The Higher Order Crack Tip Fields for FGMs Spherical Shell with Reissner's Effect

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Abstract Based on the theory of shells considering the transverse shear deformation or Reissner's effect, the crack tip fields are investigated for a cracked spherical shell made of isotropic functionally graded materials (FGMs). The elastic modulus and Poisson's ratio of the FGMs are assumed to be the linear function of x and a constant, respectively. The governing equations, i.e. the system of the tenth order partial differential equations with variable coefficients are first derived. Then, the eigen-expansion method is employed to the system, and the higher order crack tip fields of the cracked spherical shell are obtained. As the in-homogeneity parameter approaches to zero, the solutions degenerate to the corresponding fields of isotropic homogeneous spherical shell with Reissner's effect.

Keywords crack tip fields, Reissner's effect, spherical shell, FGMs

1. Introduction

As the gradient can be tailored to meet specific needs and the macroscopic interfaces of traditional composites are eliminated, the functionally graded materials have been widely applied in engineering. However, due to the limitations of the manufacture technology, a large number of micro-cracks cannot be avoided in functionally graded plates and shells, which would seriously endanger the security of these structures. Therefore, the fracture analysis for functionally graded shells is necessary.

It is well known that Kirchhoff classical theory does not consider the transverse shear deformation and has some limitations in the fracture analysis [1], so Reissner's theory [2] is often adopted.

Considering the effect of transverse shear, F.Delale[3] investigated the problem for spherical cap containing a through crack. For the plates and shells of homogeneous materials, the higher order crack tip fields are obtained based on Reissner's theory by Liu Chuntu[4]. The corresponding field for FGMs shell has been given only for which the material gradient is along thickness direction [5]. In this paper, the crack tip fields are studied for functionally graded spherical shell with Reissner's effect by the eigen-expansion method.

2. Basic equations

The effect of Poisson's ratio on stress intensity factor (SIF) is far less than that of elastic modules [6]. Therefore, Poisson's ratio is assumed as a constant, and the elastic modules is assumed as the linear function of spatial coordinates x as

$$E = E(x) = E_0(1+\beta x), \qquad \mu = const \tag{1}$$

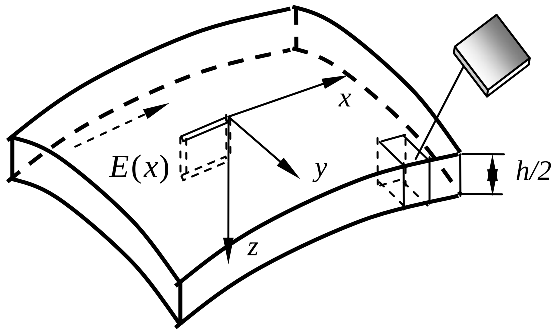

Fig.1 The functionally graded spherical shell

where, E_0 is elastic modulus at $x=0$, β is non-homogeneous coefficients. *Z*–axis is along the thickness direction, the radius is *R* and the thickness is *h* as shown in Fig.1.

The basic equations for Reissner's spherical shell are

$$\begin{cases} \beta(\mu\frac{\partial\varphi_y}{\partial y}+\frac{\partial\varphi_x}{\partial x})h^3+(1+\beta x)(\mu\frac{\partial^2\varphi_y}{\partial x\partial y}+\frac{\partial^2\varphi_x}{\partial x^2})h^3-\frac{1}{2}(1+\beta x)(\mu-1)(\frac{\partial^2\varphi_x}{\partial y^2}+\frac{\partial^2\varphi_y}{\partial x\partial y})h^3 \\ -5(1+\beta x)(\mu-1)h(\frac{\partial w}{\partial x}-\varphi_x)=0 \\ -\frac{1}{2}\beta(\mu-1)(\frac{\partial\varphi_x}{\partial y}+\frac{\partial\varphi_y}{\partial x})h^3-\frac{1}{2}(1+\beta x)(\mu-1)(\frac{\partial^2\varphi_x}{\partial x\partial y}+\frac{\partial^2\varphi_y}{\partial x^2})h^3+(1+\beta x)(\mu\frac{\partial^2\varphi_x}{\partial x\partial y} \\ +\frac{\partial^2\varphi_y}{\partial y^2})h^3-5(1+\beta x)(\mu-1)h(\frac{\partial w}{\partial y}-\varphi_y)=0 \\ 5hE_0(1+\beta x)(\nabla^2 w-\frac{\partial\varphi_x}{\partial x}-\frac{\partial\varphi_y}{\partial y})+12\kappa(1+\mu)\nabla^2\psi+5E_0\beta h(\frac{\partial w}{\partial x}-\varphi_x)=0 \\ (1+\beta x)^2\nabla^4\psi+\kappa E_0 h(1+\beta x)^3\nabla^2 w-2\beta(1+\beta x)\frac{\partial(\nabla^2\psi)}{\partial x}-2\beta^2\mu\frac{\partial^2\psi}{\partial y^2}+2\beta^2\frac{\partial^2\psi}{\partial x^2}=0 \end{cases} \tag{2}$$

