On Mixed-Mode Fracture

Christopher Harvey[1], Liangliang Guan[1], Huimin Xie[2], Simon Wang[1,*]

[1] Department of Aeronautical and Automotive Engineering, Loughborough University, Loughborough, LE11 3TU, UK
[2] Department of Engineering Mechanics, Tsinghua University, Beijing 100084, China
* Corresponding author: s.wang@lboro.ac.uk

Abstract This paper reports the authors' recent work on mixed-mode fracture in fiber-reinforced laminated composite beams and plates. The work considers the so-called one-dimensional fracture which propagates in one-dimension and consists of only mode I and mode II fracture modes. Fracture interfaces are assumed to be either rigidly or cohesively bonded. Analytical theories are developed within the contexts of both classical and first-order shear deformable laminated composite theories. When a rigid interface is assumed for brittle fracture, there are two sets of orthogonal pure modes in classical theory, and there is only one set of orthogonal pure modes in shear-deformable theory. A mixed-mode fracture is partitioned by using these orthogonal pure modes. The classical and shear deformable partitions can be regarded as either lower or upper bound partitions for 2D elasticity, and hence approximate 2D elasticity partition theories are developed by 'averaging' the classical and shear deformable partitions. When cohesively bonded interfaces are assumed for adhesively joined interfaces, the classical and shear deformable theories give the same pure modes. Approximate partition theories are also developed for 2D elasticity. Numerical investigations demonstrate excellent agreement with the corresponding analytical theories. Experimental data considered shows that the failure locus is strongly linear.

Keywords Composite, Energy release rate, Failure locus, Mixed-mode fracture, Orthogonal pure modes

1. Introduction

Delamination is a major concern in the application of laminated composite materials. Although it occurs often together with other fracture modes such as fiber breakage, matrix cracking and intra-laminar cracking, pure delamination is always an important research topic which provides insight and understanding of lamina interfacial mechanics, and it often occurs in one-dimensional delamination. A delamination is called one-dimensional when its crack front propagates only in one direction. Familiar examples are through-width delamination in laminated composite beams, circular ring shape delamination in laminated composite plates and shells, etc., as shown in Fig. 1. A distinct feature of one-dimensional delamination is that it usually consists of only the mode I and mode II fracture modes without any mode III. The study of one-dimensional delamination is of great importance for several reasons. It is the most fundamental problem in the fracture mechanics of materials. It is often used in experimental tests, such as the double cantilever beam (DCB), end-loaded split (ELS) and end-notched flexure (ENF) tests, to obtain the critical energy release rate (ERR) or toughness of a lamina interface in either pure mode I or mode II delamination. In the case of a mixed mode, it is often used to investigate delamination propagation criteria. Moreover, many practical cases of delamination in structures made of fiber-reinforced laminated composites can be approximated as one-dimensional. For example, the separation of stiffeners and skins in stiffened plate or shell panels made of laminated composite materials can be approximated as one-dimensional through-width delamination, and the separation of two material layers in laminated composite plates and shells in a drilling process can be approximated as one-dimensional circular ring-type delamination, etc.

Because of its importance, one-dimensional delamination has attracted the attention of many researchers including many of the world leaders in the areas of fracture mechanics and composite materials. The primary goal is to develop analytical theories to determine pure delamination modes

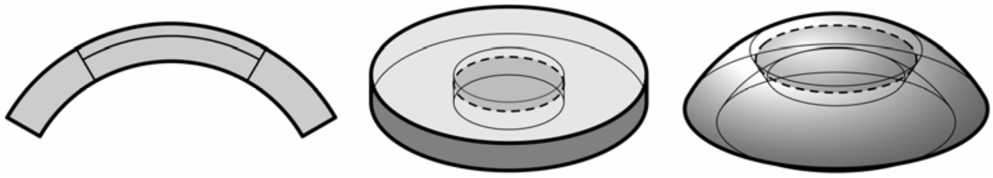

Figure 1. Some examples of one-dimensional fracture.

and then to partition a mixed mode into pure modes. Delamination propagation criteria can then be established by using the partition together with experimental data. The through-width delamination in a DCB made of isotropic material with rigidly bonded interface can be considered to be the 'simplest' one-dimensional delamination. Although it seems to be a straightforward matter to determine the pure modes and to partition a mixed mode, it has been proved to be an extremely complex and sophisticated problem. There has been a lot of confusion on the matter during the last 25 years. Ref. [1] may be the earliest work on the 'simplest' problem by Williams. A mixed-mode partition theory [1] was developed based on classical beam theory. Ref. [2] reported a combined numerical and analytical theory by Schapery and Davidson based on combined classical beam theory and 2D elasticity. It disagrees with the Williams' theory [1] and concludes that classical beam theory does not provide quite enough information to obtain an analytical decomposition of the mixed-mode ERR into its opening and shearing mode components. Hutchinson-Suo reported their work in Ref. [3] in which the mixed-mode ERR is calculated based on the classical beam theory but the partition of ERR is calculated based on stress intensity factors from 2D elasticity. Their theory [3] agrees well with the theory in Ref. [2] and claims that Williams' theory [1] contains conceptual errors. To respond to this claim, Williams reported some experimental work in Ref. [4] showing that Williams' theory [1] is in a better agreement with the test results than Hutchinson-Suo theory [3]. This has caused a lot of confusion, which has affected many academic researchers and design engineers until today. A great deal of research effort has been made during the last two decades to resolve the confusion. Among many others, the following significant works are referenced here. Ref. [5] reported a mixed-mode partition theory for laminated composite beams with rigidly bonded interface based on first-order shear-deformable beam theory, which gives different mixed-mode partitions to those from Williams' theory [1] and the Hutchinson-Suo theory [3]. The same theory as that in Ref. [5] was derived in Refs. [6, 7] but these are based on classical beam theory, which caused yet more confusion. Recently, the authors have developed analytical theories for one-dimensional delamination in laminated composite beams and plates by using a novel methodology [8–13]. All the confusion is explained. This paper reports some of the major results in Refs. [8–13].

