Boundary Fractures and Indentation Tests

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Abstract
The paper presents an evaluation of the factors influencing fracture initiation at the boundary of a rigid test plate that are used to estimate the in-situ deformability characteristics of a geologic medium. The paper outlines the techniques that are used to perform in situ plate load tests and focuses on the problem of boundary fracture generation at the edges of the geologic medium. If the mechanical behaviour of the rock mass can be assumed to display brittle elastic behaviour, computational methods based on boundary element techniques can be used to examine the mode of crack extension within the elastic geomaterial. The process of fracture generation can influence the extent of the region being evaluated and, more importantly, this can adversely affect the theoretical relationships for the interpretation of test plate data. In most instances the boundary crack may not be visible; this is especially true if the plate load test is conducted with some nominal embedment. This paper discusses issues associated with the interpretation of plate load tests conducted as a validation of experimental data determined from plate load tests. The methodology for the correct interpretation of plate load tests conducted on brittle elastic materials requires knowledge of additional parameters governing the mechanical behaviour of the rock; this involves laboratory evaluation of fracture toughness data. The paper presents results concerning the influence of axisymmetric boundary fractures on the estimated deformability characteristics of the rock mass.

Keywords plate load tests, brittle edge fracture, boundary elements, interpretation of fields tests

1. Introduction

The evaluation of the effective geomechanical characteristics of complex and heterogeneous geological materials is best accomplished through static load tests that are conducted in-situ. A technique that has been used extensively in this connection is the plate loading test where a plate of known dimensions and flexural rigidity is maintained in contact with the surface of the geological medium under examination and is then subjected to an axial loading [1,2]. As the elastic stiffness of the geomaterial increases large loads are required to attain measurable test plate deflections. When plate load tests are conducted in galleries and adits, the loads needed to indent the test plate can be achieved through reaction against the walls of the gallery or enclosure. When plate load tests are performed on large open surfaces this facility is not available and recourse must be made to provide the test loads through a self stressing reaction system. The method of cable jacking introduces the reactive loads through an anchor region located in the medium that is being tested. The method was first proposed by Zienkiewicz and Stagg [3] and presents a simpler test configuration than that involving anchor piles and a bracing frame to accommodate the remoteness of the anchoring loads from the plate location. The influence of the anchor load on the resulting net settlement of the test plate was first examined by Selvadurai [4], who examined the problem of the interaction between a smoothly indenting plate and a Mindlin force [5] located at a finite depth from test plate. The analysis was subsequently extended to cover distributed anchor loads [6], transverse isotropy of the rock mass [7], flexibility of the test plate [8-11] and creep effects of the geologic medium [12].
In this paper we examine the problem of crack extension in a brittle elastic geologic medium during the indentation of the brittle elastic half-space by a cylindrical punch with a smooth flat contact surface. The paper discusses a procedure for locating the point of nucleation of the crack within the brittle elastic solid and employs a boundary element technique to locate the progress of crack evolution as the force on the loading device is increased [13]. The numerical results illustrate how the extent of crack development influences the load vs. displacement relationship for the rigid test plate. The development of boundary fracture is characteristic of any indentation problem involving brittle elastic materials and sharp-edged indenters. Results are developed for geomechanical investigations that are carried out both at the surface of a geomaterial and at depth. The work can also be extended to include flexibility of the plate that is applying the indentation loads.

2. Theoretical Results-Indentation

The theoretical concepts that are used in the interpretation of plate loading tests conducted on brittle elastic geologic media are invariably based on the validity of the theory of elasticity. The analysis is frequently restricted to assumptions of isotropy of the rock mass, While this is considered to be a limitation for in situ testing, the characterization of elastic materials that are generally anisotropic (with 21 independent elastic constants), orthotropic (with 9 independent elastic constants), or transversely isotropic (with 5 independent elastic constants) [14, 15] is regarded as a difficult exercise even under highly controlled laboratory conditions [16]. The best that can be accomplished in an in situ plate loading test is to arrive at an effective deformability modulus of the region in which the plate load test is conducted. The simplest idealization that permits the use of an effective property is the assumption of isotropy of the tested region. It is relatively clear that if the geologic medium possesses dominant stratification then the deformability should be interpreted appropriately. The theoretical concepts can also be extended to include both transverse isotropy of the rock mass and elastic inhomogeneity of the geologic medium [17-20]; however, the inverse analysis for the elasticity parameter identification in these situations cannot be conducted using only the results of plate load tests. Even with the restrictions of isotropic and homogeneous behaviour of the rock mass, the results of a plate load test can only provide an overall estimate for the deformability of the rock mass that can include both the elastic constants encountered in the isotropic elastic model. The theoretical analysis of the plate load test involving no reactive anchor forces can be conducted by formulating the mixed boundary value problem of the indentation of an isotropic elastic halfspace by a rigid test plate. In order to formulate the mathematical problem, it is also necessary to identify the contact conditions that can be present at the interface between the test plate and the geomaterial. This largely depends on the condition of the test plate and the procedures used to either make the interface completely smooth or completely frictional, which will inhibit relative slip between the plate and the geomaterial. Finally, the extent of the geomaterial region that is tested is assumed to be large in comparison to the dimensions of the plate, enabling the region to be approximated by an elastic halfspace region. Reviews of contact problems of special interest to in situ plate loading tests are given in [21-25]. The axisymmetric mixed boundary value problem associated with the smooth indentation of a halfspace by a rigid circular test plate (Figure 1) is described by the boundary conditions

