Singular Thermo-Elastic Stress Analysis of An Irregular Shaped Inclusion

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Abstract This paper develops a super element that simulates the elastic behavior around an inclusion corner. The super inclusion corner element is finally incorporated with standard four-node hybrid-stress elements to constitute an *ad hoc* hybrid-stress finite element method for thermo-elastic stress analysis of an irregular inclusion in isotropic materials under thermal and mechanical loadings. In the numerical analysis, generalized stress intensity factors at the inclusion corner are systematically calculated for various material stiffness ratio and dimensions of the inclusion in a plate subjected to thermo-mechanical loadings.

Keywords Thermo-mechanical stress, Hybrid finite element, Inclusion, Super inclusion corner element, Stress intensity factor

1. Introduction

Much attention has been paid to inclusion problems by many researchers since Eshelby's first solution to the ellipsoidal inclusion problems. The application background is found in microstructures, composite material structures and others.

As the stress intensity at an inclusion corner is governed by the corner surrounding material properties, the corner geometry conditions and loading situations, great mathematical difficulties are usually encountered in analytical solutions. Therefore, most complicated engineering problems of inclusions have to resort to numerical methods such as the finite element method (FEM) and others. Chen [1] used the body force method to calculate stress intensity factors (SIFs). The SIFs for a dissimilar material wedge under mechanical and thermal loads were determined by using the least square method [2-4]. Path-independent conservative line integrals derived from Betti's reciprocal principle were utilized to evaluate stress intensities at the interface [5-7]. The solutions from the aforementioned methods are strongly dependent on the number of element meshes. Furthermore, it is very difficult to obtain accurate numerical results for singular stress states near the apex in dissimilar materials using the conventional finite element method, even with the help of many finite elements.

To improve the accuracy of numerical results for wedge or interface problems in the traditional finite element analysis, the analytical asymptotic solutions near the apex can be used as interpolation functions to construct a stiffness matrix for special elements containing a part or interface of a wedge. Chen [8] developed an enriched element with appropriate interpolation functions to account for the singular behavior at the junction of dissimilar materials subjected to mechanical load. Similar to Chen [8]'s work, enriched finite elements were further developed by Gadi et al. [9] and Pageau and Biggers [10]. However, numerical results with enriched finite elements are still dependent on the special element size, and the convergence of the results is not guaranteed. Therefore, more accurate numerical results require a lot of refined element meshes between the special element and standard elements. For a crack that either follows or is perpendicular to the interface, Tong et al. [11] constructed a special super element for the analysis of plane crack problems. Similarly, Tan and Meguid [12] developed a singular inclusion corner element for dissimilar material wedge problems. Mote [13], Bradford et al. [14] and Madenci et al. [15] established a special global element based on asymptotical solutions around a dissimilar material junction edge. By using the eigenfunction expansion method, Barut et al. [16] derived a special hybrid global element on the basis of exact analytical solutions of stress displacement fields under mechanical and thermal loads. In addition to the leading singular order term, a few other higher order terms were also used in constructing the special elements in Ref. [11-13, 16], which leads to more accurate numerical results

The hybrid-stress finite element method developed more than 40 years ago by Pian is now well recognized as a powerful and easy-to-use tool for solving a variety of two-dimensional linear elasticity problems containing a single or multiple singular points. To the best of the author's knowledge, the studies related to singular thermo-mechanical fields of inclusions by the hybrid-stress finite element method are absent. Moreover, a numerical solution of even a single irregular inclusion by the method could not be found in the literature, either. To predict singular stress fields around an inclusion corner under thermo-mechanical loads, a new *ad hoc* super inclusion corner element based on the numerical asymptotic solutions developed is proposed in this paper to study inclusion problems shown in Fig. 1. The validity and applicability of present approach are established through available solutions

