Numerical study of the deformations of two coplanar circular cracks during their coalescence

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Abstract

Consider two planar circular cracks embedded in an infinite linear elastic media and submitted to mode I tensile loading. Bueckner-Rice weight functions theory allows us to update the stress intensity factor when the crack fronts are slightly deformed in their plane. Using an incremental numerical method based on this theory, we study the propagation of these two cracks when they interact each other taking into account the non-linearities induced by their deformations. The advantage of this method in comparison to more standard finite element methods is that only the crack fronts have to be meshed. Using a Griffith threshold law, we notice important deformations of the crack fronts are observed and a drastically decreasing threshold loading when the fronts approach each other.

Keywords: Brittle fracture, Toughening, Finite element method, Elastic line model

The present study focuses on the coalescence phenomenon of two circular cracks. What is the critical loading to reach the coalescence? Is the crack advance facilitated due to the presence of the secondary crack? What is the shape of the cracks during their propagation? Those questions are considered in the present article. To do it accurately, the main difficulty is to calculate the three-dimensional stress intensity factors along all the fronts by taking into account the crack shape changes induced by the interaction between the cracks.

In the literature, we can find papers treating of interacting cracks but they never take into account the cracks fronts deformation during propagation (see [2 − 5] and [8]).

Here, the effects of the crack front shape changes are analized independently of the edge effects. For this purpose, two small cracks are considered, so we can make the assumption that the medium is infinite and subjected to remote loading. For this reason, methods based on integral equations are adapted here: the sole cracked area is needed. Moreover, in the present case of in-plane propagation, it is just necessary to mesh the 1D outline of the cracks. Using Bueckner-Rice formalism [7], the work of Bower and Ortiz [1] provides some examples of this approach in mode I. More recently, Lazarus [6] developped a simplified variant of their method without significant loss of accuracy. All these works only deal with a sole crack. In the present paper, we extend to two cracks in order to study their final coalescence.

1. Objectives

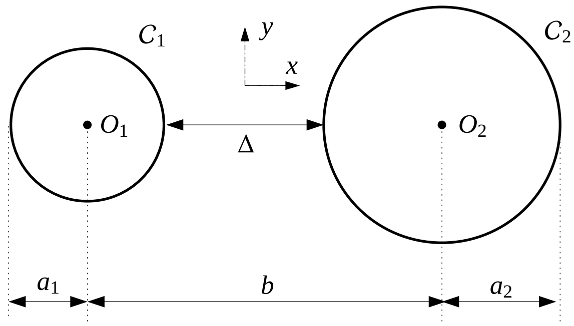


Figure 1: Two circular cracks.

Let us consider two circular coplanar cracks embedded in an isotropic elastic body (such as depicted in figure 1). The aim of this paper is to predict the in-plane propagation of these cracks subjected to remote tensile stress σ_∞ at infinity in brittle fracture. The method consists in coupling the Bueckner-Rice formalism with a propagation law starting from a configuration for which the needed quantities, namely the stress intensity factor (SIF) along the front and a certain kernel, are known. The procedure consists of different steps:

- Determination of the SIF for a given geometry:
  Knowing the geometry and the loading, how to calculate the SIF along the fronts?
- Propagation problem with a threshold:
  In brittle fracture, it is assumed that the propagation law is given by Irwin's criterion:

$$\begin{cases} K < K_c & : \text{ no propagation} \\ K = K_c & : \text{ possible propagation} \end{cases} \tag{1}$$

  For a given crack geometry, there is a critical loading σ_∞^c such as: if $\sigma_\infty < \sigma_\infty^c$ then $K(M) < K_c$, $\forall\, M$ and if $\sigma_\infty = \sigma_\infty^c$, there is at least one point M of the front that verifies $K(M) = K_c$. We want to determine this stability threshold σ_∞^c all along the propagation.

2. Numerical approach

2.1. Adimensionalization

Let us define the dimensionless problem for which the distance between cracks centers b is taken at 1. The quantities of interest become: a_1/b and a_2/b, with a loading unit $\sigma_\infty = 1$, letting $K(M) = \sqrt{b}\,\sigma_\infty\,\widehat{K}(M)$, where $\widehat{K}$ is a dimensionless quantity.

