Determination of crack surface displacements for a radial crack emanating from a semi-circular notch using weight function method

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Abstract A radial crack emanating from a semi-circular notch is of significant engineering importance. Accurate determination of key fracture mechanics parameters is essential for damage tolerance design and fatigue crack growth life predictions. The purpose of this paper is to provide an efficient and accurate closed-form weight function approach to the calculation of crack surface displacements for a radial crack emanating from a semi-circular notch in a semi-infinite plate. Results are presented for two load conditions: remote applied stress and uniform stress segment applied to crack surfaces. Based on a correction of stress intensity factor ratio, highly accurate analytical equations of crack surface displacements under the two load conditions are developed by fitting the data obtained by using the weight function method. It is demonstrated that the Wu-Carlsson closed-form weight functions are very efficient, accurate and easy-to-use for calculating crack surface displacements for arbitrary load conditions. The method will facilitate fatigue crack closure and other fracture mechanics analyses where accurate crack surface displacements are required.

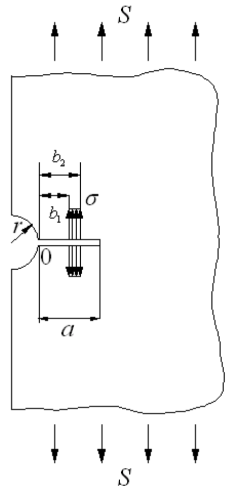
Keywords: Radial crack, Semi-circular notch, Dugdale strip-yield model, Crack surface displacement, Weight function method.

1. Introduction

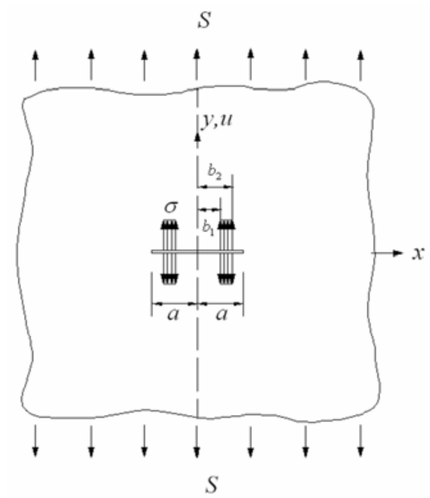
In fracture mechanics analysis and fatigue crack growth life predictions for structural components, the strip-yield model (modified Dugdale models) [1] have been employed for crack configurations like middle-crack tension [1-5] and compact specimens [6-8]. Reference [9] discussed the application of the model to analyze double radial cracks emanating from a circular hole. This crack configuration and a radial crack emanating from a semi-circular notch are among the most important crack types for aircraft structures. Stress intensity factors for such cracks under various loading conditions can be found in the literature [10-11]. However, crack surface displacements for various load conditions, preferably in analytical form, which are needed in crack-closure-based fatigue crack growth life prediction models and other fracture mechanics analyses, are rarely available.

A modified Dugdale model is proposed by Newman [1]. The modification is to leave plastically deformed materials in the wake of the crack. The primary advantage in using this model is that the plastic-zone size and crack surface displacements are obtained by superposition of two elastic problems: a crack in a plate subjected to a remote uniform stress and a uniform stress applied over a segment of the crack surfaces. In the Newman model, crack surface displacements under remote applied stresses and a segment uniform pressure (partial loading) in the immediate wake of the crack tip are both required for calculating the crack opening stress, S_{op}. The accuracy of these displacements will significantly influence S_{op}, and further, fatigue crack growth rates.

The motivation of the present paper is to explore an analytical approach, based on the closed-form weight functions, to the calculation of crack surface displacements for a radial crack emanating from a semi-circular notch in a semi-infinite plate. Results are presented for the two loading conditions: remote applied stress and uniform segment stress (partial load) applied to a segment of the crack surfaces. Based on a correction of stress intensity factor ratio, highly accurate analytical equations of the crack surface displacements are developed to fit the data from weight function method. It is envisaged that the closed-form weight function approach will provide an efficient and reliable method for calculating crack-surface displacements under arbitrary loadings. The analytical equations of the crack surface displacement can contribute to a more rapid and reliable fatigue crack growth life predictions for a radial crack emanating from a semi-circular notch in a semi-infinite plate.



