uniaxial tensile yield limit of the material, then equation (1) stands for an initial surface. Alternatively if

$$Y = Y(h) \tag{3}$$

where h is a parameter describing plastic deformation history, then equation (1) represents the evolution equation of yield/loading surface.

Triantafilou and Gibson [1] ("TG" to "abbreviated as the TG model" in the following) suggested that

$$\hat{\sigma} = \sigma_{\mathrm{e}} + 0.03\frac{\rho^*}{\rho_{\mathrm{s}}}\sigma_{\mathrm{m}} \tag{4}$$

where ρ^* denotes the density of foam, and ρ_{s} the density of cell wall of the foam, and

$$\sigma_{\mathrm{e}} = \left(\frac{3}{2}s_{ij}s_{ij}\right)^{1/2} \tag{5}$$

is the von Mises effective stress, and

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} = \sigma_{ij} - \sigma_{\mathrm{m}}\delta_{ij} = \sigma_{ij} + p\delta_{ij} \tag{6}$$

the deviate stress tensor, in which σ_{ij} represents stress tensor, and $\sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33} = 3\sigma_{\mathrm{m}} = -3p$, δ_{ij} the unit tensor. Substituting equation (4) into equation (1) obtains the TG yield/loading surface of foams.

Gibson, Ashby, Zhang and Triantafilou[2] ("GAZT" to "abbreviated as the GAZT model" in the following) put forward

$$\hat{\sigma} = \sigma_{\mathrm{e}} + 0.81\frac{\rho^*}{\rho_{\mathrm{s}}}\frac{\sigma_{\mathrm{m}}^{\;2}}{\sigma_{\mathrm{Y}}} \tag{7}$$

Substituting equation (7) into equation (1) leads to the GAZT yield/loading surface of cellular materials.

After the work of TG and GAZT the constitutive models of foams were studied by many other groups. Distinguished from the above models, researchers take other parameters, according to experiments, to describe the effect of cells rather than the relative density ρ^* / ρ_s.

Deshpande and Fleck [3] ("DF" to "abbreviated as the DF model" in the following) define a plastic Poisson's ratio as

$$\nu^p = -\frac{\dot{\varepsilon}_{11}^p}{\dot{\varepsilon}_{22}^p} \tag{8}$$

where $\dot{\varepsilon}_{11}^p$ and $\dot{\varepsilon}_{22}^p$ are plastic strain rates. In terms of equation (8) DF derived a parameter α to describe plasticity of dependent volume deformation

$$\alpha = 3\left(\frac{1/2-\nu^p}{1+\nu^p}\right)^{1/2} \tag{9}$$

where the value of α is in range $1 < \alpha < 2$. Further they suggested the generalized effective stress as follows

$$\hat{\sigma} = \left[\frac{1}{1+(\alpha/3)^2}(\sigma_e{}^2 + \alpha^2\sigma_m{}^2)\right]^{1/2} \tag{10}$$

It is evident that if $\nu^p = 1/2$, then $\alpha = 0$, this corresponds to the plasticity of independent volume deformation, i.e., the classical plasticity.

The substitution of equation (10) into equation (1) yields the DF yield/loading surface.

There are other models of constitutive law for metal foams and polymer foams, but we do not list again.

By considering isotropic hardening, based on flow rule and the above yield/loading surfaces, one can obtain the corresponding constitutive equations, which can be expressed uniformly as follows

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^e + \dot{\varepsilon}_{ij}^p = \frac{1+\nu}{E}\dot{\sigma}_{ij} - \frac{\nu}{E}\dot{\sigma}_{kk}\delta_{ij} + \frac{\dot{\hat{\sigma}}}{H(\hat{\sigma})}\frac{\partial \Phi}{\partial \sigma_{ij}} \tag{11}$$