2. Partition of mixed-mode fracture in laminated composite beams and plates with rigid interfaces

The mechanics of delamination depend on the mechanical properties of lamina interfaces. A lamina interface is considered to be a rigid interface when the interface separation is negligible before an existing delamination propagates. Otherwise, it is considered to be a non-rigid interface or as it often called, a cohesively bonded interface. Bare-bonded interfaces in the conventional manufacturing process from glass or carbon fiber epoxy pre-pregs are typical rigid interfaces because of their brittleness. While cohesively bonded interfaces are typical non-rigid interfaces which are achieved by adding adhesive layers between bare plies when manufacturing components.

2.1. Laminated composite DCBs

A laminated composite DCB with a delamination of length a is shown in Fig. 2 (a). The interface stresses in Fig. 2 (b) only show the sign convention rather than any representative distribution.

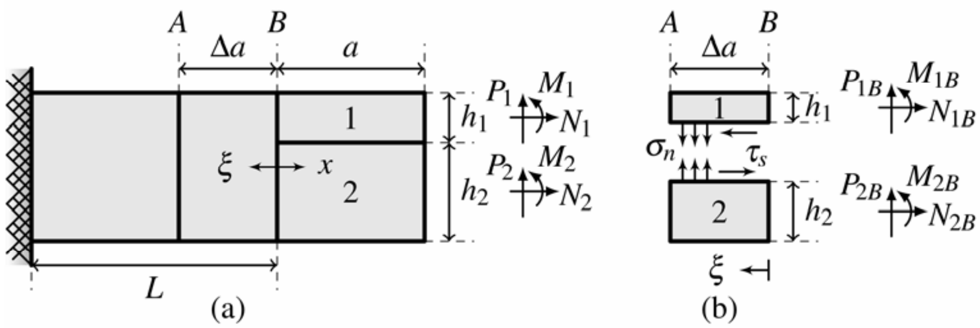

Figure 2. A laminated composite DCB and its loading conditions. (a) General description. (b) Details of the crack influence region Δa.

2.1.1. Classical beam partition theory

Using the constitutive relation in classical laminated composite beam theory, the ERR at the crack tip at location B, G is

$$G = \frac{1}{2b^2}\left(\frac{M_{1B}^{\ 2}}{D_1^*}+\frac{M_{2B}^{\ 2}}{D_2^*}-\frac{M_B^{\ 2}}{D^*}+\frac{N_{1B}^{\ 2}}{A_1^*}+\frac{N_{2B}^{\ 2}}{A_2^*}-\frac{N_B^{\ 2}}{A^*}-\frac{2B_1M_{1B}N_{1B}}{B_1^*}-\frac{2B_2M_{2B}N_{2B}}{B_2^*}+\frac{2BM_BN_B}{B^*}\right) \quad (1)$$

where subscript 'B' indicates loads at the crack tip at location B, for example, M_{1B} is the bending moment on the top sub-laminate at the crack tip. These loads are shown in Fig. 2 (b). Other quantities in Eq. (1) are

$$A_i^* = A_i - B_i^2/D_i \quad , \quad B_i^* = B_i^{\ 2} - A_iD_i \quad , \quad D_i^* = D_i - B_i^2/A_i \quad (2)$$

The range of subscript i is 1 and 2, which again refers to the upper and lower sub-laminates respectively. For the intact laminate, the subscript i is dropped. A, B and D are the equivalent extensional, coupling and bending stiffness of the DCB respectively.