where, φ_x, φ_y are the angle displacement, w is the deflection, κ is the curvature, ψ is the stress function.

3. The boundary conditions

As the crack surface is free, the boundary conditions are

$$\begin{cases} M_y=0,\ \ M_{xy}=0,\ \ Q_y=0 \\ N_y=0,\ N_{xy}=0 \end{cases} \tag{3}$$

Further, they can be expressed as

$$\begin{cases}\mu\dfrac{\partial\varphi_x}{\partial x}+\dfrac{\partial\varphi_y}{\partial y}=0, \quad \dfrac{\partial\varphi_x}{\partial y}+\dfrac{\partial\varphi_y}{\partial x}=0\\ \dfrac{\partial w}{\partial y}-\varphi_y=0, \quad \dfrac{\partial^2\psi}{\partial x^2}=0, \quad \dfrac{\partial^2\psi}{\partial x\partial y}=0\end{cases} \tag{4}$$

The crack tip stress field would be equipped with the same square root singularity as that of homogeneous materials when the material properties of different composite materials at the interfaces are continuous [7,8]. Therefore, the generalized displacements $\varphi_r, \varphi_\theta, w$ and the stress function ψ can be expressed as follows [9]

$$\varphi_x=\sum_{i=1}^{\infty}\varphi_{xi}(\theta)r^{i/2}, \ \varphi_y=\sum_{i=1}^{\infty}\varphi_{yi}(\theta)r^{i/2}, \ w=\sum_{i=1}^{\infty}w_i(\theta)r^{i/2}, \ \psi=\sum_{i=1}^{\infty}\psi_i(\theta)r^{1+i/2} \tag{5}$$

where, $\varphi_{xi}(\theta)$ 、$\varphi_{yi}(\theta)$ 、$w_i(\theta)$ are the eigen-functions of the generalized displacement components,

$\psi_i(\theta)$ are the eigen-functions of the stress function.

Substituting Eq. (5) into Eq. (4) and considering the linear independence of $r^{-3/2}$, r^{-1}, $r^{-1/2}$,…, $r^{i/2-2}$,…, the boundary conditions are

$$\begin{cases} i\mu\cos\theta\varphi_{xi}(\theta)-2\mu\sin\theta\varphi'_{xi}(\theta)+i\sin\theta\varphi_{yi}(\theta)+2\cos\theta\varphi'_{yi}(\theta)=0\\ i\sin\theta\varphi_{xi}(\theta)+2\cos\theta\varphi'_{xi}(\theta)+i\cos\theta\varphi_{yi}(\theta)-2\sin\theta\varphi'_{yi}(\theta)=0\\ \begin{cases} i\sin\theta w_{i+2}(\theta)+2\cos\theta w'_{i+2}(\theta)-2\varphi_{yi}(\theta)=0, & i=1,2,3,\ldots\\ i\sin\theta w_{i+2}(\theta)+2\cos\theta w'_{i+2}(\theta)=0, & i=1,2.\end{cases}\\ 4\psi''_i(\theta)\sin^2\theta-2i\psi'_i(\theta)\sin 2\theta+\psi_i(\theta)(i^2\cos^2\theta+4\sin^2\theta+2i)=0\\ -4\psi''_i(\theta)\sin 2\theta+4i\psi'_i(\theta)\cos 2\theta+\psi_i(\theta)\sin 2\theta(i^2-4)=0\end{cases} \tag{6}$$

where, $(\ )'=d(\ )/d\theta$, $(\ )''=d^2(\ )/d\theta^2$.