$$u_z(r,0) = \Delta, \quad \forall r \in (0,a); \quad \sigma_{zz}(r,0) = 0, \quad \forall r \in [a,\infty); \quad \sigma_{rz}(r,0) = 0, \quad \forall r \in (0,\infty) \tag{1}$$

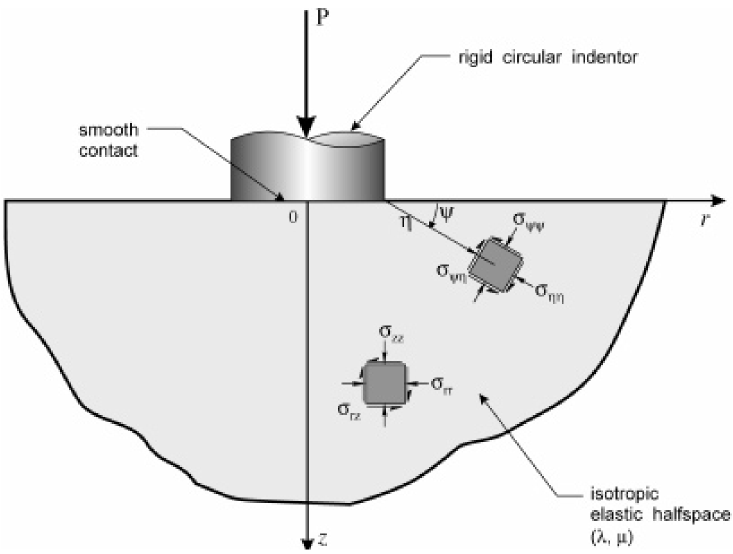


Figure 1. The classical indentation problem for a geomaterial halfspace.

where $\mathbf{u}\,(=(u_r, 0, u_z))$ and $\boldsymbol{\sigma}$ are, respectively, the axisymmetric versions of the displacement vector and the stress tensor referred to the cylindrical polar coordinate system (r,θ,z) and Δ is the displacement of the test plate. In addition, the regularity conditions require that $\mathbf{u}$ and $\boldsymbol{\sigma}$ reduce to zero as either r or $z \to \infty$. The mixed boundary value problem in elasticity defined by the set of equations (1) is a classical problem solved by Boussinesq [26] employing results of potential theory and by Harding and Sneddon [27] using the theory of dual integral equations. Details of the methods of solution are also given in [21-25] and [28, 29]. The result of interest to geomechanics is the relationship between the indentation displacement (Δ) and the corresponding axial load (P) required to achieve the indentation. This can be obtained in exact closed form as

$$\Delta = \frac{P(1-\nu)}{4\mu a} \tag{2}$$

where μ and ν are, respectively, the linear elastic shear modulus and Poisson's ratio of the geomaterial. As is evident from (2), the classical analysis of the plate load test provides only an estimate of $\mu/(1-\nu)$ and additional information is needed to determine the parameters separately. When the plate adheres to the surface of the geomaterial, the resulting boundary value problem is described by the following boundary conditions:

$$\begin{aligned} &u_z(r,0) = \Delta, \ \forall r \in (0,a) \ ; \quad \sigma_{zz}(r,0) = 0, \quad \forall r \in [a,\infty); \\ &u_r(r,0) = 0, \ \forall r \in (0,a) \ ; \quad \sigma_{rz}(r,0) = 0, \quad \forall r \in (a,\infty) \end{aligned} \tag{3}$$