A novel methodology to partition mixed-mode ERR G in Eq. (1) arises from the fact that G is of quadratic form and non-negative definite in terms of the crack tip bending moments M_{1B} and M_{2B}, and the crack tip axial forces N_{1B} and N_{2B}. An analogy of this is the positive definite kinetic energy of a vibrating structure, to which individual modal energies are attributed by using modal analysis from orthogonal natural vibration modes. A hypothesis is then made that the total ERR in a mixed-mode delamination can be partitioned into pure mode components by using orthogonal pure modes. There are two sets of fundamental orthogonal pure modes. The first set corresponds to zero relative shearing displacement just behind the crack tip (mode I) and zero crack tip opening force ahead of the crack tip (mode II). The second set corresponds to zero relative opening displacement just behind the crack tip (mode II) and zero crack tip shearing force (mode I). It is simple to derive the zero relative displacement modes first and then to find the zero force modes by applying orthogonality through Eq. (1). An alternative and more complex derivation considers the interface stresses. If the mode vector form is $\{M_{1B}, M_{2B}, N_{1B}, N_{2B}\}^T$, then the first set of fundamental orthogonal pure modes, referred to as the $\{\theta, \beta\}$ set, are found to be

$$\{\varphi_{\theta_1}\}=\begin{Bmatrix}1\\ \theta_1\\ 0\\ 0\end{Bmatrix} \ , \ \{\varphi_{\theta_2}\}=\begin{Bmatrix}1\\ 0\\ \theta_2\\ 0\end{Bmatrix} \ , \ \{\varphi_{\theta_3}\}=\begin{Bmatrix}1\\ 0\\ 0\\ \theta_3\end{Bmatrix} \ , \ \{\varphi_{\beta_1}\}=\begin{Bmatrix}1\\ \beta_1\\ 0\\ 0\end{Bmatrix} \ , \ \{\varphi_{\beta_2}\}=\begin{Bmatrix}1\\ 0\\ \beta_2\\ 0\end{Bmatrix} \ , \ \{\varphi_{\beta_3}\}=\begin{Bmatrix}1\\ 0\\ 0\\ \beta_3\end{Bmatrix} \quad (3)$$

with

$$\theta_1 = \frac{(B_2^2 - A_2 D_2)(B_1 + h_1 A_1/2)}{(B_1^2 - A_1 D_1)(B_2 - h_2 A_2/2)} \ , \ \theta_2 = -\frac{B_1 + h_1 A_1/2}{D_1 + h_1 B_1/2} \ , \ \theta_3 = \frac{(B_1 + h_1 A_1/2)(B_2^2 - A_2 D_2)}{(D_2 - h_2 B_2/2)(B_1^2 - A_1 D_1)} \tag{4}$$

$$\beta_1 = -\frac{D_2^*(D_1^* + D_1^*\theta_1 - D^*)}{D_1^*(D_2^* + D_2^*\theta_1 - D^*\theta_1)} \ , \ \beta_2 = \frac{\theta_2\left(\frac{h_2}{2D^*} - \frac{B_1}{B_1^*} + \frac{B}{B^*}\right) + \frac{1}{D_1^*} - \frac{1}{D^*}}{\theta_2\left(\frac{Bh_2}{B^*} - \frac{1}{A_1^*} + \frac{1}{A^*} + \frac{h_2^2}{4D^*}\right) - \frac{h_2}{2D^*} + \frac{B_1}{B_1^*} - \frac{B}{B^*}} \ ,$$

$$\beta_3 = \frac{\theta_3\left(\frac{h_1}{2D^*} - \frac{B}{B^*}\right) - \frac{1}{D_1^*} + \frac{1}{D^*}}{\theta_3\left(\frac{Bh_1}{B^*} + \frac{1}{A_2^*} - \frac{1}{A^*} - \frac{h_1^2}{4D^*}\right) - \frac{h_1}{2D^*} + \frac{B}{B^*}} \tag{5}$$

The second set of fundamental orthogonal pure modes, referred to as the $\{\theta', \beta'\}$ set, has the same format as that of the first set in Eq. (3), but with

$$\theta_1' = -1 \ , \ \theta_2' = \frac{\beta_2'\left(\frac{h_2}{2D^*} - \frac{B_1}{B_1^*} + \frac{B}{B^*}\right) + \frac{1}{D_1^*} - \frac{1}{D^*}}{\beta_2'\left(\frac{Bh_2}{B^*} - \frac{1}{A_1^*} + \frac{1}{A^*} + \frac{h_2^2}{4D^*}\right) - \frac{h_2}{2D^*} + \frac{B_1}{B_1^*} - \frac{B}{B^*}} \ ,$$

$$\theta_3' = \frac{\beta_3'\left(\frac{h_1}{2D^*} - \frac{B}{B^*}\right) - \frac{1}{D_1^*} + \frac{1}{D^*}}{\beta_3'\left(\frac{Bh_1}{B^*} + \frac{1}{A_2^*} - \frac{1}{A^*} - \frac{h_1^2}{4D^*}\right) - \frac{h_1}{2D^*} + \frac{B}{B^*}} \tag{6}$$

$$\beta_1' = \frac{D_2^*}{D_1^*} \ , \ \beta_2' = -\frac{A_1}{B_1} \ , \ \beta_3' = -\frac{B_2^*}{D_1^* B_2} \tag{7}$$

Any four fundamental pure modes from either the first set or the second set can be used to partition a mixed mode. The partitions are given below.

