Matched asymptotic expansions in an elastic-creeping material : a new view on the Hui-Riedel equation.

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Abstract

The work deals with the asymptotic stress-strain field around a crack tip, steadily propagating in a viscous material for antiplane conditions. A solution of this problem has been offered by Hui and Riedel, but with some unexpected features. In particular, the solution generally leads to an autonomous crack growth (independent on the loading state). This problem is revisiting here, using a multiscale asympotic analysis. Small scale yielding and low crack velocity are assumed. A small parameter ε, proportional to the crack growth rate, is introduced to switch from the inner solution (close to the crack tip) to the outer one (far field), using an asymptotic expansion of the solution. The outer solution is equivalent to the non linear elastic HRR field at the first order for ε=0, while the viscosity appears at the second order. Close to the crack tip, the viscous effects arise at the first order and the corresponding asymptotic field is governed the elastic field associated to the crack velocity, while the non linear term, corresponding to the nonlinear elasticity emerges at the second order . This is a basic difference with the Hui-Riedel solution where the two scale orders are merged. The matching conditions allow to link the far and close fields, and to correct the paradox whereby the crack velocity should not depend to the far field governed by the loading (except for perfect plasticity ($n \to \infty$) where the solution remains autonomous).

Keywords Hui-Riedel solution, creep, steadily growing crack, singularity, matched asymptotic expansions.

1. Introduction

An antiplane asymptotic solution for a steadily slowly growing crack in an elastic-non linear viscous medium has been suggested by Hui and Riedel [1]. A power law creep is considered. For uniaxial tension, the Norton law has the following form :

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{\mu} + B\sigma^n \qquad (1)$$

For $n > 3$ some paradox events arise, in particular the solution is autonomous, independent on the remote loading (this phenomenon is described by Bui as an analogy to the “soliton” in non linear waves problems [2]). Some authors have corrected this paradox, but with substantive changes to the law or introducing a threshold [3,4]. Keeping the original Norton law, a new antiplane shear analysis is offered here, using a matched asymptotic expansion method. Higher expansion terms of the stress function will allow to connect the « inner solution » (near the viscous crack tip) to the « outer solution » (corresponding to the far HRR field).

2. Initial problem formulation.

Figure 1 shows the crack, embedded inside the body Ω, located in the (x,z) plane at y=0. The stress and strain tensors in antiplane conditions are :

$$\tau_i = \sigma_{3i}, \gamma_i = 2\varepsilon_{3i}, \quad i = 1,2 \quad (2)$$

The equivalent stress is introduced as $\tau_e = (\tau_1^2 + \tau_2^2)^{1/2}$, so that the material law is (with $\bar{B} = \sqrt{3}^{(n+1)} B$) :

$$\dot{\gamma}_i = \frac{\dot{\tau}_i}{\mu} + \bar{B}\tau_e^{n-1}\tau_i, \quad i = 1,2 \quad (3),$$

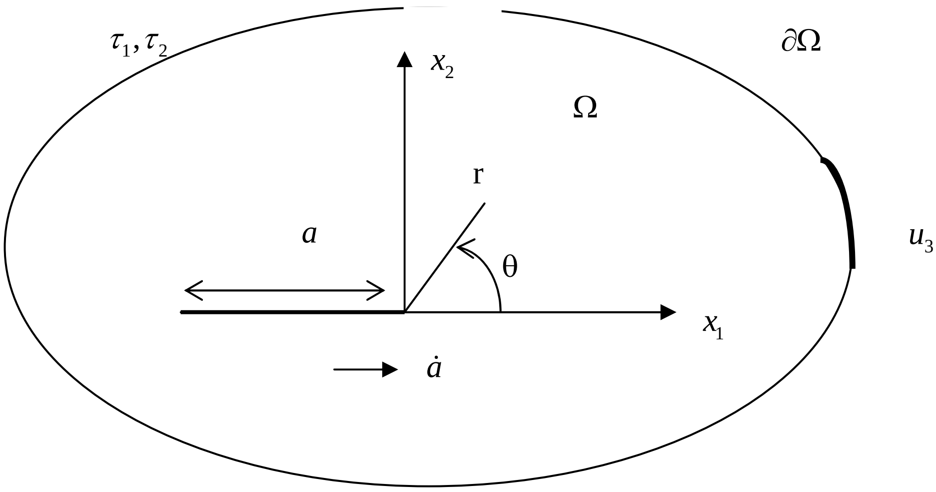

Figure 1 Crack steadily moving with velocity $\dot{a}$ under shear load in mode III.