where $\dot{\varepsilon}_{\boldsymbol{ij}}$ represents strain rate tensor, $\dot{\varepsilon}_{ij}^e$ the elastic strain rate one, $\dot{\varepsilon}_{ij}^p$ the plastic strain rate one, $\dot{\sigma}_{ij}$ the stress rate one, E and ν the Young's modulus and Poisson's ratio, and $H(\hat{\sigma})$ the hardening modulus, which can be approximately calibrated through a simple stress-strain relation, e.g. $H(\hat{\sigma}) = \mathrm{d}\sigma / \mathrm{d}\varepsilon^p$ represents the hardening modulus at the stress amplitude value as $\hat{\sigma}$, respectively. From equations (1) and (4), (7), (10) and (11), we have

$$\frac{\partial \Phi}{\partial \sigma_{ij}} = \begin{cases} \dfrac{3}{2}\dfrac{1}{\sigma_e}s_{ij} + 0.09\dfrac{\rho^*}{\rho_s}\delta_{ij}, & \text{TG model} \\ \dfrac{3}{2}\dfrac{1}{\sigma_e}s_{ij} + 0.18\dfrac{\rho^*}{\rho_s}\dfrac{\sigma_m}{\sigma_Y}\delta_{ij}, & \text{GAZT model} \\ \dfrac{1}{[1+(\alpha/3)^2]\hat{\sigma}}\left(\dfrac{3}{2}s_{ij} + 3\left(\dfrac{\alpha}{3}\right)\sigma_m\delta_{ij}\right), & \text{DF model} \end{cases} \tag{12}$$

The form of rate $\dot{\hat{\sigma}}$ can be easily found from the previous definitions of generalized effective stresses.

Coupling the constitutive equation (11) and the deformation geometry equationsand equilibrium equationsand appropriate boundary conditions, the elasto-plastic analysis for the foam materials can be carried out.

3. Generalized cohesive force model and solution

Due to the nonlinearity of equations we can find that the elasto-plastic analysis of crack problems of metal foams is very difficult, the exact analytical solution is almost not available. But the application of some simple physical models can simplify the solving dramatically. It is well-known that the Dugdale model [7], or the Dugdale-Barenblatt model [7, 8], or the cohesive force model is very effective in the study of plastic fracture of conventional structural materials. A similar work is the so-called BCS model [9, 10]. We extend the Dugdale model for conventional structural materials into the foam materials, the statement is as follows.

3.1 Generalized cohesive force model for cellular/foam materials --plane strain state

Assume that an infinite plane of foam material with a Griffith crack subjected a uniform tension $\sigma^{(\infty)}$ at infinity, refer to Fig.1, in which the pulling stress $p = \sigma^{(\infty)}$, and the plastic zone around crack tip is with a narrow band type, whose length is denoted by d, but its value is unknown at moment to be determined.

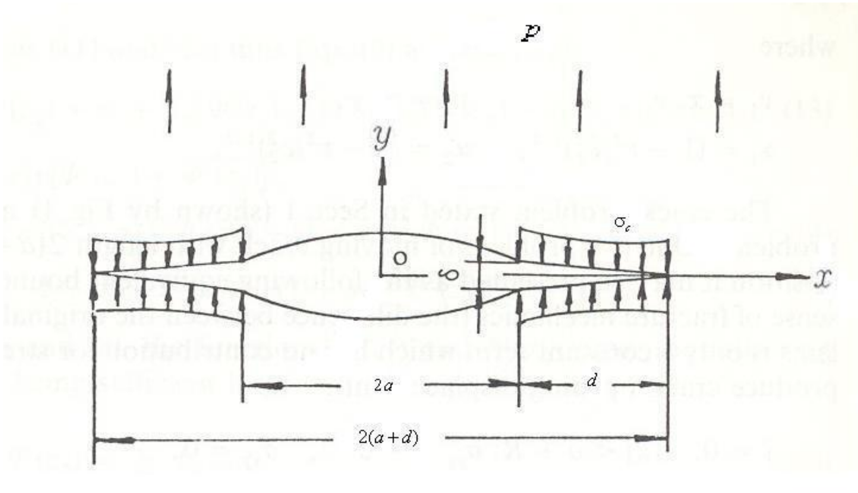

Fig. 1 The generalized cohesive force model for infinite specimen

According to the yield criteria of TG, GAZT, DF models respectively, the corresponding Dugdale plastic zone near the crack tip leads to

$$y = 0,\ a < |x| < a + d:\ \sigma_{yy} = \beta\sigma_{\mathrm{Y}},\ \sigma_{xy} = 0 \quad (13)$$