$$G_{IE} = c_{IE}\left(M_{1B} - \frac{M_{2B}}{\beta_1} - \frac{N_{1B}}{\beta_2} - \frac{N_{2B}}{\beta_3}\right)\left(M_{1B} - \frac{M_{2B}}{\beta_1'} - \frac{N_{1B}}{\beta_2'} - \frac{N_{2B}}{\beta_3'}\right) \tag{8}$$

$$G_{IIE} = c_{IIE}\left(M_{1B} - \frac{M_{2B}}{\theta_1} - \frac{N_{1B}}{\theta_2} - \frac{N_{2B}}{\theta_3}\right)\left(M_{1B} - \frac{M_{2B}}{\theta_1'} - \frac{N_{1B}}{\theta_2'} - \frac{N_{2B}}{\theta_3'}\right) \tag{9}$$

where

$$c_{IE} = G_{\theta_1}\left[\left(1 - \frac{\theta_1}{\beta_1}\right)\left(1 - \frac{\theta_1}{\beta_1'}\right)\right]^{-1} \ , \ c_{IIE} = G_{\beta_1}\left[\left(1 - \frac{\beta_1}{\theta_1}\right)\left(1 - \frac{\beta_1}{\theta_1'}\right)\right]^{-1} \tag{10}$$

and

$$G_{\theta_1} = \frac{1}{2b^2}\left[\frac{1}{D_1^*} + \frac{\theta_1^2}{D_2^*} - \frac{(1+\theta_1)^2}{D^*}\right] \ , \ G_{\beta_1} = \frac{1}{2b^2}\left[\frac{1}{D_1^*} + \frac{\beta_1^2}{D_2^*} - \frac{(1+\beta_1)^2}{D^*}\right] \tag{11}$$

The partitions in Eqs. (8) and (9) use both sets of orthogonal pure modes. The partition theory in Ref. [1] only gives the $\{\theta_1', \beta_1'\}$ pure modes correctly. The partition theories derived in Refs. [6,7] is equivalent to using only the first set of pure modes to partition a mixed-mode. The methodologies used in Refs. [6,7] are not able to find the second set of pure modes. The partitions are easily reduced for isotropic materials. With a thickness ratio $\gamma = h_2/h_1$ now introduced, they are

$$G_{IE} = c_{IE}\left(M_{1B} - \frac{M_{2B}}{\beta_1} - \frac{N_{1Be}}{\beta_2}\right)\left(M_{1B} - \frac{M_{2B}}{\beta_1'}\right) \tag{12}$$

$$G_{IIE} = c_{IIE}\left(M_{1B} - \frac{M_{2B}}{\theta_1} - \frac{N_{1Be}}{\theta_2}\right)\left(M_{1B} - \frac{M_{2B}}{\theta_1'} - \frac{N_{1Be}}{\theta_2'}\right) \tag{13}$$

where c_{IE} and c_{IIE} are still given by Eq. (10) and $N_{1Be} = N_{1B} - N_{2B}/\gamma$. The pure mode relationships are now as follows:

$$\theta_1 = -\gamma^2 \ , \ \theta_2 = -\frac{6}{h_1} \ , \ \beta_1 = \frac{\gamma^2(3+\gamma)}{1+3\gamma} \ , \ \beta_2 = \frac{2(3+\gamma)}{h_1(\gamma-1)} \text{ for } \gamma \neq 1 \ , \ \beta_2 = 1 \text{ for } \gamma = 1 \tag{14}$$

$$\theta_1' = -1 \ , \ \theta_2' = -\frac{6(1+\gamma)}{h_1(1+\gamma^3)} \ , \ \beta_1' = \gamma^3 \tag{15}$$

The isotropic G_{θ_1} and G_{β_1} for use in Eq. (10) are

$$G_{\theta_1} = \frac{24\gamma}{Eb^2h_1^3(1+\gamma)} \ , \ G_{\beta_1} = \frac{72\gamma(1+\gamma)}{b^2Eh_1^3(1+3\gamma)^2} \tag{16}$$

2.1.2. Shear deformable beam partition theory

In the absence of crack tip shear forces, the total ERR G in a mixed-mode fracture is still given by Eq. (1) within the context of the first order shear deformable laminated composite beam theory. However, the two sets of fundamental orthogonal pure modes now coincide at the first set, i.e. the $\{\theta, \beta\}$ set and the partitions of the total G are given by

$$G_{IT} = c_{IT}\left(M_{1B} - \frac{M_{2B}}{\beta_1} - \frac{N_{1B}}{\beta_2} - \frac{N_{2B}}{\beta_3}\right)^2 \ , \ G_{IIT} = c_{IIT}\left(M_{1B} - \frac{M_{2B}}{\theta_1} - \frac{N_{1B}}{\theta_2} - \frac{N_{2B}}{\theta_3}\right)^2 \tag{17}$$

where

$$c_{IT} = G_{\theta_1}\left(1 - \frac{\theta_1}{\beta_1}\right)^{-2} \ , \ c_{IIT} = G_{\beta_1}\left(1 - \frac{\beta_1}{\theta_1}\right)^{-2} \tag{18}$$