4. The higher order crack-tip field

Substituting Eq. (5) into Eq. (2) and Eq. (6) and utilizing the linear independence of $r^{-3/2}$, r^{-1}, $r^{-1/2}$,…, $r^{i/2-2}$,…, the system of ordinary differential equations are obtained. Solving the system, we can obtain the results

$$
\begin{cases}
\varphi_{x1}(\theta) = C_{11}\sin\frac{\theta}{2} + C_{12}\cos\frac{\theta}{2} + C_{13}\sin\frac{3\theta}{2} + C_{14}\cos\frac{3\theta}{2} \\
\varphi_{x2}(\theta) = C_{23}\sin\theta + C_{24}\cos\theta \\
\varphi_{x3}(\theta) = \frac{1}{12h^2(\mu+1)^2}[-3\beta h^2(\mu+1)^2 C_{13}\sin\frac{5\theta}{2} - 3\beta h^2(\mu+1)^2 C_{14}\cos\frac{5\theta}{2} + (48\beta\mu^2 h^2 C_{13} - \\
\quad 96\beta\mu h^2 C_{13} + 112\beta h^2 C_{13} - 16\beta h^2 C_{11} + 16\beta\mu^2 h^2 C_{11} + 12h^2 C_{32} + 16\mu^2 h^2 C_{33} - 48h^2 C_{33} \\
\quad +80\mu^2 B_{11} + 12\mu^2 h^2 C_{32} + 24\mu h^2 C_{32} - 32\mu h^2 C_{33} - 80B_{11})\sin\frac{3\theta}{2} + (96\beta\mu h^2 C_{14} - 80B_{11} \\
\quad -48\beta\mu^2 h^2 C_{14} - 112\beta h^2 C_{14} - 16\beta h^2 C_{12} + 16\beta\mu^2 h^2 C_{12} - 12h^2 C_{31} - 48h^2 C_{34} + 80\mu^2 B_{11} \\
\quad +16\mu^2 h^2 C_{34} - 12\mu^2 h^2 C_{31} - 24\mu h^2 C_{31} - 32\mu h^2 C_{34})\cos\frac{3\theta}{2} + (24\beta h^2 C_{13} + 24\beta\mu^2 h^2 C_{13} \\
\quad +48\beta\mu h^2 C_{13} - 12h^2 C_{33} - 12h^2\mu^2 C_{33} - 24h^2\mu C_{33})\sin\frac{\theta}{2} + (24\beta h^2 C_{14} + 24\beta\mu^2 h^2 C_{14} + \\
\quad 48\beta\mu h^2 C_{14} + 12h^2 C_{34} + 12h^2\mu^2 C_{34} + 24h^2\mu C_{34})\cos\frac{\theta}{2}] \\
\varphi_{x4}(\theta) = (-12h^2 C_{42} + 4h^2 C_{44} + 20\mu B_{21} + 4\mu h^2 C_{44} + 2\beta\mu h^2 C_{22} - 20B_{21} - 2\beta h^2 C_{22} + 4\mu h^2 C_{42} \\
\quad -2\beta h^2 C_{23} + 2\beta\mu h^2 C_{23})\sin 2\theta + (-12h^2 C_{41} - h^2 C_{43} - \mu^2 h^2 C_{43} - \beta\mu^2 h^2 C_{21} - \beta\mu h^2 C_{24} \\
\quad +5\mu^2 B_{22} - 8\mu h^2 C_{41} - 2\mu h^2 C_{43} + 4\mu^2 h^2 C_{41} - \beta h^2 C_{24} - \beta\mu h^2 C_{21})\cos 2\theta - \mu^2 h^2 C_{43} - \\
\quad \beta h^2 C_{24} - 5B_{22} + 4h^2 C_{41} - \beta\mu^2 h^2 C_{21} - \beta\mu h^2 C_{24} - \beta\mu h^2 C_{21} + 5\mu^2 B_{22} \\
\quad \ldots\ldots
\end{cases} \tag{7}
$$