2.2. Rice incremental formulae

Suppose that the crack geometry is slighty perturbed in its plane and consider a point $M_i \in C = C_1 \bigcup C_2$.

Let us set $\alpha = (1\,,\,2)$ and $\beta = (1\,,\,2)\,,\,\beta \neq \alpha$.

Then Rice's first formula reads ([7]):

$$\begin{aligned} \delta\widehat{K}_\alpha(M_i) &= \frac{1}{2\pi}\ VP\int_C \frac{W(M_k,M_i)}{D^2(M_k,M_i)}\widehat{K}(M_k)[\delta a(M_k) - \delta_* a(M_k)]ds(M_k) \\ &= \frac{1}{2\pi}\ VP\int_{C_\alpha} \frac{W_{\alpha\alpha}(M_i,M_k)}{D^2_{\alpha\alpha}(M_i,M_k)}\widehat{K}_\alpha(M_k)\delta a_\alpha^{(\alpha)}(M_k)ds(M_k) \\ &+ \frac{1}{2\pi}\int_{C_\beta} \frac{W_{\alpha\beta}(M_i,M_k)}{D^2_{\alpha\beta}(M_i,M_k)}\widehat{K}_\beta(M_k)\delta a_\alpha^{(\beta)}(M_k)ds(M_k) \end{aligned} \tag{2}$$

with:

$$\delta a_{\alpha}^{(\gamma)}(M_k) = \delta a_\gamma(M_k) - (\delta a_\alpha(M_i)\overrightarrow{n_\alpha}(M_i)).\overrightarrow{n_\gamma}(M_k) \quad , \quad \gamma = 1, 2$$

and where $W_{\alpha\gamma}(M_i, M_k)$ is a kernel expressing the effect of the advance of $M_i \in C_\alpha$ over the SIF at point $M_k \in C_\gamma$. Rice's second formula can be written of the form:

- if $M_i \in C_\alpha$ and $M_k \in C_\alpha$ (points belong to the same crack front):

$$\begin{aligned} \delta W_{\alpha\alpha}(M_i, M_k) &= \frac{D^2(M_i, M_k)}{2\pi} \; VP \int_C \frac{W(M_i, M)W(M; M_k)}{D^2(M_i; M)D^2(M; M_k)} \left[\delta a(M) - \delta_{**}a(M)\right] ds(M) \\ &= \frac{D^2_{\alpha\alpha}(M_i, M_k)}{2\pi} \; VP \int_{C_\alpha} \frac{W_{\alpha\alpha}(M_i, M)W_{\alpha\alpha}(M_k, M)}{D^2_{\alpha\alpha}(M_i, M)D^2_{\alpha\alpha}(M_k, M)} \delta a^{(\alpha)}_{\alpha\alpha}(M) ds(M) \\ &+ \frac{D^2_{\alpha\alpha}(M_i, M_k)}{2\pi} \int_{C_\beta} \frac{W_{\alpha\beta}(M_i, M)W_{\alpha\beta}(M_k, M)}{D^2_{\alpha\beta}(M_i, M)D^2_{\alpha\beta}(M_k, M)} \delta a^{(\beta)}_{\alpha\alpha}(M) ds(M) \end{aligned} \tag{3}$$

- if $M_i \in C_\alpha$ and $M_k \in C_\beta$ (points belong to different crack fronts):

$$\begin{aligned} \delta W_{\alpha\beta}(M_i, M_k) = \frac{D^2_{\alpha\beta}(M_i, M_k)}{2\pi} \; VP \int_{C_\alpha} \frac{W_{\alpha\alpha}(M_i, M)W_{\alpha\beta}(M, M_k)}{D^2_{\alpha\alpha}(M_i, M)D^2_{\alpha\beta}(M, M_k)} \delta a^{(\alpha)}_{\alpha\beta}(M) ds(M) \\ + \frac{D^2_{\alpha\beta}(M_i, M_k)}{2\pi} \; VP \int_{C_\beta} \frac{W_{\alpha\beta}(M_i, M)W_{\beta\beta}(M_k, M)}{D^2_{\alpha\beta}(M_i, M)D^2_{\beta\beta}(M_k, M)} \delta a^{(\beta)}_{\alpha\beta}(M) ds(M) \end{aligned} \tag{4}$$

with: $\delta a^{(\gamma)}_{\alpha\beta}(M) = \delta a_\gamma(M) - \overrightarrow{V_{i,k}}(M).\overrightarrow{n_\gamma}(M)$, where $\overrightarrow{V_{i,k}}$ is a geometric transformation such as: $\delta a^{(\gamma)}_{\alpha\beta}(M_i) = \delta a^{(\gamma)}_{\alpha\beta}(M_k) = 0$.