When crack tip shear forces P_{1B} , P_{2B} are present, the following two terms need to be added to the mode I ERR in Eq. (17):

$$G_P = \frac{(H_1P_{2B} - H_2P_{1B})^2}{2b^2H_1H_2(H_1+H_2)} \ , \ \alpha_{\theta_1}\Delta G_{\theta_1 P} = \frac{\alpha_{\theta_1}}{b^2}\left(\frac{P_{1B}}{H_1} - \frac{P_{2B}}{H_2}\right)\left[\frac{H_1H_2}{H_1+H_2}\left(\frac{1}{D_1^*} + \frac{\theta_1^2}{D_2^*} - \frac{(1+\theta_1)^2}{D^*}\right)\right]^{1/2} \tag{19}$$

where H_1 and H_2 are the through-thickness shear stiffnesses and

$$\alpha_{\theta_1} = \frac{M_{2B}\beta_2 + N_{1B}\beta_1 - M_{1B}\beta_1\beta_2}{\beta_2(\theta_1 - \beta_1)} + \frac{N_{2B}\beta_1}{\beta_3(\theta_1 - \beta_1)} \tag{20}$$

In the case of layered isotropic DCBs, these partitions reduce to

$$G_{IT} = c_{IT}\left(M_{1B} - \frac{M_{2B}}{\beta_1} - \frac{N_{1Be}}{\beta_2}\right)^2 \ , \ G_{IIT} = c_{IIT}\left(M_{1B} - \frac{M_{2B}}{\theta_1} - \frac{N_{1Be}}{\theta_2}\right)^2 \tag{21}$$

The mode I contribution from crack tip shear forces reduces to

$$G_P = \frac{(\gamma P_{1B} - P_{2B})^2}{2b^2h_1k^2G_{xz}\gamma(1+\gamma)} \ , \ \alpha_{\theta_1}\Delta G_{\theta_1 P} = \frac{4\sqrt{3}\alpha_{\theta_1}(\gamma P_{1B} - P_{2B})}{b^2h_1^2(1+\gamma)(k^2G_{xz}E)^{1/2}} \tag{22}$$

2.1.3. 2D elasticity partition theory

One averaged partition theory is obtained by averaging the classical and shear deformable partitions. This partition has been found to give an excellent approximation to the partition from 2D elasticity. The mode I and II components of the ERR from the averaged partition theory denoted by G_I and G_{II} respectively. They are

$$G_I = \left(G_{IE} + G_{IT}\right)/2 + G_P + \alpha_{\theta_1}\Delta G_{\theta_1 P} \quad , \quad G_{II} = \left(G_{IIE} + G_{IIT}\right)/2 \tag{23}$$

2.1.4. Local and global partition theories

When ERR is calculated right at the crack tip, i.e. using an infinitesimally small region around the crack tip, it is called a local calculation. When it is calculated using a finite small region, it is called a global calculation. In terms of the finite element method (FEM), an infinitesimally small region means one element length in a very fine mesh, whilst a finite small region means multiple element lengths. When global ERR calculation is used, the above three local partition theories, i.e. the classical, shear-deformable and 2D partition theories give the same partitions as that of the local classical partition theory. That is, the classical partition theory unifies the three theories in a global partition. The differences between the three local theories arise from the differences of the crack tip stresses in the three theories. However, the global distribution of interfacial stresses is governed by the classical beam and plate theory.

2.2. Clamped-clamped laminated composite beams

A clamped-clamped composite laminated beam with a symmetric delamination is considered. The loads P_1 and P_2 are applied at the mid-span. The pure mode I mode in the first set of orthogonal pure modes in classical beam theory, i.e. the $\{\theta, \beta\}$ set, is given by

$$P_2/P_1 = \theta_P = -B_2^*\left(2B_1 + h_1 A_1\right)/\left[B_1^*\left(2B_2 - h_2 A_2\right)\right] \tag{24}$$

Its orthogonal pure mode II mode $P_2/P_1 = \beta_P$ is too complex to be presented here algebraically. The second set of orthogonal pure modes in classical beam theory, i.e. the $\{\theta', \beta'\}$ set is given by

$$\theta'_P = P_2/P_1 = -1 \quad , \quad P_2/P_1 = \beta'_P = D_2^*/D_1^* \tag{25}$$

Within the context of shear deformable beam theory, the expressions for $P_2/P_1 = \theta_P$ pure mode I and $P_2/P_1 = \beta_P$ pure mode II are too complex to be presented here algebraically. However, when the through-thickness shear effect is not excessively large, they are very close to those in classical beam theory.

2.3. Clamped circular layered isotropic plates

A clamped circular layered isotropic plate with a central delamination and central loads P_1 and P_2 are considered. The first set of orthogonal pure modes in classical plate theory are found to be

$$P_2/P_1 = \theta_P = \theta_1 \quad , \quad P_2/P_1 = \beta_P = \beta_1 \tag{26}$$

where θ_1 and β_1 are given in Eq. (14). The corresponding ERRs are given by

$$G_{\theta_P} = 3P_1^2\gamma\left(1-\nu^2\right)/\left[2Eh_1^3\pi^2\left(1+\gamma\right)\right] \quad , \quad G_{\beta_P} = 9P_1^2\gamma\left(1-\nu^2\right)\left(1+\gamma\right)/\left[2Eh_1^3\pi^2\left(1+3\gamma\right)^2\right] \tag{27}$$

The second set of pure mode I and II modes are the same as those in Eq. (15). In the first order shear deformable plate theory, the first set of pure modes is approximately pure and the second set disappears.