The equilibrium equations are $\nabla_i \tau_i = 0$, where the summation convention holds. The stress function is then introduced, so that the previous equation is automatically fulfilled :

$$\tau_1 = -\frac{\partial \Psi}{\partial x_2}, \quad \tau_2 = \frac{\partial \Psi}{\partial x_1} \quad (4)$$

The crack is assumed to grow steadily so that in the moving coordinate system the fields remain constant, which involves :

$$\frac{\partial \Psi}{\partial t} = -\dot{a}\frac{\partial \Psi}{\partial x_1} \quad (5)$$

Using the compatibility of the strain rates, the following equation holds everywhere inside the body Ω:

$$-\frac{\dot{a}}{\mu}\frac{\partial \Psi}{\partial x_1}\Delta\Psi + \bar{B}\nabla_i(\tau_e^{n-1}\nabla_i\Psi) = 0 \quad (6)$$

3. Rescaling the problem.

Dimensionless variables will be used now ($\bar{x}_1 \equiv x_1 / a, \bar{x}_2 \equiv x_2 / a, \bar{\Psi}(\bar{x}_1, \bar{x}_2) \equiv \Psi / \mu$). Furthermore, a small parameter $\varepsilon = \frac{\dot{a}}{\bar{B}a\mu^n}$ is introduced, so that the Hui-Riedel relation may be written as :

$$-\varepsilon\,\Delta\frac{\partial\bar{\Psi}}{\partial\bar{x}_1} + \nabla_i\left[\bar{\tau}_e^{n-1}\nabla_i\bar{\Psi}\right] = 0 \quad (7)$$

with the boundary conditions : $\frac{\partial\bar{\Psi}}{\partial\bar{x}_1} = 0, \quad -1 < \bar{x}_1 < 0, \bar{x}_2 = 0$ on the crack lips and $\tau_i(\partial\Omega) = \tau_0$ for the remote loading. Henceforth, the bar will be removed from all the following notations, to lighten the notations. The key ideas are firstly to distinguish two observation scales (the fields in the crack tip vicinity and the far fields), secondly to expand the stress function at higher orders for each scale, and finally to match these asymptotic expansions. This method has been already applied for instance in elastic-plastic materials [5] or for a series of cracks in the frame of elasticity [6,7].

4. Asymptotic expansions of the stress function.

It is supposed that the stress function is asymptotically expanded, and that each i^{th} expansion term is weighed by the parameter ε^{α_i}.

Far from the crack tip, $r >> \varepsilon$ in Ω_o, the expansion will be designed by “outer”, in the vicinity of the crack tip in Ω_i, it will be called “inner”.

4.1. Outer expansion

It is assumed that the stress function may be expanded as :

$$\Psi(x_1, x_2) = \varepsilon^{\alpha_1}\Psi^{(1)}(x_1, x_2) + \varepsilon^{\alpha_2}\Psi^{(2)}(x_1, x_2) + \varepsilon^{\alpha_3}\Psi^{(3)}(x_1, x_2) + \ldots \quad (8)$$

Developing with the previous relation the equivalent stress, we find that :

$$\tau_e^{n-1} = \varepsilon^{\alpha_1(n-1)}(\nabla_i\Psi^{(1)}.\nabla_i\Psi^{(1)} + 2\varepsilon^{\alpha_2-\alpha_1}\nabla_i\Psi^{(1)}.\nabla_i\Psi^{(2)} + \varepsilon^{2(\alpha_2-\alpha_1)}\nabla_i\Psi^{(2)}.\nabla_i\Psi^{(2)} + ...)^{(n-1)/2} \quad (9)$$