(a)



(b)

Figure.1 Crack geometry. (a) A radial crack emanating from a semi-circular notch in a semi-infinite plate. (b) Middle crack in an infinite plate

2. Theoretical procedures

The weight function method [10] is a very powerful method for the determination of key fracture mechanics parameters for cracks under arbitrary load conditions, e.g. stress intensity factors and crack surface displacements. The high efficiency of this method lies in that the weight function is a property of crack geometry only, and is independent of loading. For given crack configuration, once the weight function is obtained, fracture mechanics parameters for the same crack configuration under arbitrary loadings can be easily determined through a simple quadrature.

For stress intensity factors [10]:

$$K = f\sigma\sqrt{\pi a} \tag{1}$$

$$f = \int_0^a \frac{\sigma(x/a)}{\sigma} \cdot \frac{m(a,x)}{\sqrt{\pi a r}} \mathrm{d}x \tag{2}$$

where $m(a, x)$ is the weight function for the given crack geometry, which can be derived from the crack opening displacements $u(a, x)$ for a reference load case.

$$m(a,x) = \frac{E^{'}}{K(a)} \frac{\partial u(a,x)}{\partial a} \tag{3}$$

where $E^{'}=E$ for plane stress and $E^{'} = E/(1-\eta^2)$ for plane strain.

For the corresponding crack surface displacements [10]:

$$u(a,x) = \frac{\sigma}{E^{'}} \int_{a_0}^{a} [f(s)\sqrt{\pi s}] \cdot m(s,x)\mathrm{d}s \tag{4}$$

From the above equations, it is seen that the central issue in the weight function method is the determination of weight function $m(a, x)$ for the crack geometry in consideration. Various approaches have been used for determining $m(a, x)$. One effective way is to derive $m(a, x)$ through a reference load case. With this procedure, systematic derivations for the weight functions $m(a, x)$ for a large number of two-dimensional crack configurations have been given by Wu and Carlsson [10], which can be readily utilized for the present analysis.

For the edge crack configuration, which is applicable to the present case, i.e. a radial crack emanating from a semi-circular notch in a semi-infinite plate, the stress intensity factor and the corresponding crack face loading for the reference load case are expressed in polynomials of the type:

$$f_r(\frac{a}{r}) = \sum_{i=0}^{I} \alpha_i (\frac{a}{r})^i \tag{5}$$

$$\frac{\sigma_r(\frac{x}{r})}{\sigma} = \sum_{m=0}^{M} S_m (\frac{x}{r})^m \tag{6}$$

where r is the notch radius, and the coefficients α_i and S_m for the present crack configuration are given in Section 3.

The weight function was expressed in closed-form by Wu and Carlsson [10], as

$$m(a,x) = \sqrt{\frac{r}{2\pi a}} \sum_{i=1}^{4} \beta_i (\frac{a}{r}) \cdot (1-\frac{x}{a})^{i-\frac{3}{2}} \tag{7}$$

where

$$\left.\begin{aligned}
&\beta_1(\frac{a}{r}) = 2.0 \\
&\beta_2(\frac{a}{r}) = \frac{[4\frac{a}{r}f_r^{'}(\frac{a}{r}) + 2f_r(\frac{a}{r}) + \frac{3}{2}F_2(\frac{a}{r})]}{f_r(\frac{a}{r})} \\
&\beta_3(\frac{a}{r}) = \frac{\left\{\frac{a}{r}F_2^{'}(\frac{a}{r}) + \frac{1}{2}[5F_3(\frac{a}{r}) - F_2(\frac{a}{r})]\right\}}{f_r(\frac{a}{r})} \\
&\beta_4(\frac{a}{r}) = \frac{[\frac{a}{r}F_3^{'}(\frac{a}{r}) - \frac{3}{2}F_3(\frac{a}{r})]}{f_r(\frac{a}{r})}
\end{aligned}\right\} \tag{8}$$

where

$$\left.\begin{aligned}
&F_1(\frac{a}{r}) = 4f_r(\frac{a}{r}) \\
&F_2(\frac{a}{r}) = \frac{15\left[\sqrt{2}\pi\phi(\frac{a}{r}) - E_1(\frac{a}{r})\cdot F_1(\frac{a}{r})\right] - E_3(\frac{a}{r})\cdot F_1(\frac{a}{r})}{15E_2(\frac{a}{r}) - 3E_3(\frac{a}{r})} \\
&F_3(\frac{a}{r}) = \frac{E_2(\frac{a}{r})\cdot F_1(\frac{a}{r}) - 3\left[\sqrt{2}\pi\phi(\frac{a}{r}) - E_1(\frac{a}{r})\cdot F_1(\frac{a}{r})\right]}{15E_2(\frac{a}{r}) - 3E_3(\frac{a}{r})}
\end{aligned}\right\} \tag{9}$$