This mixed boundary value problem can be examined by appeal to the theory of integral equations where the problem can be reduced to the solution of the Hilbert problem involving singular integral equations. The elasticity problem of adhesive contact between a plate and an elastic halfspace region was examined by Mossakovskii [30] and Ufliand [31] and the exact closed form result is given by

$$\Delta = \frac{P(1-2\nu)}{4\mu a \ln(3-4\nu)} \tag{4}$$

The Hilbert problem approach accounts for the oscillatory form of the stress singularity at the boundary of the rigid plate. Selvadurai [32] also examined the mixed boundary value problem defined by (3) but by replacing the oscillatory form of the stress singularity by a regular $(a^2 - r^2)^{-1/2}$ type singularity, thus reducing the problem to the solution of a Fredholm integral equation of the second-kind. It was shown that the difference between the exact result based on the Hilbert problem formulation and the Fredholm integral equation formulation is less than 0.5% when $\nu = 0$ and the results converge when $\nu = 1/2$. A further classical development is to consider that the entire surface of the halfspace is composed of an inextensible membrane, in which case the bonded boundary condition is automatically satisfied in the indentation zone and the shear tractions are non zero beyond the indented zone. The load-displacement relationship of the indenter can be obtained from the result for the problem of a rigid disc embedded in an elastic infinite space [33, 34]: i.e.

$$\Delta = \frac{P(3-4\nu)}{16\mu a(1-\nu)} \tag{5}$$

It should be noted that in the limit of material incompressibility, (2) and (4) reduce to the same result. The analysis can be extended to include Coulomb friction at the contact zone [35] and the influence of depth of embedment of the test plate [36, 37].

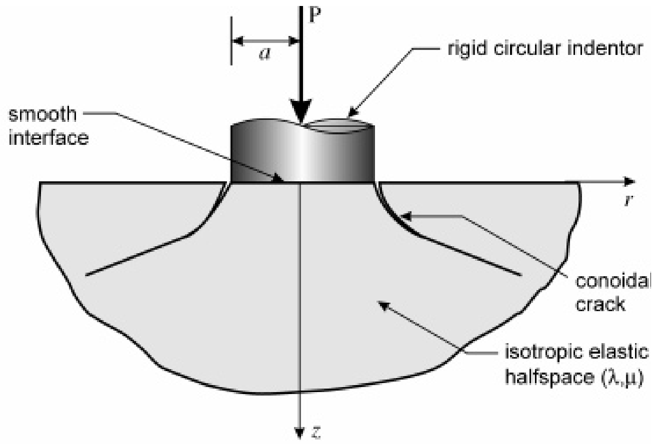


Figure 2. Conoidal boundary fractures emanating from the edge of the indented region.

3. Computational Results-Indentation Fracture

The region at the outer boundary of the test plate is subjected to singular stress fields due to the mixed boundary conditions imposed by the indentation. It can be shown that even when indentation is made by flexible flat test plates, the edges of the indented geomaterial will experience stress concentrations that are singular. The boundary of the indenter is therefore a location where indentation fracture can initiate. The objective of this paper is to demonstrate the influences of fracture development on the load displacement relationship for a rigid test plate. We consider the axisymmetric indentation of the surface of an isotropic elastic half-space region by a smooth flat rigid indenter of radius a (Figure 2). The process of crack initiation and crack extension is most conveniently handled using a computational approach that can model the quasi-static crack extension process. The analysis of crack extension during indentation can be performed via a variety of computational schemes. These can include either finite element methods or boundary integral equation methods or combinations of these. The application of finite element techniques to fracture extension is well established; it requires the specification of criteria both for the initiation of crack extension and for the location of the orientation of the crack path. These relationships applicable to brittle elastic fracture initiation and extension are available in the literature on fracture mechanics [38]. In modelling crack extension via the finite element method, re-meshing is an important feature that ensures accuracy of both the local and global stress fields. Adaptive re-meshing techniques have been used quite effectively to examine crack extension in brittle geomaterials such as concrete and rock [39]. An alternative to re-meshing involves extensive graded mesh refinement in the vicinity of the singular crack tip element and allows crack extension to take place at element boundaries. Alternative schemes, such as the boundary element method, provide greater flexibility when examining the crack extension process. The primary advantage of integral equation-based concepts such as the boundary element method or the displacement discontinuity method is that the domain rearrangement resulting from the crack extension process requires only an incremental change in the boundary element mesh or along the displacement discontinuity line of the crack extension. We shall illustrate here the application of the boundary element scheme to examine the process of quasi-static conoidal crack extension in the geomaterial originating at the boundary indenter. The application of boundary element schemes to problems in fracture mechanics originated with the work of Cruse and Wilson [40] and has been extended by a number investigators [41-43] to include a variety of problems including cracks with frictional interfaces. The review [44] gives a comprehensive survey of research related to boundary element formulations in fracture mechanics. Further details of the application of boundary element techniques to crack indentation problems are given in [13] and summarized here for completeness.

3.1. Governing Equations

We examine the class of axisymmetric problems where fracture extension in brittle elastic media satisfies Hooke's Law and the corresponding Navier equations: i.e.