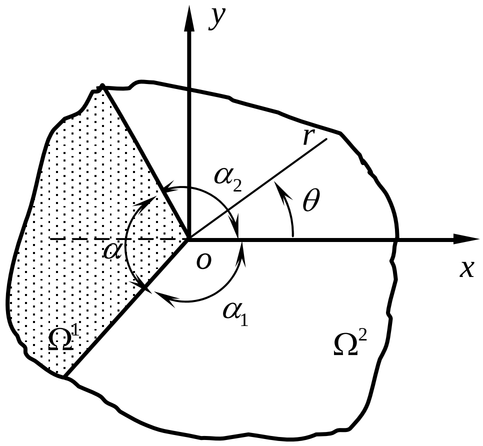


Figure 1 Local coordinate system near the inclusion corner-tip

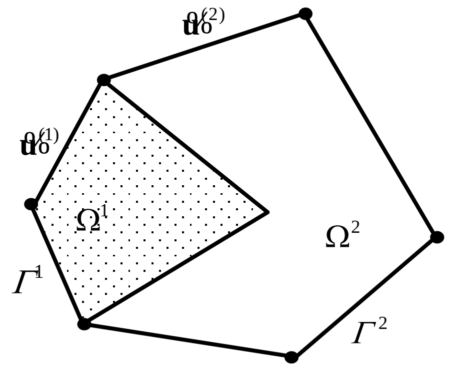


Figure 2 A super *n*-sided polygonal inclusion corner element

2. The hybrid variational functionals for thermo-elasticity involving an inclusion

Let a super *n*-sided polygonal element centered at the inclusion corner is taken as the complementary region (C-region) which contains inclusion domain Ω^2 with outer boundary Γ^2 and its surrounding matrix domain Ω^1 with outer boundary Γ^1, as shown in Fig.2.

Under appropriate continuity conditions, the stiffness matrix for a super inclusion element of dissimilar material wedge subjected to thermo-mechanical loads is written as [16]:

$$\begin{aligned}\Pi = &\sum_{k=1}^{2}\left\{-\frac{1}{2}\int_{\Gamma^k} {}^{m}_{\lambda}\boldsymbol{\sigma}^{(k)T}\boldsymbol{n}^T\, {}^{m}_{\lambda}\boldsymbol{u}^{(k)}dS - \int_{\Gamma^k} {}^{m}_{\lambda}\boldsymbol{\sigma}^{(k)T}\boldsymbol{n}^T\, {}^{t}_{\lambda+c}\boldsymbol{u}^{(k)}dS\right\} \\ &+\sum_{k=1}^{2}\left\{\int_{\Gamma^k} {}^{m}_{\lambda}\boldsymbol{\sigma}^{(k)T}\boldsymbol{n}^T\tilde{\boldsymbol{u}}^{(k)}dS\right\}+\Pi_0\end{aligned} \tag{1}$$

in which ${}^{m}_{\lambda}(\text{g}^{(k)})$ represent the unknown stress and displacement fields due to the mechanical load and ${}^{\ t}_{\lambda+c}\boldsymbol{u}^{(k)}$ are the known initial displacement components from the uniform temperature variation ΔT. The left subscripts $\lambda+c$ represent the summation of singular and non-singular components of the variables. Π_0 represents the total potential, given as an initial value, associated with the displacements and stresses under thermal load. Matrix $\boldsymbol{n}$ contains 3×2 components of the unit outward normal to boundary Γ^k. $\boldsymbol{u}^{(k)}$ are displacements under thermo-mechanical loads.

Introducing the coordinate system transformation matrices $(\text{g}) = \boldsymbol{Z}_{\text{g}}(\text{g})$ and $\boldsymbol{u} = \boldsymbol{U\beta}$, $\boldsymbol{\sigma} = \boldsymbol{P\beta}$, $\tilde{\boldsymbol{u}}^{(k)} = \boldsymbol{L}\boldsymbol{q}_{\text{s}}$, where matrix $\boldsymbol{L}$ is the linear interpolation function, and vector $\boldsymbol{q}_{\text{s}}$ is the nodal displacement on the boundary segment Γ^k of the super corner element, then we have

$$\Pi = -\frac{1}{2}\,{}^{m}\boldsymbol{\beta}^{T}\boldsymbol{H}\,{}^{m}\boldsymbol{\beta} - {}^{m}\boldsymbol{\beta}^{T}\,{}^{t}\boldsymbol{f} + {}^{m}\boldsymbol{\beta}^{T}\boldsymbol{G}\boldsymbol{q}_{\text{s}} + \Pi_0 \tag{2}$$