The Taylor expansion of the previous expression is then :

$$\tau_e^{n-1} = \varepsilon^{\alpha_1(n-1)}\sigma^{(1)} + \varepsilon^{\alpha_2+\alpha_1(n-2)}\sigma^{(2)} + \varepsilon^{2\alpha_2+\alpha_1(n-3)}\sigma^{(3)} + O(\varepsilon^{\alpha_3+\alpha_1(n-2)}) \quad (10)$$

where $\sigma^{(1)},\sigma^{(2)},\sigma^{(3)}$ are functions depending on $\nabla_i\Psi^{(1)},\nabla_i\Psi^{(2)},n$. Using the relation (10), the equilibrium equation is therefore in Ω_o :

$$\begin{aligned}&-\Delta\frac{\partial}{\partial x_1}(\Psi^{(1)} + \varepsilon^{\alpha_2-\alpha_1}\Psi^{(2)} + \varepsilon^{\alpha_3-\alpha_1}\Psi^{(3)} + ...) + \\ &\varepsilon^{\nu}\nabla_i\left((\sigma^{(1)} + \varepsilon^{\alpha_2-\alpha_1}\sigma^{(2)} + \varepsilon^{2(\alpha_2-\alpha_1)}\sigma^{(3)} + ...)\nabla_i(\Psi^{(1)} + \varepsilon^{\alpha_2-\alpha_1}\Psi^{(2)} + ...)\right) = 0\end{aligned} \quad (11)$$

where $\nu = \alpha_1(n-1)-1$.

The first term in the relation (11) is relative to the viscous behaviour near the crack tip. The second term relative to the non linear behaviour far from the crack tip must be dominant here. Thus :

$$\nu = \alpha_1(n-1)-1 < 0, \quad \forall n \geq 1 \quad (12)$$

The outer equilibrium equation at the first order is then :

$$\nabla_i\left(\sigma^{(1)}\nabla_i\Psi^{(1)}\right) = 0 \quad (13)$$

The following terms order is $\varepsilon^{\alpha_2+\alpha_1(n-2)-1}$ It is assumed now that :

$$\alpha_2 + \alpha_1(n-2) - 1 = 0 \quad (14)$$

This assumption will be explained in a next section, and justified by matching considerations. Therefore, the final equilibrium for the outer expansion, at the first order is :

$$-\Delta\frac{\partial}{\partial x_1}\Psi^{(1)} + \nabla_i\left(\sigma^{(1)}\nabla_i\Psi^{(2)} + \sigma^{(2)}\nabla_i\Psi^{(1)}\right) = 0 \quad (15)$$

with $\sigma^{(1)} = \left\|\nabla\ \Psi^{(1)}\right\|^{n-1}, \sigma^{(2)} = (n-1)\left\|\nabla\ \Psi^{(1)}\right\|^{n-3}\nabla_i\Psi^{(1)}\nabla_i\Psi^{(2)}$

4.2. Inner expansion

In the neighbouring of the crack tip, a new variable designed as : $y_i = \dfrac{x_i}{\varepsilon^{\beta}}$ is used as a microscope focal. This variable change is one of the clue to explain the paradox of the autonomous solution (in the original analysis $\beta = 1$). The inner expansion is then assumed as :

$$\Psi(x_1,x_2) = \varepsilon^{\beta_1}\varphi^{(1)}(y_1,y_2) + \varepsilon^{\beta_2}\varphi^{(2)}(y_1,y_2) + \varepsilon^{\beta_3}\varphi^{(3)}(y_1,y_2) + ... (16)$$

The equilibrium equation in Ω_i is :

$$\begin{aligned}&-\varepsilon^{\delta}\Delta\frac{\partial}{\partial y_1}(\varphi^{(1)}+\varepsilon^{\beta_2-\beta_1}\varphi^{(2)}+\varepsilon^{\beta_3-\beta_1}\varphi^{(3)}+...)+\\&\nabla_i\left((\tau^{(1)}+\varepsilon^{\beta_2-\beta_1}\tau^{(2)}+\varepsilon^{2(\beta_2-\beta)}\tau^{(3)}+...)\nabla_i(\varphi^{(1)}+\varepsilon^{\beta_2-\beta_1}\varphi^{(2)}+...)\right)=0\end{aligned}\qquad(17)$$

where $\delta=\beta\ (n-2)+\beta_1(1-n)+1$. We claim that :

$$\delta=\beta\ (n-2)+\beta_1(1-n)+1<0\quad(18)$$

In fact, this parameter cannot be positive, otherwise the relation (17) would be the same as equ. (13) at the first order (note that in the Hui-Riedel analysis $\delta=0$). At the order ε^{δ}, the equilibrium condition holds :

$$-\Delta\frac{\partial}{\partial y_1}\varphi^{(1)}=0,\quad in\ \Omega_i\quad(19)$$