These formulae give us the first order perturbation of the SIF and kernels, knowing the perturbation δa and the initial SIF and W. Here comes the natural idea of an iterative procedure to predict the propagation. For this purpose, it's necessary to start from a configuration for which the quantities $\widehat{K_\alpha}$ and $W_{\alpha\gamma}$ are known. We assume that for two circular cracks which are distant enough, the SIF and functions $W_{\alpha\gamma}$ are those for single crack that is :

$$\begin{cases} \widehat{K_\alpha}(M) = 2\sqrt{a/\pi} \\ W_{\alpha\alpha} = 1 \\ W_{\alpha\beta} = 0 \end{cases} \tag{5}$$

This situation will serve as starting point of our method, corresponding to two circular cracks of size $a_0/b << 1$.

2.3. Propagation

Assumption is made that propagation is governed by the SIF so that we have:

$$\delta a\,(M) = \delta a_{max} \left[\frac{K(M)}{K_{max}}\right]^\beta \tag{6}$$

where δa_{max} is a small given quantity.

It is presumed here that this law simulates brutal fracture if $\beta \gg 1$ and fatigue propagation otherwise.
Subjected to Irwin's criteria, cracks are supposed to propagate in a quasistatic way under a remote loading σ_∞, varying

at each numeric step such as: $\underset{M\in C}{max} K(M)/K_c = 1$. This condition ensures that $K(M) < K_c, \forall M \in C$ and that there is always an “active” part of the front. The loading is thus recalculated at each numerical step:

$$\sigma_\infty \sqrt{b} = \frac{K_c}{\underset{M\in C}{max}\widehat{K}} \tag{7}$$

We can define here the real nondimensionalized loading as follows:

$$\sqrt{b}\,\frac{\sigma_\infty}{K_c} = \frac{1}{\underset{M\in C}{max}\widehat{K}(M)} \tag{8}$$

3. Determination of the SIF along two coplanar circular cracks of same radius $a_1 = a_2 = a$

In this section, the SIF values obtained for two coplanar penny-shaped cracks in interaction are presented. Unfortunately, to our knowledge, no 3D analytical solution exists for this problem that should serve as benchmarks. For weakly interacting cracks (that is the SIF remains close to their value for one single crack), Isida et al. [4], Fabrikant [3], Kachanov and Laures [5], Chen and Lee [2] and Zhan and Wang [8] provide numerical results that are in agreement with each other. We thus believe that those values must be correct and shall serve to validate our code and test the influence of our numerical parameters a_0/b, N and $\delta a/a$ (section 3.1). For closely spaced cracks, the numerical approximations are more questionable and few studies exist. Among them those of Fabrikant [3], Kachanov and Laures [5] and Zhan and Wang [8] will serve to compare with our simulations (section 3.2).

3.1. Weak interaction

Figures 2 reflect the influence of different parameters: the number of nodes N on each front, the initial adimensionless radius a_0/b and the crack advance $\delta a/a$. In this section, cracks are subjected to a uniform advance defined as follows: $\dfrac{\delta a}{a} = \gamma\ min\left(a\,,\,\dfrac{\Delta}{2}\right)$.
All the figures 2 represent K_{max}/K_0 as a function of $\Delta/2a$ in the same y-range to make easier comparisons. Moreover it shall be noticed that the SIF of all points within the frame are less diffent than 10 % from the ones for a single isolated crack.

The initial cracks should not be too small because of the incremental nature of the method, numerical errors would accumulate. Typically, one shall choose a_0/b between 0.05 and 0.1.

There is a few dependence on the number of points, provided that $N > 100$. Moreover we notice that $K(s)$ presents some irregularities for $N < 160$. Thus values of $N \geq 160$ shall be used. Since the CPU depends on N, we shall be reasonable. Typically $N = 160$ seems a good compromise.