3. Partition of mixed-mode fracture in layered isotropic DCBs with non-rigid interfaces

3.1. Classical beam partition theory

The mode I ERR G_{IE} is considered first. The interface normal stress σ_n is found to be

$$\sigma_n = -Eh_1^3\gamma^3 / \left[3(1+\gamma)^3\right]\left[\overline{w}^{(4)} + 3(1-\gamma)/(2h_1\gamma)\overline{u}^{(3)}\right] \tag{28}$$

where $\overline{w} = \overline{w}_1 - \overline{w}_2$ and $\overline{u} = \overline{u}_2 - \overline{u}_1$ are the relative opening and shearing displacements at interface. The mode I ERR is then found by using *J*-integral.

$$G_{IE} = \lim_{da\to 0}\left\{\frac{1}{da}\int_0^{da}\int_0^{\overline{w}} \sigma_n d\overline{w}dx\right\} = \int_0^{\overline{w}_B} \sigma_{nB} d\overline{w}_B \tag{29}$$

Substituting Eq. (28) into Eq. (29) gives

$$G_{IE} = G_{IE}^L + (1+3\gamma)/[b(1+\gamma)^3](P_{2B} - \beta_1 P_{1B})\overline{w}_B^{(1)} \tag{30}$$

The first term $G_{IE}^L = G_{IT}$ in Eq. (20) and the $\overline{w}_B^{(1)}$ in the second term is the relative crack tip rotation. It is seen that Eq. (30) is not completely analytical due to the second term. It is more important to note that the second set of orthogonal pure modes is not present. The mode II ERR can be considered similarly. The interface shear stress τ_s is found to be

$$\tau_s = \tau_{sP} + \tau_{s\sigma} + \tau_{s\overline{u}} \tag{31}$$

with

$$\tau_{sP} = 3\left(\gamma^2 P_{1B} + P_{2B}\right)/\left[2bh_1\gamma(1+\gamma)\right] \ , \ \tau_{s\sigma} = 3(1-\gamma)/(2h_1\gamma)\int_0^x \sigma_n dx \ , \ \tau_{s\overline{u}} = Eh_1\gamma\overline{u}^{(2)}/\left[4(1+\gamma)\right] \tag{32}$$

The mode II ERR G_{IIE} is then calculated by using *J*-integral.

$$G_{IIE} = G_{IIE}^L + \int_0^{\overline{u}_B} \tau_{sP} d\overline{u}_B \tag{33}$$

The first term $G_{IIE}^L = G_{IIT}$ in Eq. (20) and the second term can be calculated for a given cohesive law.

3.2. Shear deformable beam partition theory

It is simple to verify that the mode II ERR G_{IIT} remains the same as the G_{IIE} in Eq. (33). However, the mode I ERR G_{IT} needs reconsideration. The governing equation for the interface normal stress σ_n is

$$\sigma_n^{(2)} - \lambda^2\sigma_n = \alpha\left(\overline{w}^{(4)} + 3(1-\gamma)/(2h_1\gamma)\overline{u}^{(3)}\right) \tag{34}$$

where $\lambda = (1+\gamma)/(\gamma h_1)\left(3k^2G_{xz}/E\right)^{1/2}$ and $\alpha = k^2G_{xz}h_1\gamma/(1+\gamma)$. By using the method of parameter variation, the solution to Eq. (34) is found.

$$\begin{aligned}\sigma_n &= c_1e^{\lambda x} + c_2e^{-\lambda x} + \alpha\left[\lambda^2\overline{w} + \overline{w}^{(2)} + 3\overline{u}^{(1)}(1-\gamma)/(2h_1\gamma)\right] \\ &+ \alpha\lambda^3/2\left(e^{\lambda x}\int_0^x \overline{w}e^{-\lambda x}dx - e^{-\lambda x}\int_0^x \overline{w}e^{\lambda x}dx\right) + 3\alpha\lambda^2(1-\gamma)/(4h_1\gamma)\left(e^{\lambda x}\int_0^x \overline{u}e^{-\lambda x}dx + e^{-\lambda x}\int_0^x \overline{u}e^{\lambda x}dx\right)\end{aligned} \tag{35}$$

The two integration constants c_1 and c_2 are determined using the conditions $\sigma_n(\Delta a) = \sigma_n^{(1)}(\Delta a) = 0$.

$$c_1 = -\alpha\lambda^3/2\int_0^{\Delta a} \overline{w}e^{-\lambda x}dx - 3\alpha\lambda^2(1-\gamma)/(4h_1\gamma)\int_0^{\Delta a} \overline{u}e^{-\lambda x}dx \tag{36}$$

$$c_2 = \alpha\lambda^3/2\int_0^{\Delta a}\overline{w}e^{\lambda x}dx - 3\alpha\lambda^2(1-\gamma)/(4h_1\gamma)\int_0^{\Delta a}\overline{u}e^{\lambda x}dx \quad (37)$$