where

$$\boldsymbol{H} = \sum_{k=1}^{2}\frac{1}{2}\int_{\Gamma^k}\left[\boldsymbol{P}^{(k)T}\boldsymbol{Z}_{\sigma}^{(k)T}\boldsymbol{n}^{\text{T}}\boldsymbol{Z}_{u}^{(k)}\boldsymbol{U}^{(k)} + \boldsymbol{U}^{(k)\text{T}}\boldsymbol{Z}_{u}^{(k)T}\boldsymbol{n}\boldsymbol{Z}_{\sigma}^{(k)}\boldsymbol{P}^{(k)}\right]\text{d}\,S\,,\quad \boldsymbol{G} = \sum_{k=1}^{2}\int_{\Gamma^k}\boldsymbol{P}^{(k)T}\boldsymbol{Z}_{\sigma}^{(k)T}\boldsymbol{n}\boldsymbol{L}\,\text{d}\,S$$

$${}^{t}\boldsymbol{f}^{(k)} = \sum_{k=1}^{2}\int_{\Gamma^k}\boldsymbol{P}^{(k)T}\boldsymbol{Z}_{\sigma}^{(k)T}\boldsymbol{n}^{\text{T}}\boldsymbol{Z}_{u}^{(k)}\,{}^{\ t}_{\lambda+c}\boldsymbol{u}_{p}^{(k)}dS$$

Setting $\delta\Pi = 0$ and noting that $\delta\Pi_0 = 0$, we determine that:

$${}^{m}\boldsymbol{\beta} = \boldsymbol{H}^{-1}\left\{\boldsymbol{R}\boldsymbol{q}_{\text{s}} - {}^{t}\boldsymbol{f}\right\} \tag{3}$$

and thus

$$\Pi = \frac{1}{2}\boldsymbol{q}_{\text{s}}^{T}\boldsymbol{G}^{T}\boldsymbol{H}^{-1}\boldsymbol{G}\boldsymbol{q}_{\text{s}} - \boldsymbol{q}_{\text{s}}^{T}\boldsymbol{G}^{T}\boldsymbol{H}^{-1}\,{}^{t}\boldsymbol{f} + \frac{1}{2}\,{}^{t}\boldsymbol{f}^{T}\boldsymbol{H}^{-1}\,{}^{t}\boldsymbol{f} + \Pi_0 \tag{4}$$

From Eq.(4), we have the following matrices:

$$\boldsymbol{K}_s = \boldsymbol{G}^{T}\boldsymbol{H}^{-1}\boldsymbol{G}\,,\quad {}^{t}\boldsymbol{F} = \boldsymbol{G}^{T}\boldsymbol{H}^{-1}\,{}^{t}\boldsymbol{f} \tag{5}$$

where $\boldsymbol{K}_s$ is the stiffness matrix of the super corner element and ${}^{t}\boldsymbol{F}$ represents the nodal force due to thermal load on the boundary segment Γ^k. This *ad hoc* element is used to model the near-field region and is combined with the conventional standard four-node hybrid-stress elements in the far-field region.

3. Definition of the stress intensity factor (SIF)

The singular stress field around the inclusion corner apex under thermo-mechanical loads can usually be expressed in a form of singular terms as:

$$\sigma_{\theta\theta} = \sum_{n=1}^{N+M} K_n r^{\lambda_n} f_{\theta\theta}^{n}(\theta) + {}_{\mathrm{c}}^{\mathrm{t}}\sigma_{\theta\theta} \tag{6}$$

$$\sigma_{r\theta} = \sum_{n=1}^{N+M} K_n r^{\lambda_n} f_{r\theta}^{n}(\theta) + {}_{\mathrm{c}}^{\mathrm{t}}\sigma_{r\theta} \tag{7}$$

where (r,θ) is a local polar coordinate system centered at the inclusion corner apex, and the axis of $\theta = 0$ is the bisector of the two wedge apexes shown in Fig.1; *N* represents the total number of complex singularity orders λ_n between -1 and 0, and *M* is the number of real singularity orders λ_n; $f_{\theta\theta}^{n}(\theta)$ and $f_{r\theta}^{n}(\theta)$ are the notch angular variation of normal stress fields and shear angular variations associated with λ_n, respectively; ${}_{\mathrm{c}}^{\mathrm{t}}\sigma_{\theta\theta}$ and ${}_{\mathrm{c}}^{\mathrm{t}}\sigma_{r\theta}$ are regular stresses caused by a thermal load.