Based on the constitutive laws of listed by (11) with (4), (7), (10) respectively, where σ_{Y} represents the uniaxial tensile yield limit of foam materials, and the parameter β describing cellular/foam materials behaviour and stress state, for plane stress case:

$$\beta=\begin{cases}\dfrac{1}{1+0.03\rho^{*}/\rho_{s}}, \text{ TG model}\\ \dfrac{\left(1+1.44\rho^{*}/\rho_{s}\right)^{1/2}-1}{1+0.03\rho^{*}/\rho_{s}}, \text{ GAZT model}\\ \left[\dfrac{1+\left(\alpha/3\right)^{2}}{1+\left(2\alpha/3\right)^{2}}\right]^{1/2}, \text{ DF mod}\end{cases} \tag{14}$$

Though there is a new unknown quantity d, the nonlinear problems is linearized, so the equations are reduced to biharmonic equation:

$$\nabla^{2}\nabla^{2}U=0 \tag{15}$$

where $\partial^{2}U/\partial^{2}x=\sigma_{yy}$, $\partial^{2}U/\partial^{2}y=\sigma_{xx}$, $\partial^{2}U/\partial x\partial y=-\sigma_{xy}$ .The complex representations of U is the stress potential function, u_{i} the displacement vector and σ_{ij} is similar to that given in conventional structural materials.

The corresponding boundary conditions are

$$\begin{cases}\sqrt{x^{2}+y^{2}}\to\infty:\ \sigma_{yy}=\sigma^{(\infty)},\ \sigma_{xx}=\sigma_{xy}-0;\\ y=0,\ |x|<a:\ \sigma_{yy}=0,\ \sigma_{xy}=0;\\ y=0,\ a<|x|<a+d:\ \sigma_{yy}=\beta\sigma_{\mathrm{Y}},\ \sigma_{xy}=0\end{cases} \tag{16}$$

The linearization problem of cellular/foam materials is concluded to solve the boundary value problem.

Similar to the classical plastic fracture theory, the solutions are

$$d=a\left[\sec\left(\frac{\pi\sigma^{(\infty)}}{2\beta\sigma_{\mathrm{Y}}}\right)-1\right] \tag{17}$$

and

$$\delta_{\mathrm{t}}=\mathrm{CTOD}=\frac{8a\beta\sigma_{\mathrm{Y}}}{\pi E}\ln\sec\left(\frac{\pi\sigma^{(\infty)}}{2\beta\sigma_{\mathrm{Y}}}\right) \tag{18}$$

in which equation (17) determines the plastic zone size, and equation (18) gives the crack tip opening displacement (these two equations were first given by Fan et [5]), respectively. The both formulas contain the parameter β describing behaviour of foam materials. The variations of the crack tip opening displacement versus applied stress for different values of the new material constant α for DF model are shown in Figs. 2 in which the foam material constants $E=0.271\mathrm{GPa}$, $\sigma_{Y}=0.811\mathrm{MPa}$.