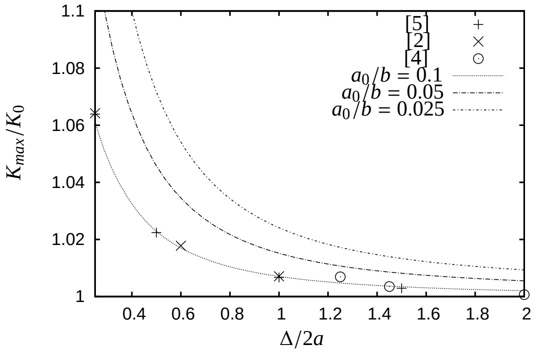
Once again due to the incremental nature of the method, we notice on figure (2c) that $\delta a/a$ shall be not too small but enough to use the first order perturbation formulae. Typically, $\delta a/a \in (0.025 - 0.1)$ is acceptable.

We shall use $a_0/b = 0.1$, $N = 160$, $\delta a/a = 0.025$ in the sequel.

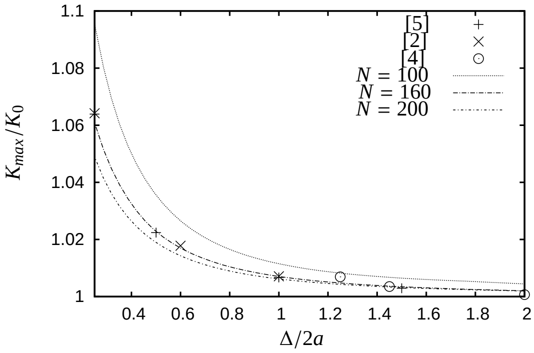
3.2. Strong interaction

In the sequel, let's define: $l = \min(a, \Delta)$.
It shall be noticed that the method is unstable for some set of parameters. It is linked to the incremental nature of the method and to the amplification of K as soon as some angular points appear along the crack front. For instance, one can notice on figure (3c) that for $N = 160$, $(a_0/b = 0.1$, $\delta a/l = 0.01$ the value of K_{max} diverges. In the sequel, we consider those calculations as ill and arrange to find well suited set of parameters. A systematic study of numerical stability is under consideration and will be published in the future.

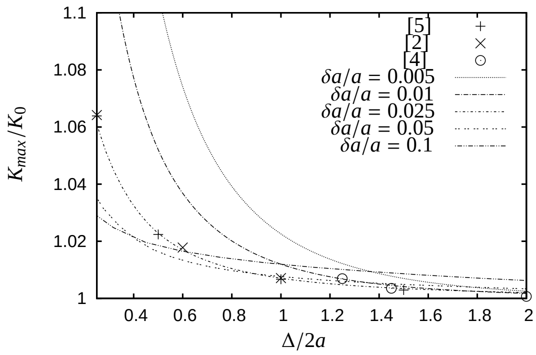
From those results, we can conclude that the method is able to give qualitatively correct values of K but quantitatively, is quiet sensitive to the numerical parameters. In particular for cracks as close as $\Delta/2a < 10^{-4}$, a relative dispersion (standard deviation/mean value) can be observed of approximately 100 % by choosing reasonable parameters ($N = 100 - 200$, $a_0/b = 0.1 - 0.2$, $\delta a/l = 0.01 - 0.05$). For higher values of $\Delta/2a$, the dispersion decreases. It is of 50 % for $\Delta/2a \sim 10^{-2}$, 10 % for $\Delta/2a \sim 10^{-1}$, 1 % for $\Delta/2a \sim 0.5$, 0.1 % for $\Delta/2a \sim 1$.