Then, mode I ERR is found using *J*-integral.

$$\begin{aligned} G_{IT} &= -\int_0^{\overline{w}_B}\lambda^2 M_{nm}/bd\overline{w}_B + \alpha\left(\overline{w}_B^{(1)}\right)^2/2 + 3\alpha(1-\gamma)/(2h_1\gamma)\int_0^{\overline{w}_B}\overline{u}_B^{(1)}d\overline{w}_B \\ &= \int_0^{\overline{w}_{B\sigma_I}}\sigma_I d\overline{w}_B + \sigma_I\left(\overline{w}_B - \overline{w}_{B\sigma_I}\right) + \alpha\left(\overline{w}_B^{(1)}\right)^2/2 \end{aligned} \quad (38)$$

Note that the first term in Eq. (38) is calculated from a given interface cohesive law with $\sigma_I = -\lambda^2 M_{nm}/b + 3\alpha(1-\gamma)/(2h_1\gamma)\overline{u}_B^{(1)}$ in which

$$M_{nm} = (1+3\gamma)/(1+\gamma)^3\left(\beta_1 M_{1B} - M_{2B}\right) + h_1\gamma^2(1-\gamma)/[2(1+\gamma)^3]N_{1Be} \quad (39)$$

and

$$\overline{u}_B^{(1)} = -\left(6\gamma^2 M_{1B} + 6M_{2B} + \gamma^2 h_1^2 N_{1Be}\right)/\left(Ebh_1^2\gamma^2\right) \quad (40)$$

In the case of a rigid interface the first two terms in Eq. (40) disappear and the third term reduces to $G_{IT}^L = G_{IT}$ in Eq. (20). For a non-rigid interface the first term in Eq. (38) is calculated based on the given cohesive law and the second and third terms are not able to be determined analytically. However, for most of practical engineering problems with hard interfaces the third term in Eq. (38) can be replaced by G_{IT} in Eq. (20). Therefore, $\overline{w}_B$ in Eq. (38) can be calculated by using a given interface cohesive law and the following:

$$G_{IT}^L = \int_{\overline{w}_{B\sigma_I}}^{\overline{w}_B}\left(\sigma_{nB} - \sigma_I\right)d\overline{w}_B \quad (41)$$

Therefore, the second term in Eq. (38) is found and the mode I ERR G_{IT} for a hard interface is obtained analytically.

3.3. 2D elasticity partition theory

A DCB under crack tip bending moments M_{1B} and M_{2B} is considered here. Refer to Ref. [12] for general loading conditions. By using the two sets of fundamental orthogonal pure modes, i.e. $\{\theta, \beta\}$ in Eq. (14) and $\{\theta', \beta'\}$ in Eq. (15), approximate orthogonal pure mode I and mode II modes are

$$\theta_N(k_{er}) = \theta_{N2} + 1/2(\theta_{N1} - \theta_{N3})\log k_{er} + 1/2(\theta_{N1} - 2\theta_{N2} + \theta_{N3})(\log k_{er})^2 \quad (42)$$

$$\beta_N(k_{er}) = \beta_{N2} + 1/2(\beta_{N1} - \beta_{N3})\log k_{er} + 1/2(\beta_{N1} - 2\beta_{N2} + \beta_{N3})(\log k_{er})^2 \quad (43)$$

where $k_{er} = k/E$ is the ratio of interface stiffness to Young's modulus. θ_{N1}, θ_{N2}, θ_{N3}, β_{N1}, β_{N2}, β_{N3} are functions of the two sets of fundamental orthogonal pure modes. Detailed expressions for them are given in Ref. [12]. A mixed mode can be partitioned using this pair of pure modes.

4. Numerical and experimental assessments

The partition theories presented above have been extensively validated by using FEM simulations and in general excellent agreement has been observed [8–13]. Here, one example is presented for an isotropic DCB with non-rigid interface. The geometric dimensions of the DCB are length $L = 110\ \text{mm}$, width $b = 1\ \text{mm}$, total thickness $h_1 + h_2 = 2\ \text{mm}$ and crack length $a = 10\ \text{mm}$. The material Young's modulus is $E = 1\ \text{GPa}$. The loading conditions are $M_{1B} = 1\ \text{Nm}$ and $M_{2B} = 0$. Mixed-mode partitions from the present 2D elasticity theory and Abaqus FEM are recorded in Table 1 for various thickness ratios γ and interface stiffness to Young's modulus ratios k_{er}. An excellent agreement is observed between the analytical and FEM results. Note that G_I / G in Table 1 is a percentage.

Table 1. Comparison between present 2D elasticity partition theory and FEM results.