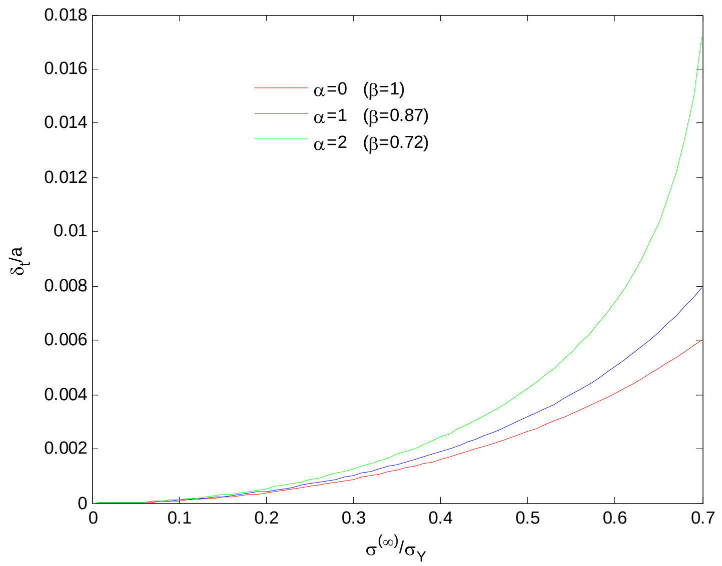

Fig. 2 Variation of values of δ_t versus applied stress for infinite specimen

3.2 Generalized cohesive force model for cellular/foam materials --plane strain state

The discussion is similar to Subsection 3.1, the only differences lie in that the boundary conditions (16) are replaced and the parameter β (given by (14)) is replaced by β' (given by the following (20))

$$y = 0,\ a < |x| < a + d:\ \sigma_{yy} = \beta' \sigma_Y,\ \sigma_{xy} = 0 \tag{19}$$

And

$$\beta' = \begin{cases} \dfrac{1}{1 - 2v + 0.02(\rho^* / \rho_s)(1+v)}, & \text{TG model} \\ \dfrac{[(1-2v)^2 + 1.44(\rho^* / \rho_s)(1+v)^2]^{1/2} - (1-2v)}{0.72(\rho^* / \rho_s)(1+v)^2}, & \text{GAZT model} \\ \left[\dfrac{1 + (\alpha/3)^2}{(1-2v)^2 + (2\alpha/3)^2(1+v)^2}\right]^{1/2}, & \text{DF model} \end{cases} \tag{20}$$

The solutions are similar to those in Subsection 3.1, the only difference is that the parameter β should be replaced by β'

$$d = a\left[\sec\left(\frac{\pi\sigma^{(\infty)}}{2\beta'\sigma_Y}\right) - 1\right] \tag{21}$$

$$\delta_t = \text{CTOD} = \frac{8a\beta'\sigma_Y}{\pi E/(1-v^2)} \ln\sec\left(\frac{\pi\sigma^{(\infty)}}{2\beta'\sigma_Y}\right) \tag{22}$$

4. The development of the generalized cohesive force model

In the applications, the finite size specimens are particular important, which are discussed as following.

4.1 Central crack specimen of finite width

4.1.1 Plane stress state

The approximate analytic solution is obtained as follows

$$\delta_{\mathrm{t}} = \mathrm{CTOD} = \left[\left(\frac{2W}{\pi a}\right)\tan\left(\frac{\pi a}{2W}\right)\right]\frac{8a\beta\sigma_{\mathrm{Y}}}{\pi E}\ln\sec\left(\frac{\pi\sigma^{(\infty)}}{2\beta\sigma_{\mathrm{Y}}}\right) \tag{23}$$

4.1.2 Plane strain state

The solution is

$$\delta_{\mathrm{t}} = \mathrm{CTOD} = \left[\left(\frac{2W}{\pi a}\right)\tan\left(\frac{\pi a}{2W}\right)\right]\frac{8a\beta'\sigma_{\mathrm{Y}}}{\pi E/(1-\nu^{2})}\ln\sec\left(\frac{\pi\sigma^{(\infty)}}{2\beta'\sigma_{\mathrm{Y}}}\right) \tag{24}$$

4.2 Single edge crack specimen with finite width

Another significant specimen is single edge crack specimen, shown as below:

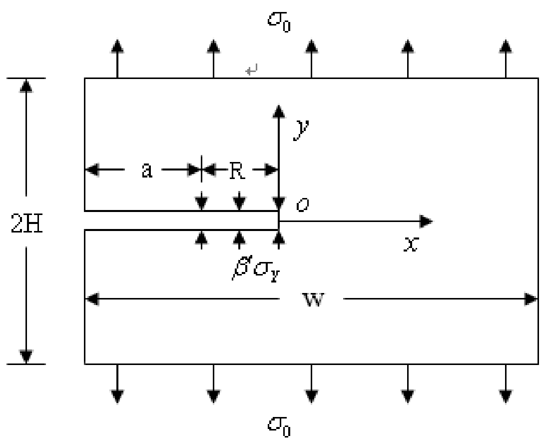

Fig. 3 Single edge crack specimen under tension

4.2.1 Plane stress state

The approximate analytic solution is obtained as follows

$$\delta_{\mathrm{t}} = \mathrm{CTOD} = \frac{4}{\pi}\left[\left(\frac{2W}{\pi a}\right)\tan\left(\frac{\pi a}{2W}\right)\right]\frac{8a\beta\sigma_{\mathrm{Y}}}{\pi E}\ln\sec\left(\frac{\pi\sigma^{(\infty)}}{2\beta\sigma_{\mathrm{Y}}}\right) \tag{25}$$