(a) $N = 160$, $\delta a/a = 0.025$

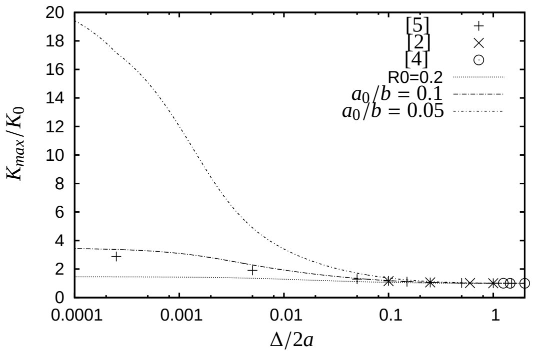


(b) $\delta a/a = 0.025$, $a_0/b = 0.1$

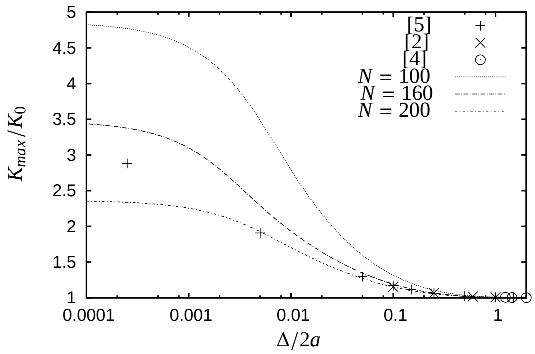


(c) $\delta a/a$ ($N = 160$, $a_0/b = 0.1$

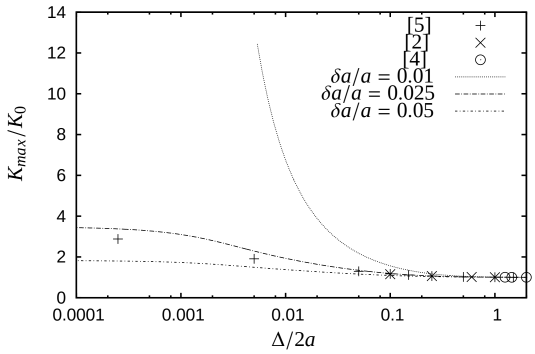
Figure 2: Dependance on the initial size a_0/b (a), on the number N of points in the mesh (b), on the crack advance (c)



(a) $N = 160$, $\delta a/a = 0.025$



(b) $\delta a/l = 0.025$, $a_0/b = 0.1$



(c) $\delta a/l = 0.025$, $a_0/b = 0.1$

Figure 3: Dependance on the initial size a_0/b (a), on the number N of points in the mesh (b), on the crack advance (c)

Nevertheless we achieve to obtain very similar results than previous authors Fabrikant [3], Kachanov and Laures [5] and Zhan and Wang [8] by choosing $a_0/b = 0.1$, $N = 160$, $\delta a/a = 0.025$ (see figure 4 where *phi* is the polar angle). We shall use those parameters as reference in the sequel.

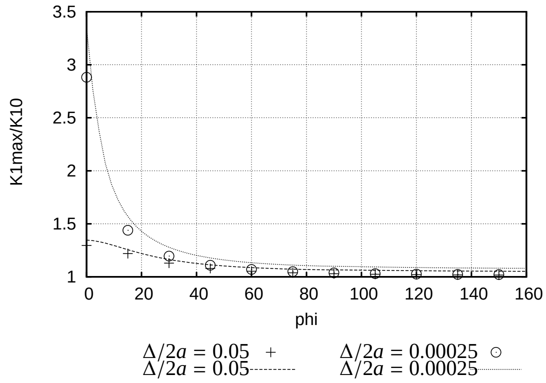


Figure 4: Points corresponds to the values of Kachanov and Laures [5] given in table 1. Lines correspond to our simulations for $a_0/b = 0.1$, $N = 160$, $\delta a/l = 0.025$.

4. Propagation of two circular cracks in brittle fracture

We present here our results for simulations in brittle fracture. For numerical purposes, Irwin's law can be remedied by a Paris type law provided to choose an exponent β large enough. In practice, above $\beta = 30$, results are very close and become independent of β. That is why the sole case $\beta = 30$ is presented here. Crack deformation is consequent so that we had to set up a remesh procedure to redistribute nodes.