Transferring the expression (19) into (17), the remaining terms are of order 1 and $\varepsilon^{\delta+\beta_2-\beta_1}$. Both of them must be considered, otherwise the solution regress to the HRR field or to the solution of (19). Therefore :

$$\delta+\beta_2-\beta_1=0\quad(20)$$

Finally, at the first order the inner equilibrium equation is (with $\tau^{(1)}=\left\|\nabla\ \varphi^{(1)}\right\|^{n-1}$) :

$$-\Delta\frac{\partial}{\partial y_1}\varphi^{(2)}+\nabla_i\left(\tau^{(1)}\nabla_i\varphi^{(1)}\right)=0,\quad in\ \Omega_i\quad(21)$$

It may be noticed that at the micro-scale, contrary to the macroscopic scale, the Laplace operator is relative to the second term of the stress function expansion and the non linear term is relative to the first one. Starting from now, it is necessary to solve the equations (15) and (21).

5. Stress functions solutions

5.1. Singularity analysis

From the equation (13), the singular HRR field emerges with :

$$\Psi^{(1)}(x_1,x_2)=K_1r^s f_1(\theta)+more\ regular\ terms\ ,\quad(22)$$

with $s=\dfrac{n}{n+1}$, and where K_1 will be clarified in a further section. Injecting the expression (22) inside the complete equilibrium equation (15) allows to compute the second term $\Psi^{(2)}$:

$$\Psi^{(2)}(x_1, x_2) = K'_1 r^s f_1(\theta) + K_2 r^t f_2(\theta) + \textit{more regular terms} \quad (23)$$

the exponent t is deduced from s, so that :

$$t = \frac{n-2}{n+1} \quad (24)$$

A HRR field with higher order terms has been established for far fields. To clarify the fields close to the crack tip, it is necessary to use matching conditions.

5.2. Matching outer and inner expansions.

For the inner expansion, equilibrium equations (19) and (21) have only free stress boundary conditions on the crack lips, and no outer boundary conditions since Ω_i is unbounded. The matching conditions will substitute to these latter, and involve that in the overlapping area, the inner expansion matches the outer one for small ε values. The asymptotic matching principle will then applied [8]. The first outer term (respectively the first inner term) is rewriting in inner variable (resp. outer variable), with $r = \varepsilon^\beta \rho$ (resp. $(y_1 = \dfrac{x_1}{\varepsilon^\beta}, y_1 = \dfrac{x_1}{\varepsilon^\beta})$, and the relation (22) becomes :

$$\Psi^{(1)}(x_1, x_2) = K_1 \varepsilon^{\alpha_1} \varepsilon^{\beta s} \rho^s f_1(\theta) + ... = \varepsilon^{\alpha_1} \varphi^{(1)}(x_1 / \varepsilon^\beta, x_2 / \varepsilon^\beta) + ... (25)$$

$\forall \varepsilon$, which leads to the supplementary relation :

$$\beta_1 = \alpha_1 + \beta s \quad (26)$$

When $\varepsilon \to 0, \rho \to \infty$, the first order matching holds :

$$\varphi^{(1)}(y_1, y_2) \sim K_1 \rho^s f_1(\theta) \quad (27)$$

Let’s tackle now the resolution of (19).

$$\varphi^{(1)}(y_1, y_2) = K_1 \rho^s f_1(\theta) + \tilde{\varphi}(y_1, y_2) \quad (28)$$

where $\tilde{\varphi}(y_1, y_2)$ is of order $\rho^p,\ p < s,\ \textit{for}\ \rho \to \infty$

Moreover, the classical solution of linear elasticity is valid for the relation (19) :

$$\hat{\varphi}(y_1, y_2) = K_{III} \rho^{1/2} \sin \frac{\theta}{2} \quad (29)$$

Inserting the first term of (28) into (19) :

$$\zeta(y_1, y_2) = \Delta \frac{\partial \rho^s f_1(\theta)}{\partial y_1} \quad (30)$$

and reporting (28),(29),(30) in (19), the function $\tilde{\varphi}(y_1, y_2)$ may be found, solving the problem :

$$\begin{cases} -\Delta \dfrac{\partial \tilde{\varphi}}{\partial y_1} = K_1 \zeta \ in\ \Omega_i, \\ \dfrac{\partial \tilde{\varphi}}{\partial y_2} = 0, \forall y_1 \in]-\infty, 0[, \\ \tilde{\varphi} \to \hat{\varphi}\ for\ |y| \to \infty \end{cases} \quad (31)$$