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} \quad ; \quad \mu \nabla^2 u_i + (\lambda + \mu) u_{k,ki} = 0 \tag{6}$$

and λ and μ are Lamé's constants and ∇^2 is Laplace's operator. The boundary integral equation governing axisymmetric deformations of the geomaterial region can be written as

$$c_{lk} u_k + \int_\Gamma \left\{ P_{lk}^* u_k + u_{lk}^* P_k \right\} \frac{r}{r_i} d\Gamma = 0 \tag{7}$$

where Γ is the boundary of the domain; u_k and P_k are, respectively, the displacements and tractions on Γ and u_{ik}^* and P_{ik}^* are the fundamental solutions [45,46]. In (7), c_{lk} is a constant, which can take values of either zero (within the domain), $\delta_{ij}/2$ (if the point is located at a smooth boundary) or is a function of the discontinuity at a corner and of Poisson's ratio. For axial symmetry, the displacement fundamental solutions take the forms

$$u_{rr}^* = C_1 \left\{ \frac{4(1-\nu)(\rho^2 + \bar{z}^2) - \rho^2}{2r\bar{R}} \right\} K(\bar{m}) - \left\{ \frac{(7-8\nu)\bar{R}}{4r} - \frac{(e^4 - \bar{z}^4)}{r\bar{R}^3 m_1} \right\} E(\bar{m}) \tag{8}$$

$$u_{rz}^* = C_1 \bar{z} \left\{ \frac{(e^2 + \bar{z}^2)}{2\bar{R}^3 m_1} E(\bar{m}) - \frac{1}{2\bar{R}} K(\bar{m}) \right\} \tag{9}$$

…etc., where

$$\begin{gathered} \bar{z} = (z - z_i); \quad \bar{r} = (r + r_i); \quad \rho^2 = (r^2 + r_i^2); \quad m_1 = 1 - \bar{m} \\ e^2 = (r^2 - r_i^2); \quad \bar{R}^2 = \bar{r}^2 + \bar{z}^2; \quad \bar{m} = \frac{4rr_i}{\bar{R}^2}; \quad C_1 = \frac{1}{4\pi\mu(1-\nu)} \end{gathered} \tag{10}$$

and $K(\bar{m})$ and $E(\bar{m})$ are complete elliptic integrals of the first and second-kind and (r, z) and (r_i, z_i) correspond to the coordinates of the field and source points respectively. The relevant fundamental solutions for P_{lk}^* can be obtained by manipulating results of the types (8) and (9). Upon discretization of the boundary Γ, the integral equation can be expressed in the form of a boundary element matrix equation

$$[\mathbf{D}]\{\mathbf{U}\} = [\mathbf{T}]\{\mathbf{P}\} \tag{11}$$

where $[\mathbf{D}]$ and $[\mathbf{T}]$ are obtained, respectively, by integration of the displacement and traction fundamental solutions. When considering the discretization of the boundary Γ of the domain, quadratic elements can be employed quite effectively; the variations of the displacements and tractions within an element can be described by

$$\left. \begin{matrix} u_i \\ P_i \end{matrix} \right\} = \sum_{n=0}^{3} a_n \zeta^n \tag{12}$$

where ζ is the local coordinate.. Then modeling cracks that occur at the boundaries or within the interior of the elastic geomaterial, it is necessary to modify (12) to take into consideration the $1/\sqrt{\zeta}$ type locally two-dimensional stress singularity at the crack tip. In contrast to finite element approaches that use quarter-point elements, here we utilize the singular traction quarter-point boundary elements [40] where the tractions can be expressed in the form

$$P_i = \frac{c_0}{\sqrt{r}} + c_1 + c_2\sqrt{r} \tag{13}$$

where $c_i (i = 0,1,2)$ are constants. The Mode I and Mode II stress intensity factors that will be used in the estimation of crack growth can be determined by applying a displacement correlation technique, which makes use of the nodal displacement at four locations A, B, E and D and the crack tip (Figure 3).

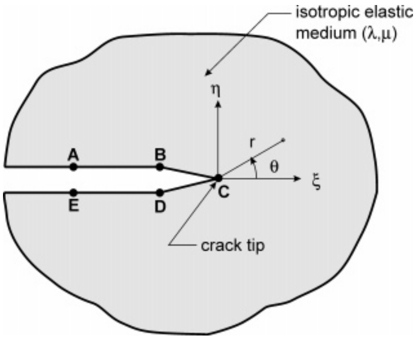


Figure 3. The crack tip geometry and the node locations.