In the derivation the conformal mapping

$$z = \omega(\zeta) = \frac{2W}{\pi}\arctan\left\{\sqrt{1-\zeta^2}\tan\left(\frac{\pi a}{2W}\right)\right\} - a \tag{26}$$

is used to transform region of the specimen at the physical plane (i.e., the $z-$plane) onto the upper half-plane at the mapping plane (i.e., the $\zeta-$plane).

4.2.2 Plane strain state

The solution for plane strain is

$$\delta_{\mathrm{t}} = \mathrm{CTOD} = \frac{4}{\pi}\left[\left(\frac{2W}{\pi a}\right)\tan\left(\frac{\pi a}{2W}\right)\right]\frac{8a\beta'\sigma_{\mathrm{Y}}}{\pi E}\ln\sec\left(\frac{\pi\sigma^{(\infty)}}{2\beta'\sigma_{\mathrm{Y}}}\right) \tag{27}$$

The numerical results are shown in Fig.4.

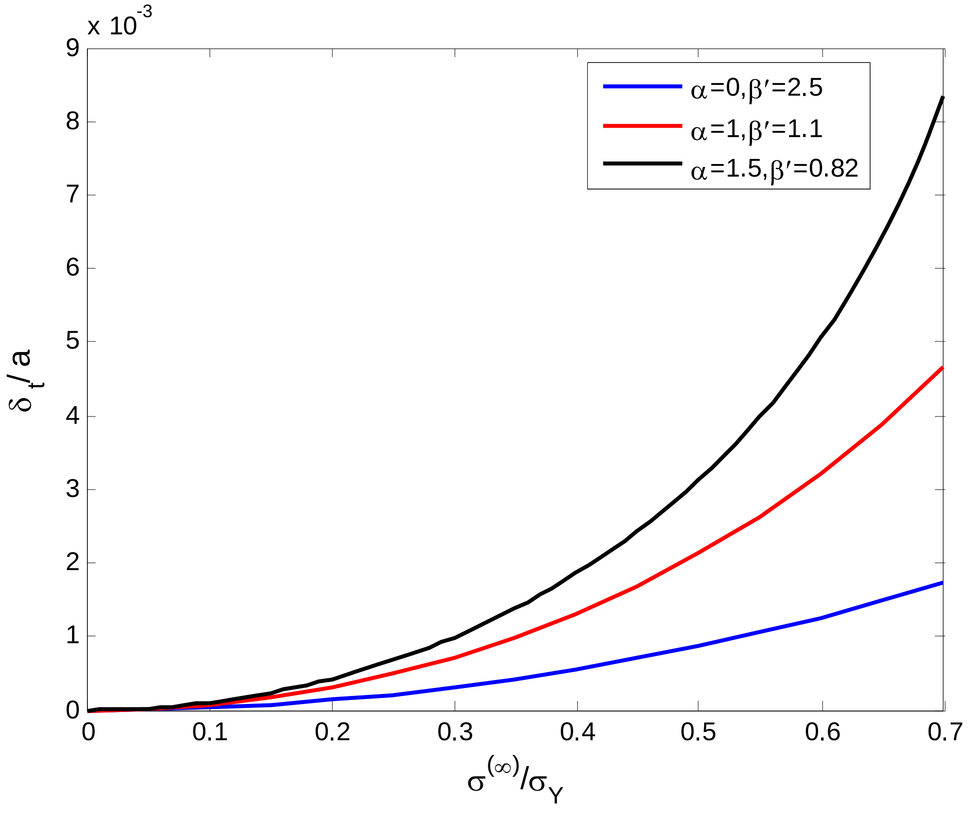

Fig.4Variation of values of δ_t versus applied stressfor finite specimen with a/W =0.3

4.3. Pure bending specimen of central crack

The pure bending specimen is significant in applications.