According to Chen's work [17], the singularity orders λ_n have only two roots, i.e., λ_1 and λ_2; When Dundur's composite parameters α and β [18] meet the condition $\beta(\alpha-\beta)>0$, λ_1 and λ_2 are always real within the range of $-1<\mathrm{Re}(\lambda_1,\lambda_2)<0$. On the moment, expressions (6) and (7) can be rewritten as:

$$\sigma_{\theta\theta} = K_1 r^{\lambda_1} f_{\theta\theta}^{1}(\theta) + K_2 r^{\lambda_2} f_{\theta\theta}^{2}(\theta) + {}_{\mathrm{c}}^{\mathrm{t}}\sigma_{\theta\theta} \tag{8}$$

$$\sigma_{r\theta} = K_1 r^{\lambda_1} f_{r\theta}^{1}(\theta) + K_2 r^{\lambda_2} f_{r\theta}^{2}(\theta) + {}_{\mathrm{c}}^{\mathrm{t}}\sigma_{r\theta} \tag{9}$$

Defining $f_{\theta\theta}^{1}(\theta)$ and $f_{r\theta}^{2}(\theta)$ in such a way that $f_{\theta\theta}^{1}(\theta_0)=1$ and $f_{r\theta}^{2}(\theta_0)=1$, where θ_0 can be chosen arbitrarily, then we have

$$K_1 = \frac{1}{[1-f_{r\theta}^{1}(\theta_0)\cdot f_{\theta\theta}^{2}(\theta_0)]} \lim_{r\to 0} r^{-\lambda_1} [\sigma_{\theta\theta} - {}_{\mathrm{c}}^{\mathrm{t}}\sigma_{\theta\theta} - f_{\theta\theta}^{2}(\theta_0)\cdot(\sigma_{r\theta} - {}_{\mathrm{c}}^{\mathrm{t}}\sigma_{r\theta})]\Big|_{\theta=\theta_0} \tag{10}$$

$$K_2 = \frac{1}{[1-f_{r\theta}^{1}(\theta_0)\cdot f_{\theta\theta}^{2}(\theta_0)]} \lim_{r\to 0} r^{-\lambda_2} [\sigma_{r\theta} - {}_{\mathrm{c}}^{\mathrm{t}}\sigma_{r\theta} - f_{r\theta}^{1}(\theta_0)\cdot(\sigma_{\theta\theta} - {}_{\mathrm{c}}^{\mathrm{t}}\sigma_{\theta\theta})]\Big|_{\theta=\theta_0} \tag{11}$$

Once the values of K_1 and K_2 are obtained, the singular stress fields at every θ can be solved from Eqs.(8) and (9).

4. Numerical results and discussions

The present method is used to analyze the singular stresses around the corner apex A of a rectangular inclusion subjected to uniform temperature change ΔT as shown in Fig.Fig.3. Fig.4 shows a configuration for a super 8-node quadrilateral inclusion corner element. For the solution of the singular stress fields, around the square inclusion corner apex A, defined in Eqs.(8) and(9), the stress intensity factors K_1 and K_2 should be first determined. Generally speaking, any component of the stresses at any angle θ may be used as the compared object to determine the stress intensity factors. However, for simplicity, herein we only use the stresses at $\theta = 0^{\circ}$ and $\theta = 180^{\circ}$, that is, the stresses at points on the bisector of the vertex angle in region Ω^2 (for $\theta = 0^{\circ}$) and Ω^1 (for $\theta = 180^{\circ}$).