The stress intensity factors are given by

$$\left.\begin{matrix} K_{\mathrm{I}} \\ K_{\mathrm{II}} \end{matrix}\right\} = \frac{\mu}{(k+1)}\sqrt{\frac{2\pi}{l_0}} \left\{\begin{matrix} 4[u_\eta(B) - u_\eta(D)] + [u_\eta(E) - u_\eta(A)] \\ 4[u_\xi(B) - u_\xi(D)] + [u_\xi(E) - u_\xi(A)] \end{matrix}\right. \tag{14}$$

where $k = (3 - 4\nu)$ and l_0 is the length of the crack tip element ξ and η are the local coordinates at the crack tip.

3.2. Modelling of Crack Extension

The boundary element approach can be used to examine the crack extension during indentation. The stress state necessary to initiate crack nucleation can be obtained by using integral results for the stress state associated with the mixed boundary value problem defined by (1). The results presented by Harding and Sneddon [27] can be used for this purpose. The axisymmetric stress state is

$$\begin{aligned} \sigma_{zz}(r,z) &= -\frac{4\mu\Delta}{\pi a}\left(\frac{\lambda+\mu}{\lambda+2\mu}\right)\left\{J_1^0 + \xi J_2^0\right\} \\ \sigma_{\theta\theta}(r,z) &= -\frac{4\lambda\mu\Delta}{\pi a(\lambda+2\mu)}\left\{J_1^0\right\} - \frac{4\mu^2\Delta}{\rho(\lambda+2\mu)}\left\{J_0^1 - \frac{(\lambda+\mu)}{\mu}J_1^1\right\} \\ \sigma_{rz}(r,z) &= -\frac{4\mu\Delta}{\pi a}\left(\frac{\lambda+\mu}{\lambda+2\mu}\right)\xi J_2^1 \end{aligned} \tag{15}$$

etc…., where

$$\begin{aligned} J_n^m &= \int_0^\infty p^{(n-1)}\sin(p)e^{-p\xi}J_m(\rho p)\,dp \\ r^2 &= 1+\xi^2; \quad \tan\theta = \xi^{-1}; \quad R^2 = (\rho^2 + \xi^2 - 1)^2 + 4\xi^2 \\ \tan\phi &= \frac{2\xi}{(\rho^2 + \xi^2 - 1)} \end{aligned} \tag{16}$$

$J_m(x)$ is the Bessel function of the first kind of order m and the dimensionless coordinates are $\rho = r/a$ and $\xi = z/a$. The maximum local tensile stress within the elastic geomaterial, in the vicinity of the boundary of the indenter (Figure 1) can be obtained through a computer based search technique. The location of the point of maximum tensile stress will be characterized by the local coordinates η_0 and ψ_0 and will depend only on Poisson's ratio. This technique allows the location of the orientation and length of a starter crack and the boundary element meshing is structured to accommodate this starter crack and a semi-infinite domain (Figure 4).

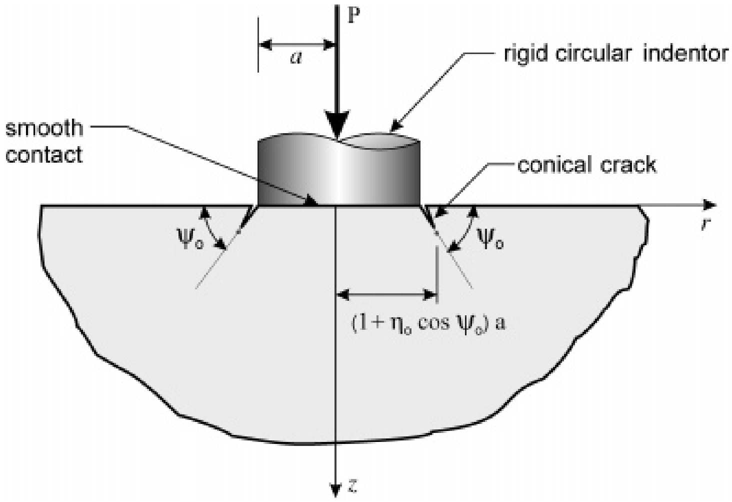


Figure 4.The location and orientation of starter crack.