$$
\begin{cases}
\varphi_{y1}(\theta) = -\frac{1}{\mu+1}[-(C_{14} + \mu C_{14})\sin\frac{3\theta}{2} + (C_{13} + \mu C_{13})\cos\frac{3\theta}{2} + (-4\mu C_{14} + 12C_{14} \\
\quad +C_{12} + \mu C_{12})\sin\frac{\theta}{2} + (-4\mu C_{13} + 12C_{13} - C_{11} - \mu C_{11})\cos\frac{\theta}{2}] \\
\varphi_{y2}(\theta) = C_{21}\sin\theta + C_{22}\cos\theta \\
\varphi_{y3}(\theta) = C_{34}\sin\frac{\theta}{2} + C_{33}\cos\frac{\theta}{2} + C_{31}\sin\frac{3\theta}{2} + C_{32}\cos\frac{3\theta}{2} - \frac{1}{4}\beta C_{14}\sin\frac{5\theta}{2} + \frac{1}{4}\beta C_{13}\cos\frac{5\theta}{2} \\
\varphi_{y4}(\theta) = C_{43}\sin 2\theta + C_{44}\cos 2\theta + C_{42} \\
\quad \ldots\ldots
\end{cases} \tag{8}
$$

$$
\begin{cases}
\psi_1(\theta) = A_{11}\sin\frac{\theta}{2} + A_{12}\cos\frac{\theta}{2} + A_{13}\sin\frac{3\theta}{2} + A_{14}\cos\frac{3\theta}{2} \\
\psi_2(\theta) = A_{21} + A_{24}\cos 2\theta \\
\psi_3(\theta) = A_{31}\sin\frac{\theta}{2} + A_{32}\cos\frac{\theta}{2} + \frac{A_{11}}{4}\beta\sin\frac{3\theta}{2} + \frac{A_{12}}{4}\beta\cos\frac{3\theta}{2} + A_{34}\cos\frac{5\theta}{2} + A_{33}\sin\frac{5\theta}{2} \\
\psi_4(\theta) = A_{41}\sin 3\theta + A_{42}\cos 3\theta + A_{43}\sin\theta + A_{34}\cos\theta \\
\quad \ldots\ldots.
\end{cases} \tag{9}
$$

$$
\begin{cases}
w_1(\theta) = B_{11}\sin\dfrac{\theta}{2}, \\
w_2(\theta) = B_{22}\cos\theta, \\
w_3(\theta) = \dfrac{1}{20E_0h(\mu+1)}[(-40E_0hC_{13} + 5E_0\beta hB_{11} - 96\kappa\mu A_{11} - 48\kappa\mu^2 A_{11} - 48\kappa A_{11} + 5E_0\beta h\mu B_{11} \\
\qquad +40E_0h\mu C_{13})\sin\dfrac{\theta}{2} + (-96\kappa\mu A_{12} - 48\kappa\mu^2 A_{12} - 48\kappa A_{12} - 40E_0hC_{14} + 40E_0h\mu C_{14})\cos\dfrac{\theta}{2} \\
\qquad +(20hE_0B_{32} + 20hE_0\mu B_{32})\sin\dfrac{3\theta}{2} + (20hE_0B_{31} + 20hE_0\mu B_{31})\cos\dfrac{3\theta}{2}] \\
w_4(\theta) = B_{42}\sin 2\theta + B_{41}\cos 2\theta + \dfrac{5hE_0(C_{21} + C_{24} - \beta B_{22}) - 48A_{21}(\mu+1)\kappa}{2E_0h} \\
\qquad \ldots\ldots
\end{cases}
\quad (10)
$$

where: A_{ij} , B_{ij} , C_{ij} are the undetermined coefficients.

Substituting Eq. (7)-(10) into Eq.(5), the generalized displacement fields of FGMs spherical shell are obtained.

5. Conclusion

For a functionally graded spherical shell with Reissner effect, the higher order crack tip fields which are similar to the Williams' solutions of crack problems in homogenous materials are obtained. As the in-homogeneity parameter $\beta \to 0$, the solutions degenerate to the corresponding fields of isotropic homogeneous spherical shell with Reissner's effect. Obviously, these results provide the theoretical basis for experimental investigation and engineering application.

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