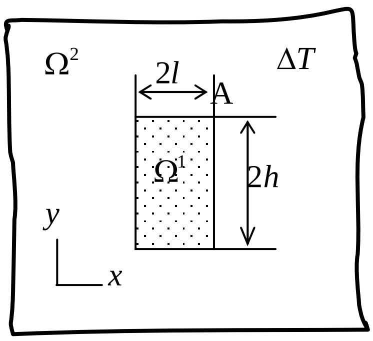


Fig.3 A rectangular inclusion in a infinite matrix under thermal loading

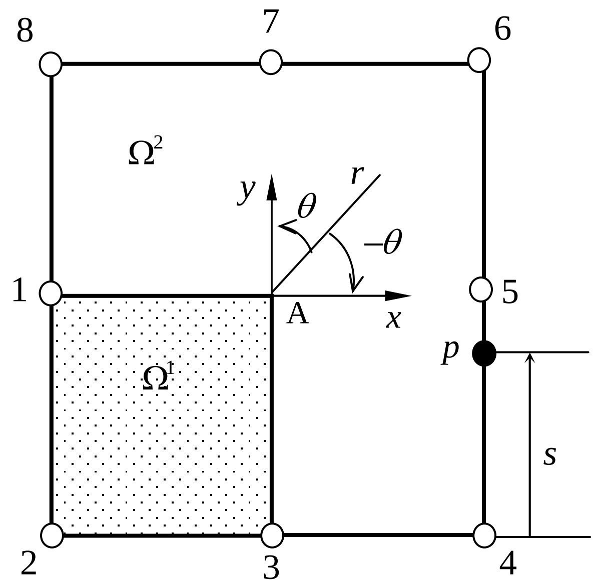


Fig.4 Configuration of a super 8-node quadrilateral inclusion corner element

An example for a square inclusion ($l = h$) is given herein. In the numerical analysis, To model the infinite plate, its width and height are all set to be$10l$; a quarter of plate is used for element divisions due to the symmetry of its geometry and loading, and 332 4-noded stress-based element and one 8-noded super elements are utilized. The material parameters of elastic modulus$E_1 : E_2 = 100$ and Poisson ratio $\nu_1 = \nu_2 = 0.3$ are employed. The singular stresses away from the apex A along the boundaries of the inclusion are analyzed with different the unknown parameters$\boldsymbol{\beta}s$. They are plotted in Fig. 5. In the Figures, for the sake of comparison, the numerical results from the commercial software ANSYS package are also shown. It is shown that the singular stresses $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ rapidly increase with the decrease of the distance away from the inclusion corner apex A along the boundary ($\theta = -135^{\circ}$) of the square inclusion, which is well recognized; When the number of $\boldsymbol{\beta}s$ meets the LBB condition: greater than equal to the number of freedom degrees of nodes in the super element minus the rigid modes (=3 in plane deformation), the present numerical results are in good agreement with the solutions of the ANSYS. Fig.6 tells us a fact that the influence of thermal expansion coefficient α_2 on the dimensionless stress factor F_1 is limited to very small the range of $\alpha_2 < 20\times 10^{-6}$. Figs.7 and 8 give the relationships between the dimensionless stress factor F_1 and material parameters. From Fig.7, it can be seen that F_1

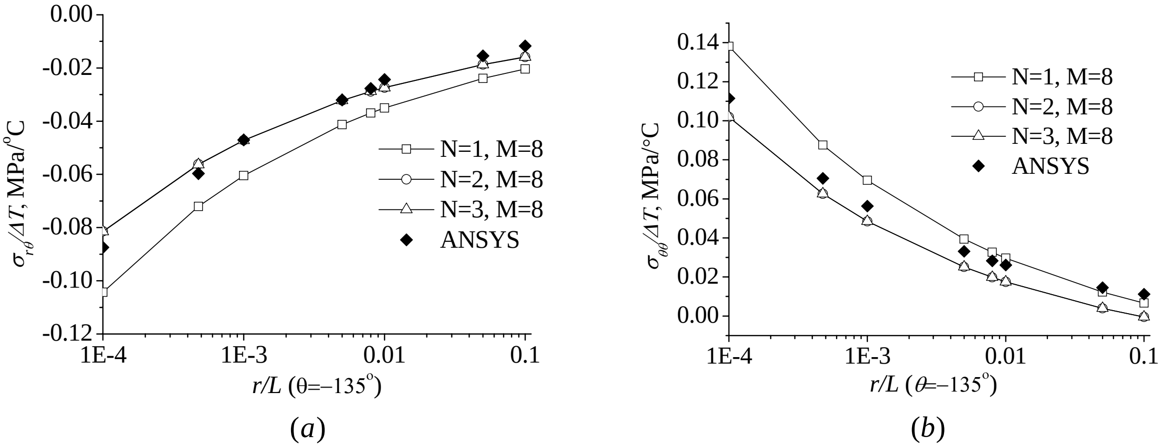


Fig.5 The singular stresses away from the apex A (r/L) along the boundary ($\theta = -135^{\circ}$) of a square inclusion: (*a*) σ_{θ} / ΔT ; (*b*) $\sigma_{r\theta}$ / ΔT