3.2. Onset and Orientation of Crack Extension

The onset of crack extension can be based on a number of criteria applicable to brittle geomaterials. An elementary criterion for onset of crack extension is the attainment of a critical value of the Mode I stress intensity factor; i.e.

$$K_{\mathrm{I}} = K_{\mathrm{IC}} \tag{17}$$

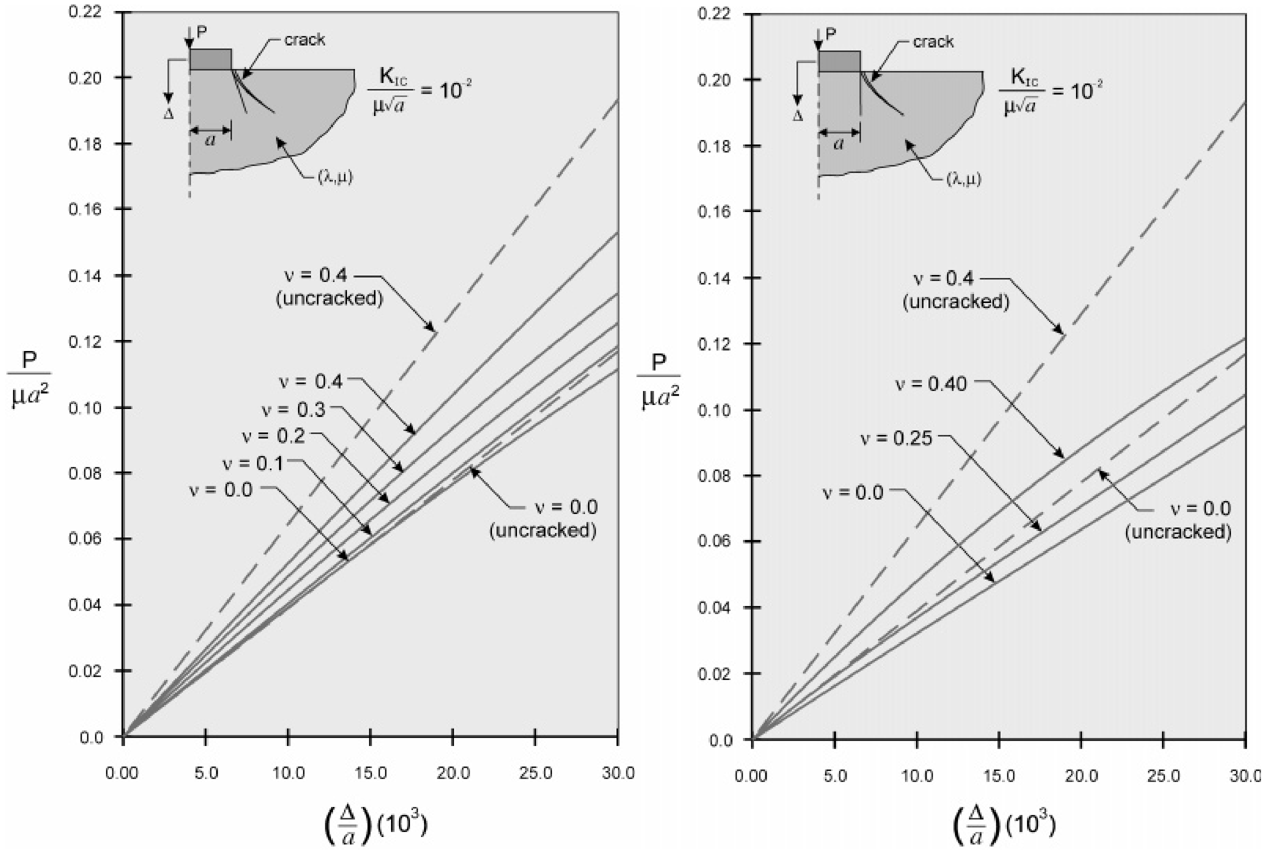
The orientation of crack extension has to take into account the influence of both stress intensity factors. The criterion used is that proposed by Erdogan and Sih [47]. The maximum stress criterion assumes that the crack will extend in the plane that is normal to the maximum stress $\sigma_{\psi\psi}$ shown in Figure 4b and according to

$$K_{\mathrm{I}} \sin\psi + K_{\mathrm{II}}(3\cos\psi - 1) = 0 \tag{18}$$

Other criteria, such as crack extension along paths where $K_{\mathrm{II}} = 0$, are possible [48] but in this study the criterion (18) is used.

4. Numerical Results

The objective of the study is to examine the extent to which the load displacement of the rigid indenter is influenced by the development of boundary fracture. The results can be presented in relation to a load-displacement relationship for the rigid indenter, taking into account the parameters that control the crack initiation and extension process described previously and documented in detail in [13]. Figure 5a illustrates the results for the load-displacement relationship for the case where the orientation of the starter crack is determined from procedures outlined in Section 3.2. Figure 5b illustrates similar results derived by assuming a priori that the starter crack is oriented normal to the boundary of the indented surface. Results for the load-displacement relationships for the uncracked situations are also presented for purposes of comparison. In both instances, cracking of the halfspace region will lead to a reduction in the elastic stiffness of the rigid indenter.