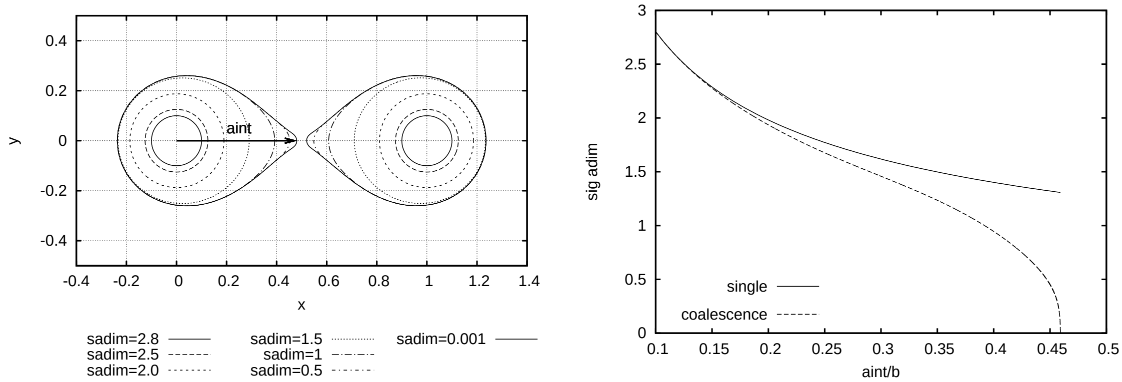
Figure (5a) shows the successive positions of the fronts for different values of the dimensionless loading $\sigma_\infty \frac{\sqrt{b}}{K_c}$. When the cracks are distant, threshold is reached for the entire set of points because the SIF values are almost uniform along the fronts.

When a/b reaches about $1/4$, interaction between cracks leads to an increase in SIF of points near the opssite crack. In consequence, the threshold is only achieved at these nodes whence a pronounced front deformation. It should be noted that coalescence couldn't be reached because of the values larger and larger of the SIF. Indeed, SIF values are asymptotically infinite at the vinicity of the "interaction area".

Figure (5b) represents the real loading in terms of cracks advance, characterized by the dimensionless quantity a_{int}/b for the case of coalescence and for the isolated crack. It can be noticed that the loading strongly decreases during propagation and tends to almost disappear when cracks are close to one another.

5. Conclusion and perspectives

The purpose of this work was to apply Lazarus's numerical code to study the coalecence of circular cracks.
To validate the code, we compared the SIF values, obtained for different configurations, with those found in the literature. Good agreements with Fabrikant [3], Kachanov and Laures [5] and Zhan and Wang [8] were achieved. After validation with literature, brittle propagation was experimented. It shoud be emphasized that each front was highly perturbed by the presence of the other crack. Our simulations showed cracks with a strongly elongated profile within the "interaction area". It can also be observed a significant decrease of the fracture loading as soon as the interaction between cracks was felt.
The possibility of extending the code to more complex geometries is considered.



(a) Successive positions of the crack

(b) Critical loading during propagation

Figure 5: $\beta = 30$, $N = 160$, $\delta a_{max}/l = 0.001$, $a_0/b = 0.1$, remesh each 1000 numerical cycles

Acknowledgements

[1] A. F. Bower, M. Ortiz, 1990, Solution of three-dimensional crack problems by a finite perturbation method, J. of the Mechanics and Physics of Solids 38(4), 443 480.
[2] Y. Chen, K. Y. Lee, 2002, Solution of flat crack problem by using variational principle and differential-integral equation, Int. J. of Solids and Structures 39 (23), 578797.
[3] V. Fabrikant, 1987, Close interaction of coplanar circular cracks in an elastic medium, Acta Mechanica 67, 3959.
[4] M. Isida, K. Hirota, H. Noguchi, T. Yoshida, 1985, Two parallel elliptical cracks in an infinite solid subjected to tension, Int. J. of Fracture 27, 3148.
[5] M. Kachanov, J.-P Laures, 1989, Three-dimensional problems of strongly interacting arbitrarily located penny-shaped cracks, Int. J. of Fracture 41, 289313.
[6] V. Lazarus, 2003, Brittle fracture and fatigue propagation paths of 3D plane cracks under uniform remote tensile loading, Int. J. of Fracture 122 (1-2), 2346.
[7] J. R. Rice, 1989, Weight function theory for three-dimensional elastic crack analysis, in R. P. Wei and R. P. Gangloff, eds, Fracture Mechanics: Perspectives and Directions (Twentieth Symposium), American Society for Testing and Materials STP 1020, Philadelphia, USA, pp. 2957.
[8] S. Zhan, T. Wang, 2006, Interactions of penny-shaped cracks in three-dimensional solids, Acta Mechanica Sinica 22, 341353.