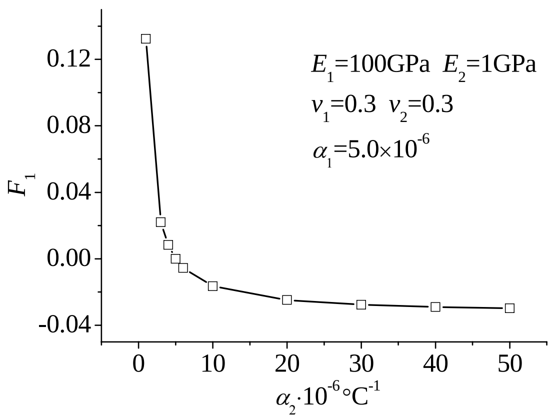
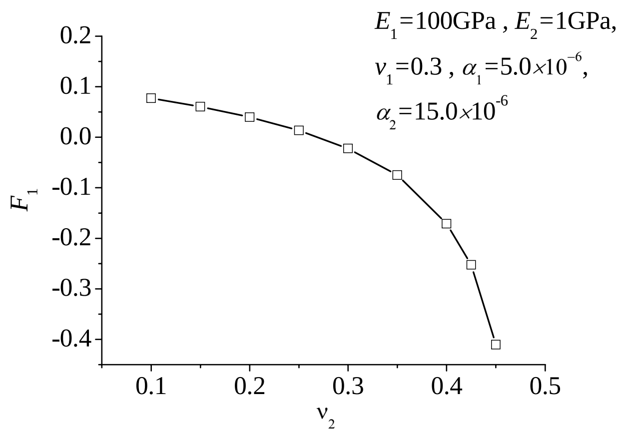


Fig.6 The effect of the thermal expansion coefficient α_2 on F_1 around the inclusion apex A



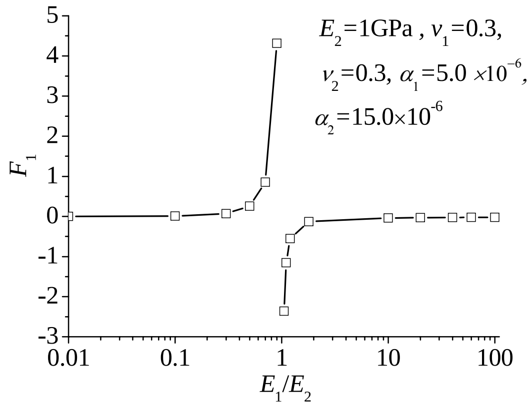


Fig.7 F_1 vs ν_2

Fig.8 F_1 vs E_1/E_2

decreases more and more rapidly with increasing Poisson ratio ν_2. However, it is shown in Fig.8 that the influence of modulus ratio out of range of $-10 < |E_1 / E_2| < 10$ on the dimensionless stress factor F_1 is so small that it can be neglected. In Figs.6-8, the dimensionless expression of F_1 is defined as

$$F_1 = \frac{K_1}{\alpha_2 E_2 \Delta T \sqrt{\pi} l^{-\lambda_1}} \tag{9}$$

5. Concluding remarks

A new finite element method was developed to analyze irregular-shaped inclusion problems under thermo-mechanical loads. The method consists of a super polygonal inclusion corner element in conjunction with standard four-node hybrid-stress elements. A benchmark example of a square inclusion problem was discussed. The present results is validated by comparison with the numerical results obtained using the conventional finite element commercial software ANSYS package. The present numerical solutions show that our method provides satisfactory results with coarse meshes and is effective and applicable to thermo-mechanical problems with multiple singular points. In addition, for square inclusion problems, the following useful conclusions are drawn:

(1) The dimensionless stress intensity factor F_1 decreases more and more rapidly with increasing Poisson ratio ν_2;

(2) The influence of modulus ratio out of range of $-10 < |E_1 / E_2| < 10$ on the dimensionless stress factor F_1 is so small that it can be neglected.

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