(a) (b)

Figure 5. Influence of geomaterial cracking on the stiffness of the rigid indenter

5. Concluding Remarks

The objective of the study is to examine the extent to which the load displacement of the rigid indenter is influenced by indentation fracture that can occur beneath the surface of the region that is being indented. In a geomechanics context, indention testing is carried out in order to determine the in situ properties of the geomaterial. Studies of this type serve two purposes; first it will alert the user to the stress levels that can lead to the development of cracking in the indented region. Modern acoustic emissions monitoring could be used to supplement the experiments. Secondly, if fractures occur it will influence the interpretation of the in situ deformability characteristics of the geomaterial region. The boundary cracking will generally lead to a lower estimate of the in situ modulus. The methodology described here is not without constraints, the most important of which is the assumption that some estimates can be made of the fracture toughness of the geomaterial as interpreted through the critical Mode I stress intensity factor. This pre-supposes that this parameter can be estimated from either laboratory tests or preferably in situ fracture tests conducted by flat jack expansion testing of surface slots cut into the rock surface.

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References

[1] Ch. Jaeger, Rock Mechanics and Rock Engineering, Cambridge University Press, Cambridge, 1972.

[2] F.G. Bell (Ed.), Ground Engineer's Reference Book, Butterworths, London, 1987.

[3] O.C. Zienkiewicz, K.G. Stagg, Cable method of in-situ rock testing, Int. J. Rock Mech. Mining Sci., 4(1967) 273-300.

[4] A.P.S. Selvadurai, (1978) The interaction between a rigid circular punch on an elastic halfspace and a Mindlin force. Mech Res Comm, 5 (1978) 57-64.

[5] R.D. Mindlin, Force at a point in the interior of a semi-infinite solid, Physics, 7(1936) 195-202

[6] A.P.S. Selvadurai, The displacement of a rigid circular foundation anchored to an isotropic elastic halfspace. Géotechnique, 29 (1979) 195-202.

[7] A.P.S. Selvadurai, The interaction between a rigid circular foundation and an anchor region located in a transversely isotropic elastic rock mass, in: P.J.N. Pells Ed. Proceedings of the International Conference on Structural Foundations on Rock, Sydney, A.A. Balkema, The Netherlands, Vol.1 (1980) 23-28.

[8] A.P.S. Selvadurai, Elastic contact between a flexible circular plate and a transversely isotropic elastic halfspace, Int J Solids Struct, 16 (1980) 167-176.

[9] A.P.S. Selvadurai, N.A. Dumont, Mindlin's problem for a halfspace indented by a flexible plate, J Elast, 105 (2011) 253-269.

[10] A.P.S. Selvadurai A contact problem for a Reissner plate and an isotropic elastic halfspace, J Theor Appl Mech, 3 (1984) 181-196.

[11] A.P.S. Selvadurai, An energy estimate of the flexural deflections of a circular foundation embedded in an elastic medium, Int J Num Analyt Meth Geomech, 3 (1979) 285-292.

[12] A.P.S. Selvadurai, Some results concerning the viscoelastic relaxation of prestress in a near surface rock anchor, Int J Rock Mech, Min Sci Geomech Abst, 16 (1979) 309-317

[13] A.P.S Selvadurai, Fracture evolution during indentation of brittle elastic solid, Mech Cohesive-Frict Mat, 5 (2000) 325-339.

[14] S.G. Lekhnitskii, Anisotropic Elasticity, Holden-Day, San Francisco, 1963.

[15] A.J.M. Spencer, Continuum Mechanics, Dover Publ, Mineola, New York, 2004.

[16] R.O. Davis, A.P.S. Selvadurai, Elasticity and Geomechanics, Cambridge University Press, Cambridge, 1996.

[17] H.A. Elliott, Axially symmetric stress distributions in aelotropic hexagonal crystals, Proc Camb Phil Soc, 45 (1949) 621-630.

[18] R.T. Shield, R.T. 1951. Notes on problems in hexagonal aelotropic materials, Proc Camb Phil Soc, 47 (1951) 401-409

[19] V.A. Sveklo, The action of a stamp on an elastic anisotropic halfspace, Prikl Math Mech, 34 (1970) 165-171.

[20] A.P.S. Selvadurai, The settlement of a rigid circular foundation resting on a halfspace exhibiting a near surface elastic non-homogeneity, Int J Num Analyt Meth Geomech, 20 (1996) 351-364.

[21] A,P.S. Selvadurai, Elastic Analysis of Soil-Foundation Interaction, Developments in Geotechnical Engineering 17, Elsevier Sci. Publ Co, 1979.

[22] G.M.L. Gladwell, Contact Problems in the Classical Theory of Elasticity, Sijthoff-Noordhoff, Alphen Aan den Rijn, The Netherlands, 1980.