with

$$\phi(\frac{a}{r}) = \frac{1}{a^2}\int_0^a (s\cdot\left[f_r(\frac{s}{r})\right]^2)\mathrm{d}s \tag{10}$$

$$E_j(\frac{a}{r}) = \sum_{m=0}^{7}\frac{2^{m+1}m!S_m(\frac{a}{r})^m}{\prod_{k=0}^{m}(1+2j+2k)}, \quad j=1,2,3 \tag{11}$$

where r is the radius of the semi-circular notch; f_r and S_m are defined in Eq. (5) and Eq. (6).

3. Crack surface displacements for a radial crack emanating from a semi-circular notch in a semi-infinite plate

For a radial crack emanating from a semi-circular notch in a semi-infinite plate, uniform remote tension is chosen as the reference load case, and the corresponding non-dimensional stress intensity factor and the reference crack face loadings are expressed in Eq. (5) and Eq. (6) with the following coefficients, respectively:

a_i : 3.4345, -6.9715, 14.5781, -19.9230, 16.9424, -8.5301, 2.3134, -0.2595. (a/r≤2.0)

S_m : 3.0643, -6.6864, 13.1677, -16.9513, 13.7335, -6.6293, 1.7273, -0.1861. (x/r≤2.0)

3.1. Uniform remote tension

The normalized stress intensity factor of a radial crack emanating from a semi-circular notch in a semi-infinite plate under uniform remote tension, $f_{B,S}$, is given in Eq. (5), as:

$$f_{B,S} = \sum_{i=0}^{7} \alpha_i (\frac{a}{r})^i \tag{12}$$

With the information of $f_{B,S}$ and $m(a, x)$, the crack surface displacements can be easily calculated from Eq. (4). Figure 2 shows the normalized crack surface displacements of a radial crack emanating from a semi-circular notch in a semi-infinite plate under uniform remote tension for non-dimensional crack length a/r = 0.05 to 1. The circular symbols in Fig. 2~3 are obtained by the crack surface displacement analytical equations, which will be explained later.

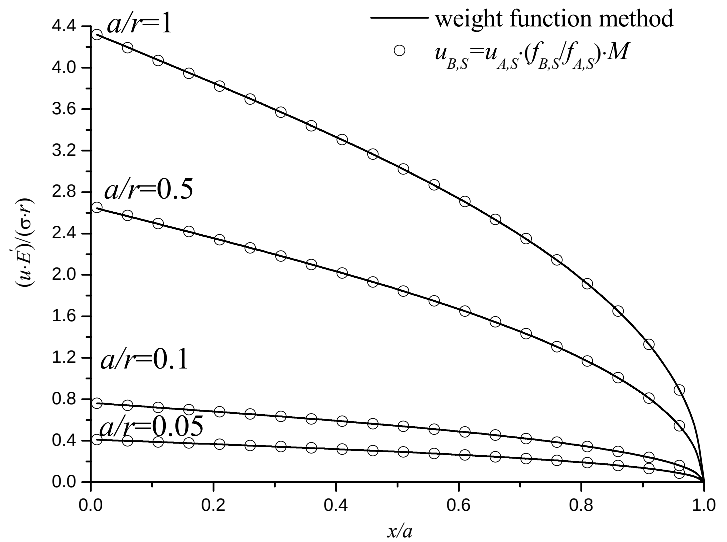


Figure.2 Non-dimensional crack surface displacements for a radial crack emanating from a semi-circular notch in a semi-infinite plate subjected to uniform remote tension.

3.2. Partial crack surfaces subject to Dugdale loading

For partial crack surfaces subject to Dugdale loading (see Fig. 1(a), b_2=a), the stress distribution is expressed in a simple form:

$$\sigma(x) = \sigma \qquad (b_1 \le x \le a) \tag{13}$$

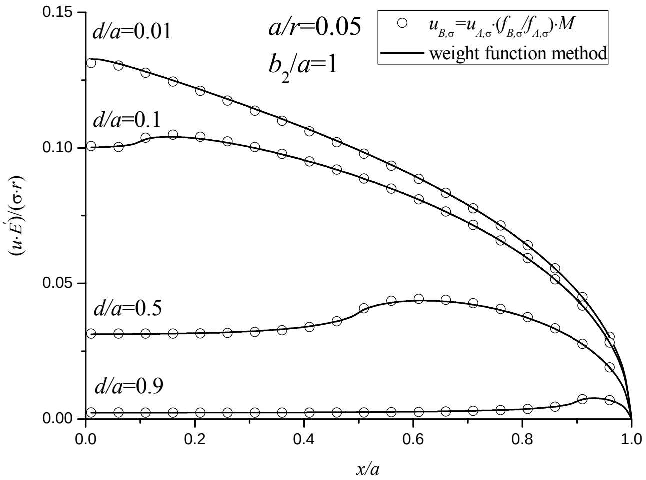
For partial crack surfaces subject to Dugdale loading, the normalized stress intensity factor of a radial crack emanating from a semi-circular notch in a semi-infinite plate is easily calculated by inserting Eq.(13) and Eq. (7) into Eq. (2), with

$$f_{B,\sigma} = \frac{\sqrt{2}}{\pi}\left[\sum_{1}^{4}\frac{1}{(2i-1)}\beta_i(\frac{a}{r})\cdot(1-\frac{b_1}{a})^{i-\frac{1}{2}}\right] \tag{14}$$

With the above $f_{B,\sigma}$-expression and $m(a, x)$, the crack surface displacements (Eq. (4)) for partial crack surfaces subject to Dugdale loading can be readily calculated from the following equation:

$$u_{B,\sigma} = \frac{\sigma}{E'\pi}\int_{a_0}^{a}(\left[\frac{1-\frac{b_1}{s}}{1-\frac{x}{s}}\right]^{\frac{1}{2}}\cdot\sum_{i=1}^{4}\frac{\beta_i(s/r)}{2i-1}\cdot(1-\frac{b_1}{s})^{i-1}\cdot\sum_{i=1}^{4}\beta_i(s/r)\cdot(1-\frac{x}{s})^{i-1})\mathrm{d}s \tag{15}$$

Figure 3 present the results, the normalized crack surface displacements of a radial crack emanating from a semi-circular notch in a semi-infinite plate with crack surfaces subject to Dugdale loading ($b_2=a$, $b_1=d$). The non-dimensional crack lengths in Fig. 3 are: a/r=0.05, 0.1, 0.5 and 1.0.



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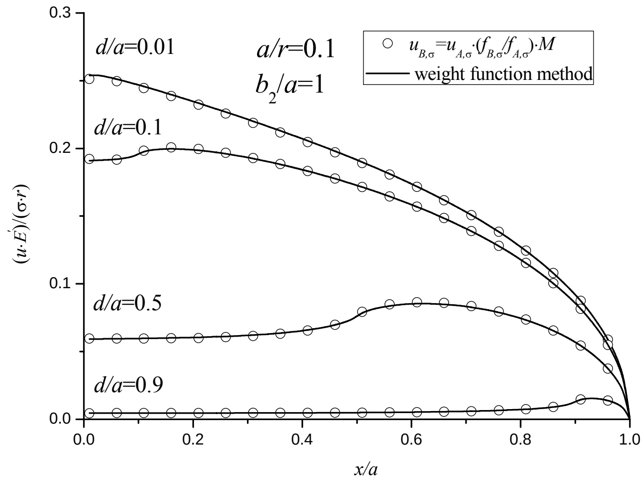

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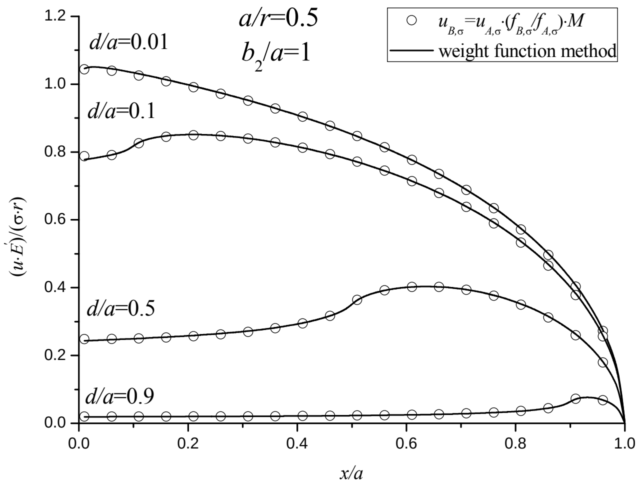