Using the same way as in (25), the expansions second terms are matched :

$$\Psi(x_1, x_2) = K_1 \varepsilon^{\alpha_1} \varepsilon^{\beta_1} \rho^s f_1(\theta) + \varepsilon^{\alpha_2} \varepsilon^{\beta s} \rho^s f_1(\theta) + . = \varepsilon^{\beta_1} \varphi^{(1)}(x_1 / \varepsilon^{\beta}, x_2 / \varepsilon^{\beta}) + \varepsilon^{\beta_2} \varphi^{(2)}(x_1 / \varepsilon^{\beta}, x_2 / \varepsilon^{\beta}) \quad (32)$$

and the supplementary condition holds :

$$\alpha_2 = \alpha_1 - \frac{1}{2}\beta + \beta s \quad (33)$$

5.3. Complementary matching using path-independent integrals.

The problem unknowns are $(\alpha_1, \alpha_2, \beta_1, \beta_2, \beta)$ and considerations about dominant terms at each scale allow us to determine four relations (14,19,27,33). The missing relation will be built up starting from energy considerations. It is well known that for HRR fields [9,10] the crack-tip fracture behaviour may be characterized by a Γ path-independent line integral, where Γ is a line circumscribing the crack tip :

$$J = \int_{\Gamma} (\omega n_1 - \sigma . n \frac{\partial u}{\partial x_1})\, ds \quad (34)$$

where ω is the material strain energy density and n the outer normal to the line Γ. For a steadily moving crack under creep conditions, a similar path-independent has been offered [11,12] :

$$C^* = \int_{\Gamma} (\omega n_1 - \sigma . n \frac{\partial u}{\partial y_1})\, ds \quad (35)$$

Developing the two integrals, respectively for the far and the near crack tip fields, the following relations hold :

$$\begin{cases} J = \varepsilon^{(n+1)\alpha_1} K_1^{n+1} I \\ C^* = \varepsilon^{2\beta_1 - \beta} K_{III}^2 \end{cases} \quad (36)$$

In the transition area, the integral values have to coincide, so that :

$$2\beta_1 - \beta = (n+1)\alpha_1 \quad (37)$$

5.4. Summary of the results.

The equations (14,19,27,33,35) may be summarized :

$$\begin{cases} \delta - \beta_1 + \beta_2 = 0 \\ \alpha_2 + \alpha_1(n-2) - 1 = 0 \\ \beta_1 = \alpha_1 + \beta s, \qquad with \quad s = \dfrac{n}{n+1} \\ \alpha_2 = \beta_2 - \beta p, \qquad with \quad p = \dfrac{n+2}{n+1} \\ 2\beta_1 - \beta = (n+1)\alpha_1 \end{cases} \quad (38)$$

with $\delta = \beta\ (n-2) + \beta_1(1-n) + 1 < 0$

which leads to the solutions :

$$\begin{cases} \beta = \beta_1 = \dfrac{(n+1)}{(n-1)} \\ \alpha_1 = \dfrac{1}{(n-1)} \\ \alpha_2 = \dfrac{1}{(n-1)} \\ \beta_2 = \dfrac{n+3}{(n-1)} \end{cases} \quad (39)$$

The parameter β depends on the hardening coefficient. For $\beta = 1$, no zoom is available (this is the case of the Hui-Riedel analysis) so that the connection between the remote fields and the viscous fields is impossible (unless $n < 3$). When $n \to \infty$ (perfect plasticity), $\beta \to 1$, there is no matching again, and therefore an autonomous solution, but in an asymptotic way, which is natural because the material yields without supplementary loading.

6. Conclusions and outlines.

The matched asymptotic expansion method affords to connect the H.R.R. far fields and the fields near a crack tip steadily moving under creep conditions, as described in the Hui-Riedel analysis. The space variable magnifying designed by ε^{β} allows to adjust the viscous dominant area size with respect to the crack velocity and the material properties. The Hui-Riedel analysis is a particular case where no zoom is used ($\beta = 1$). A significant work remains to achieve a complete solution setting up. A further step in the analysis is to used this scaling method with the time variable, so as to break the assumption $\frac{\dot{a}}{B\mu^{n}} << 1$. The study must be also completed by the angular functions resolve.

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