4.3.1 Plane stress state

The solution is

$$\delta_t = \mathrm{CTOD} = \frac{8\boldsymbol{a}\beta\sigma_{\boldsymbol{Y}}}{\pi\boldsymbol{E}} \ln\sec\left(\frac{\pi\sigma_{\boldsymbol{N}}}{2\beta\sigma_{\boldsymbol{Y}}}\right) \tag{28}$$

$$\sigma_N = 6\frac{M(a+d)}{BW^3} \tag{29}$$

in which M is bending moment, the a crack length, the W width of specimen, and

$$\boldsymbol{d} \approx \boldsymbol{a}\left[\sec\frac{\pi}{2\beta\sigma_{\boldsymbol{Y}}}\left(6\frac{\boldsymbol{Ma}}{\boldsymbol{BW}^3}\right)-1\right] \tag{30}$$

the plastic zone size, and the others are the same defined before.

4.3.2 Plane strain state

The solution for plane strain is

$$\boldsymbol{d} \approx \boldsymbol{a}\left[\sec\frac{\pi}{2\beta'\sigma_{\boldsymbol{Y}}}\left(6\frac{\boldsymbol{Ma}}{\boldsymbol{BW}^3}\right)-1\right] \tag{31}$$

and

$$\delta_t = \mathrm{CTOD} = \frac{8\boldsymbol{a}\beta'\sigma_{\boldsymbol{Y}}}{\pi\boldsymbol{E}/(1-\boldsymbol{v}^2)} \ln\sec\left(\frac{\pi\sigma_{\boldsymbol{N}}}{2\beta'\sigma_{\boldsymbol{Y}}}\right) \tag{32}$$

5. Finite element analysis

Analytic solutions also have their limitations, and numerical method is important as well. We list a part of numerical solutions conducted by finite element method.

5.1 Single edge crack under tension[14]

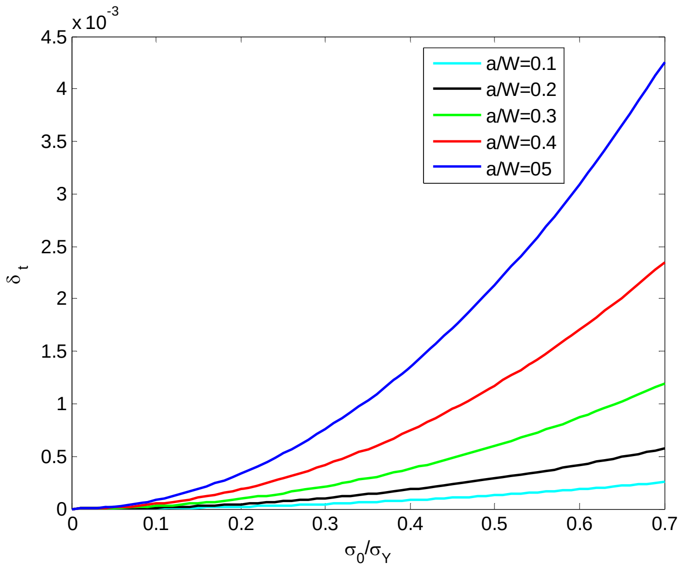

Fig. 5The CTOD of a finite size edge crack specimen of cellular/foam materials indifferent a/w

The comparison between Fig.4 and Fig.5 shows the analytic and numerical solutions are in good agreementwith each other.

5.2 Single edge crack under bending

The elasto-plastic analysis of metal foam in terms of finite element is carried out, the plastic zone around crack tip is shown in Fig.6.