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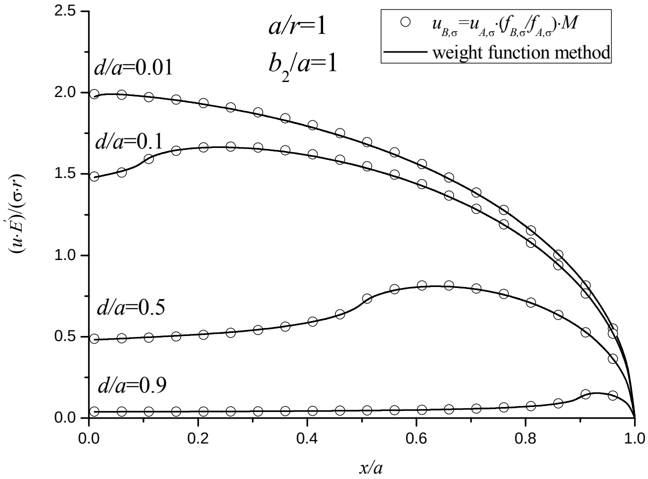

(d)

Figure.3 Non-dimensional crack surface displacements for a radial crack emanating from a semi-circular notch in a semi-infinite plate subjected to Dugdale loading in the immediate wake of crack tip, with (a) a/r=0.05. (b) a/r=0.1. (c) a/r=0.5. (d) a/r=1.0.

4. Analytical crack surface displacement equations for a radial crack emanating from a semi-circular notch in a semi-infinite plate

It has been a common practice in the literature that the crack surface displacements for one case (B) can be estimated from available crack surface displacements for a similar known case (A), by multiplying a correction factor. Recently, Tong and Wu proposed a general expression for an edge crack in a semi-infinite plate and double radial cracks at a circular hole with good success [12,13]. The correction is composed of two parts: the ratio of stress intensity factors for the two cases, f_B / f_A and a fitted correction factor M, as the following:

$$u_B = u_A \cdot \frac{f_B}{f_A} \cdot M \tag{16}$$

where the subscripts $_A$ and $_B$ represent the two load cases.

Because the crack surface displacements of a center crack in an infinite plate (see Fig. 1(b)) have exact solutions, this case (see Fig. 1(b)) is selected as load case A, a radial crack emanating from a semi-circular notch in a semi-infinite plate is selected as load case B.

For a center crack in an infinite plate under uniform remote tension, the exact crack surface displacement equation is [11]

$$u_{A.S} = \frac{2(1-\eta^2)S}{E}\sqrt{a^2 - x^2} \quad |x| \le a \tag{17}$$

The normalized stress intensity factor is

$$f_{A,S} = 1 \tag{18}$$

For a center crack in an infinite with crack surfaces subject to Dugdale loading, the exact crack surface displacement equation is [11]

$$u_{A,\sigma} = h(x) + h(-x) \quad |x| \le a \tag{19a}$$

$$h(x) = \frac{2(1-\eta^2)\sigma}{\pi E}\left[\begin{array}{l}(d-x)\operatorname{arccosh}(\frac{a^2 - dx}{a|d-x|}) + \\ \sqrt{a^2 - x^2}\arcsin(\frac{d}{a})\end{array}\right]_{d=b_1}^{d=a} \tag{19b}$$

The normalized stress intensity factor is [11]

$$f_{A,\sigma} = \left[\frac{2}{\pi}\sin^{-1}(\frac{d}{a})\right]_{d=b_1}^{d=a} \tag{20}$$

4.1. Analytical crack surface displacement equation for uniform remote tension

For a radial crack emanating from a semi-circular notch in a semi-infinite plate under uniform remote tension, the crack-surface displacement analytic equation can be expressed as

$$u_{B,S} = u_{A,S} \cdot \frac{f_{B,S}}{f_{A,S}} \cdot M_{B,S} \tag{21a}$$

where $f_{B,S}$, $f_{A,S}$ and $u_{A,S}$ are given in Eq. (12), Eq. (18) and Eq. (17), respectively. The factor $M_{B,S}$ is determined by fitting the crack opening displacement results obtained from the weight function method. The fitted expression is

$$M_{B,S} = \frac{1 + 3.52 \cdot \frac{a}{r} + 0.12 \cdot \frac{x}{a} + 0.38 \cdot (\frac{a}{r})^2 + 0.645 \cdot (\frac{x}{a})^2 + 2.15 \cdot \frac{x}{a} \cdot \frac{a}{r}}{0.774 + 2.364 \cdot \frac{a}{r} + 0.45 \cdot \frac{x}{a} + 0.44 \cdot (\frac{a}{r})^2 + 0.545 \cdot (\frac{x}{a})^2 + 3.28 \cdot \frac{x}{a} \cdot \frac{a}{r}} \quad (a/r \leq 1.0) \tag{21b}$$

Figure 2 shows that this analytical expression fits the weight function results very well. The maximum error is 0.36% (for $a/r = 0.5$).