[23] A.P.S. Selvadurai, On the mathematical modelling of certain fundamental elastostatic contact problems in geomechanics, in: Modelling in Geomechanics, (M. Zaman, G. Gioda, J.R. Booker Eds.) Ch. 13 (2000) 301-328.

[24] A.P.S. Selvadurai, The analytical method in geomechanics, Appl Mech Rev, 60 (2007) 87-106.

[25] S.M. Aleynikov, Spatial Contact Problems in Geotechnics. Boundary Element Method,

Springer-Verlag, Berlin, 2011.
[26] J. Boussinesq, Application des Potentiels á l'Étude de l'Équilibre et du Mouvement des Solides Élastiques, Gauthier-Villars, Paris, 1885.
[27] J.W. Harding, I.N. Sneddon, The elastic stresses produced by the indentation of the plane surface by a rigid flat indenter, Proc Camb Phil Soc, 41 (1945) 16-26.
[28] I.N. Sneddon, Fourier Transforms, McGraw-Hill, New York, 1951.
[29] A.P.S. Selvadurai, Partial Differential Equations in Mechanics, Vol. 2, The Biharmonic Equation, Poisson's Equation, Springer-Verlag, Berlin, 2000.
[30] V.I. Mossakovskii, The fundamental mixed boundary value problem of the theory of elasticity for a halfspace with a circular line separating the boundary condition, Prikl Math Mekh, 18 (1954) 187-196.
[31] Ia. S. Ufliand, The contact problem of the theory of elasticity for a die, circular in its plane, in the presence of adhesion, Prikl Math Mekh, 20 (1956) 578-587.
[32] A.P.S. Selvadurai, The influence of a boundary fracture on the elastic stiffness of a deeply embedded anchor plate, Int J Num Analyt Meth Geomech, 13 (1989)159-170.
[33] A.P.S Selvadurai, The load-deflexion characteristics of a deep rigid anchor in an elastic medium, Géotechnique, 26 (1976) 603-612.
[34] A.P.S. Selvadurai, T.J. Nicholas, A theoretical assessment of the screw plate test, 3rd International Conference on Numerical Methods in Geomechanics, (W. Wittke, Ed.), Aachen, Vol.3, A.A. Balkema, The Netherlands (1979) 1245-1252.
[35] D.A. Spence, Self-similar solutions to adhesive contact problems with incremental loading, Proc Roy Soc, Ser A, 305(1968) 55-80.
[36] S.C. Hunter, D. Gamblen, The theory of a rigid disc ground anchor buried in an elastic soil either with adhesion or without adhesion, J Mech Phys Solids, 22 (1975) 371-399.
[37] A.P.S. Selvadurai, The axial loading of a rigid circular anchor plate embedded in an elastic halfspace, Int J Num Analyt Meth Geomech, 17(1993) 343-353.
[38] B. R. Lawn, Fracture of Brittle Solids, Cambridge University Press, Cambridge, 1993.
[39] S.P. Shah, S.E. Swartz (Eds.), Fracture of Concrete and Rock, SEM-RILEM Int Conf. Springer-Verlag, Berlin, 1989.
[40] T. Cruse, R.B. Wilson, Boundary Integral Equation Methods for Elastic Fracture Mechanics, AFOSR-TR-0355, 1977.
[41] C.A. Brebbia, J.C.F. Telles, L.C. Wrobel, Boundary Element Techniques, Springer-Verlag, Berlin, 1984.
[42] A.P.S. Selvadurai, Matrix crack extension at a frictionally constrained fiber, J Engng Mat Tech, Trans ASME, 116 (1994) 398-402.
[43] A.P.S. Selvadurai, A. ten Busschen, Mechanics of the segmentation of an embedded fiber. Part II. Computational modelling and comparisons, J Appl Mech, Trans ASME, 62 (1995) 98-107.
[44] M.H. Aliabadi, Boundary element formulations in fracture mechanics, Appl Mech Rev, 50 (1997) 83-96.
[45] Th. Kermanidis, Numerical solution for axially symmetric elasticity problems, Int J Solids Struct, 11(1975) 495-500.
[46] M.F.F. Oliveira, N.A. Dumont, A.P.S. Selvadurai, Boundary element formulation of axisymmetric problems for an elastic halfspace, Eng Anal Bound Elem, 36 (2012)1478-1492.
[47] F. Erdogan and G.C. Sih, On the crack extension in plates under plane loading and transverse shear, J. Basic Engng, Trans ASME, 85 (1963) 297-312.
[48] A.P.S. Selvadurai, The modeling of axisymmetric basal crack evolution in a borehole indentation problem, Eng Anal Bound Elem, 21 (1998)377-383.