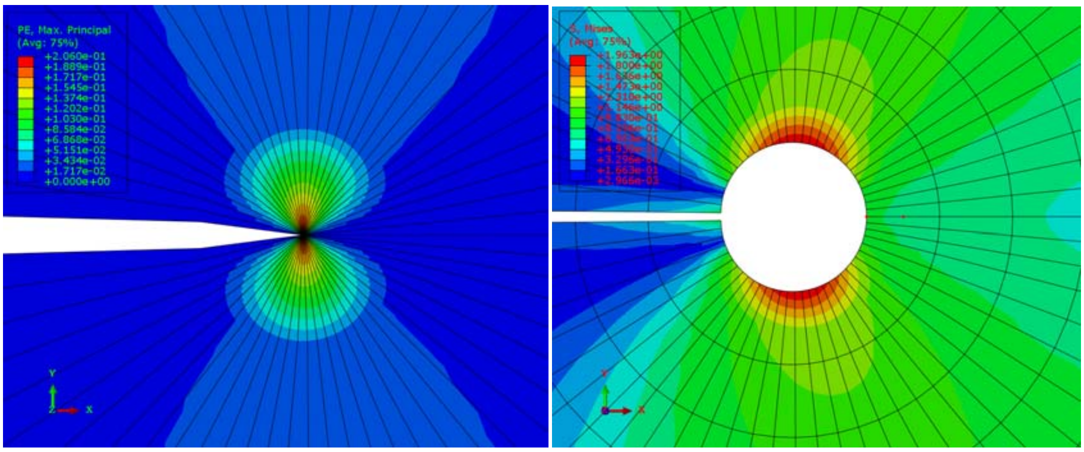

Fig.6The plastic zone around crack tip

6. Crack slowly growth

The crack growth in the metal foams is significant, we discussed two cases and introduce as below.

6.1 TG model [5,15]

In the analysis we developed the singular perturbation, see Fan et al [5]. The constitutive law for TG model is, refer to equations (15) and (16)

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}^{\mathrm{e}}_{ij} + \dot{\varepsilon}^{\mathrm{p}}_{ij} = \frac{1+v}{E}\dot{\sigma}_{ij} - \frac{v}{E}\dot{\sigma}_{kk}\delta_{ij} + \frac{\dot{\hat{\sigma}}}{H(\hat{\sigma})}\frac{\partial \Phi}{\partial \sigma_{ij}} = \frac{1+v}{E}\dot{\sigma}_{ij} - \frac{v}{E}\dot{\sigma}_{kk}\delta_{ij} + \dot{\Lambda} s_{ij}$$
$$\dot{\Lambda} = \frac{1}{H\left(\hat{\sigma}\right)}\left\{\frac{9}{4}\frac{1}{\sigma_e^2}s_{kl}\dot{\sigma}_{kl} + 0.045\frac{\rho^*}{\rho_s}\sigma_m\right\} \qquad (33)$$

The singular perturbation is taken as

$$\sigma_{ij} = \sigma_0 \xi^s \sum_{m=0}^{\infty} \sigma_{ij}^{(m)}(\theta)\xi^{-m}, \xi = \ln\frac{R}{r} \qquad (34)$$

in which σ_0 denotes the yield stress in uniaxial tension, s a parameter to be determined, $\sigma_{ij}^{(m)}(\theta)$ the expansion coefficients, R the size of crack tip, and r the distance measured from crack tip.

The structure of the plastic zones around crack tip and the stress distribution are found by further analysis, but the results are omitted due to the limitation of space.

6.2 DF model[6]

The perturbation analysis and results are also constructed, but omitted due to the limitation of space.

7. Crack fast propagation[5]

Some approximate solutions for crack propagation in the material have been obtained as well but the discussions are omitted here.

8. Fracture criteria

Because the metal foams are plastic material, the fracture presents plastic behaviour, the fracture should be used the crack tip opening criterion rather than stress intensity factor criterion, i.e.,

$$\delta_t = \delta_t^c \qquad (35)$$

in which δ_t^c is the critic value of δ_t, a material constant, measured by experimental tests. In the above the main attention of our analysis lies in the determination of the crack tip opening displacements for various specimens, because the quantity presents the fundamental importance in the plastic fracture analysis.

9. Conclusion and discussion

The above discussion gives a comprehensive introduction on the plasticity of metal foams, plastic fracture theory and some solutions for cracked specimens with infinite and finite sizes. These solutions are useful to the theoretical and experimental studies. In the work the complex analysis is developed.

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