4.2. Analytic crack surface displacement equation for Dugdale loading

For a radial crack emanating from a semi-circular notch in a semi-infinite plate with crack surfaces subject to Dugdale loading, the analytical crack-surface displacement equation is

$$u_{B,\sigma} = u_{A,\sigma} \cdot \frac{f_{B,\sigma}}{f_{A,\sigma}} \cdot M_{B,\sigma} \tag{22a}$$

where $u_{A,\sigma}$, $f_{A,\sigma}$ and $f_{B,\sigma}$ are given by Eq. (19), Eq. (20) and Eq. (14), respectively. The factor $M_{B,\sigma}$ is determined by fitting the crack opening displacement results obtained from the weight function method, Eq.(15):

$$M_{B,\sigma} = \frac{0.921 - 0.713 \cdot \frac{a}{r} - 0.728 \cdot \frac{d}{a} + 0.63 \cdot \frac{x}{a} + 2.98 \cdot \frac{a}{r} \cdot \frac{d}{a} + 1.65 \cdot \frac{a}{r} \cdot \frac{x}{a} - 2.26 \cdot \frac{d}{a} \cdot \frac{x}{a} - 0.14 \cdot (\frac{a}{r})^2 + 0.454 \cdot (\frac{d}{a})^2 + 3.1 \cdot (\frac{x}{a})^2}{0.718 - 0.287 \cdot \frac{a}{r} - 0.594 \cdot \frac{d}{a} + 0.723 \cdot \frac{x}{a} + 2.91 \cdot \frac{a}{r} \cdot \frac{d}{a} + 1.544 \cdot \frac{a}{r} \cdot \frac{x}{a} - 2.33 \cdot \frac{d}{a} \cdot \frac{x}{a} - 0.37 \cdot (\frac{a}{r})^2 + 0.392 \cdot (\frac{d}{a})^2 + 3.2 \cdot (\frac{x}{a})^2} \tag{22b}$$

Figure 3 shows some typical comparisons between the fitted expression, Eq. (22), and the weight function results. For a radial crack emanating from a semi-circular notch in a semi-infinite plate, Eq. (22) represented by the circular symbol, is in very good agreement with the results from weight function method (the solid curves) for all points along crack surface. The maximum difference occurs in the loading segment (a/r=0.05, b_1/a=0.9), and is about 2.4%.

With this analytical equation of crack surface displacements for the Dugdale loading, crack surface displacements for a segment (width Δb) pressure acting at an arbitrary location along the crack faces can readily be obtained by using the same procedure as Ref [12,13], i.e. taking the difference between two Dugdale loadings with b_2=a, b_1 and b_1+Δb in Fig.1.

$$u_{\mathrm{B},\sigma,\Delta b} = \left(u_{\mathrm{A},\sigma} \cdot \frac{f_{\mathrm{B},\sigma}}{f_{\mathrm{A},\sigma}} \cdot M_{\mathrm{B},\sigma}\right)\Bigg|_{d=b_1} - \left(u_{\mathrm{A},\sigma} \cdot \frac{f_{\mathrm{B},\sigma}}{f_{\mathrm{A},\sigma}} \cdot M_{\mathrm{B},\sigma}\right)\Bigg|_{d=b_1+\Delta b} \tag{24}$$

Equation (23) will not only significantly improve the accuracy of fatigue crack closure analysis for the specific crack geometry in consideration, but also will improve the computational efficiency for the modified strip-yield-model-based fatigue crack growth predictions by eliminating the time -consuming numerical integrations for crack surface displacements under partial loading.

5. Summary and conclusions

Crack surface displacements for a radial crack emanating from a semi-circular notch in a semi-infinite plate are determined, and analytical equations have been developed. Two load cases have been treated, i.e. uniform remote tension and crack face Dugdale loading. The weight function method was used for the analysis. The following conclusions can be drawn.

(1) The closed-form weight function method provides a powerful means for accurate determination of crack surface displacements under arbitrary load conditions.

(2) Based on a correction of stress intensity factor ratio, highly accurate analytical equations of the crack surface displacements for a radial crack emanating from a semi-circular notch in a semi-infinite plate are developed, which fit the results from the weight function method very well.

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