		Analytical ($\times 10^6$ N/m)					FEM ($\times 10^6$ N/m)				
γ	k_{er}	0.1	0.5	1	5	10	0.1	0.5	1	5	10
1	G_I	3.000	3.000	3.000	3.000	3.000	3.029	3.029	3.028	3.024	3.022
	G_I / G	57.14	57.14	57.14	57.14	57.14	57.66	57.60	57.79	58.59	59.22
3	G_I	45.30	43.75	42.99	42.18	40.17	45.12	44.09	43.48	41.47	40.29
	G_I / G	95.87	92.59	90.98	89.26	85.01	94.72	92.71	91.56	87.96	85.97
5	G_I	159.7	154.9	152.0	148.6	139.7	159.0	156.4	154.7	148.6	144.7
	G_I / G	99.05	96.06	94.24	92.16	86.61	97.80	96.39	95.50	92.44	90.59
7	G_I	381.9	371.5	364.3	355.8	332.0	380.7	375.4	371.8	358.2	349.1
	G_I / G	99.65	96.94	95.06	92.83	86.63	98.59	97.46	96.72	94.07	92.40
9	G_I	748.0	728.9	714.5	697.0	647.6	746.3	736.6	729.8	703.7	686.0
	G_I / G	99.83	97.29	95.36	93.03	86.43	98.92	97.95	97.30	94.92	93.39

One example of experimental assessments is also presented here. Since the specimens in the tests were manufactured without adhesive layers [4] the laminar interfaces are considered to be rigid. Five partition theories, i.e. Williams theory [1], Suo-Hutchinson theory [3], Wang-Harvey classical, shear deformable and 2D theories, are assessed in Fig. 3. Although the Suo-Hutchinson and Wang-Harvey 2D partition theories are considered to be most accurate, the Wang-Harvey classical theory agrees the best with experimental data. It is suggested that the propagation of mixed-mode delamination on rigid interfaces is governed by the global partition as the global partitions of shear deformable and 2D partition theories are the same as the classical partitions.

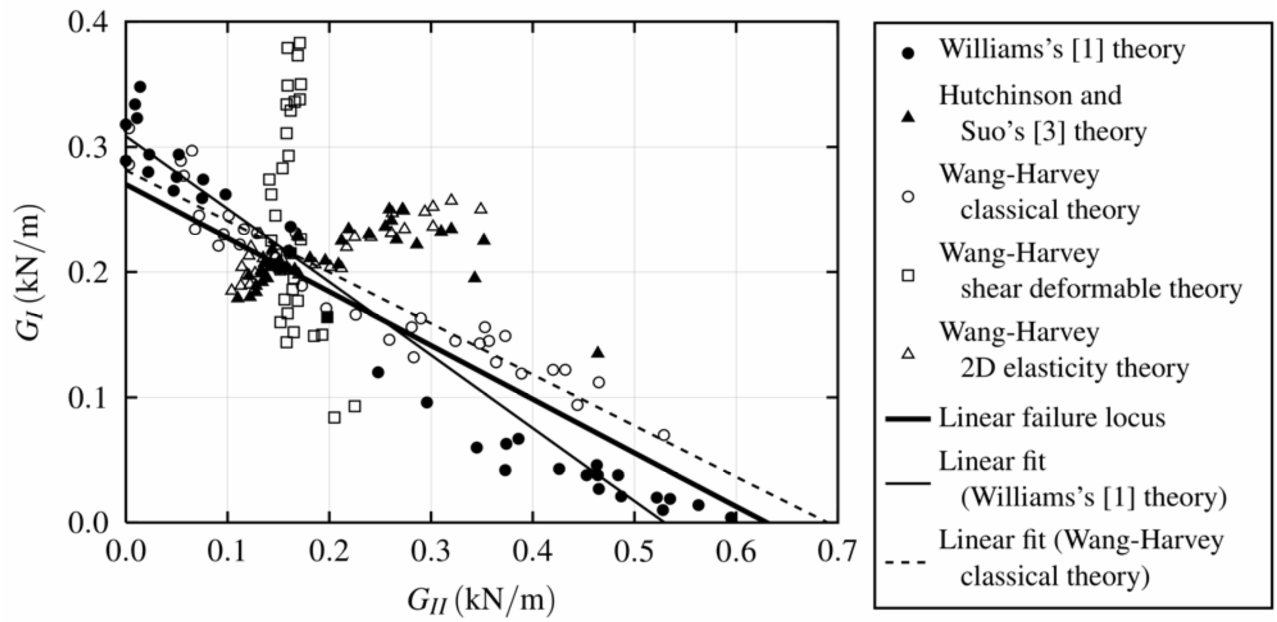

Figure 3. A comparison of various partition theories and the linear failure locus for epoxy-matrix/carbon-fiber composite specimens.

5. Conclusions

The present work discovers the most fundamental fracture modes – the two sets of orthogonal pure modes. A mixed-fracture mode can be superimposed or partitioned by these most fundamental pure modes. The two sets co-exist in classical laminated composite beams and plates and coincide in shear deformable beams and plates for rigid interfaces. When non-rigid interfaces considered the two sets coincide in both classical and shear deformable theories. By using these two sets of pure modes, a mixed-mode can also be partitioned based on 2D elasticity theory. The novel methodology is rooted in the mechanics of material and operated by a powerful mathematical method. It is capable of studying delamination in curved laminated composite beams and shells as well. It is also capable of studying general and buckling driven delamination consisting of all opening, shearing and tearing modes.

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