T-stress evaluation in nonhomogeneous materials under thermal loading by means of interaction energy integral method

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Abstract This paper addresses finite element evaluation of the non-singular T-stress in nonhomogeneous materials under steady-state thermal loads by means of interaction energy integral method. The interaction energy integral method developed in this paper can solve the T-stress with high accuracy and efficiently in nonhomogeneous materials. The interaction energy integral method in conjunction with the extended FEM is used to solve several representative examples to show its validity. It can be found that the present method is efficiency to calculate the T-stress in nonhomogeneous materials.

Keywords Interaction energy integral, T-stress, Interface, Nonhomogeneous materials

1. Introduction

Stress intensity factors (SIFs) play a significant role in linear elastic fracture mechanics as they characterize the crack-tip stress and strain fields. Apart from the SIFs, the T-stress [1], which is the nonsingular term, has become another key parameter in fracture mechanics because it has been found the T-stress affects the crack growth direction, shape and size of the plastic zone, crack-tip constraint and fracture toughness greatly [2-4].
Many researchers evaluated T-stress when the material is subject to mechanic loading [5-8], however, there are few researches on T-stress when the material is subject to thermal loading. Sladek and Sladek [9] used the conservation integral method to evaluate the T-stress and the stress intensity factors in stationary thermoelasticity. Dag [10] used J integral to evaluate the mix-mode SIFs and the T-stress in FGMs under thermal loading. KC and Kim [11] and Kim and KC [12] have studied the SIFs and T-stress in FGMs under thermal loading using the interaction energy integral method. However, all of the above paper did not consider the situation that the materials contain interfaces.

2. Interaction energy integral formulation

The traditional J-integral given by Rice [13] is

$$J = \lim_{\Gamma_0 \to 0} \int_{\Gamma_0} (W\ \delta_{1j} - \sigma_{ij} u_{i,1}) n_j d\Gamma \tag{1}$$

where W is the strain energy density given by

$$W = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}^{m} = \frac{1}{2}\sigma_{ij}(\varepsilon_{ij}^{t} - \varepsilon_{ij}^{th}) \tag{2}$$

and n_j is the outward normal vector to the contour Γ_0, as shown in Fig. 1, and δ_{ij} is the Kronecker delta. In Eq. (2), ε_{ij}^{m} is the mechanical strain, ε_{ij}^{t} denotes the total strain, $\varepsilon_{ij}^{th} = \alpha\Delta\theta\delta_{ij}$ refers to thermal strain, α represents the thermal expansion coefficient and $\Delta\theta = \theta - \theta_0$ denotes temperature change with θ_0 is the initial temperature.

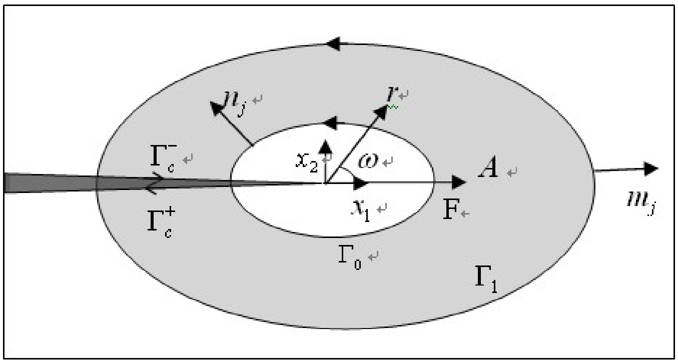

Fig. 1. Schematic illustration of the contour integrals and domain integrals.

The J-integral can be deduced to an equivalent domain integral (EDI) by using the divergence theorem, that is

$$J = \int_A (\sigma_{ij} u_{i,1} - W\ \delta_{1j})_{,j}\, q dA + \int_A (\sigma_{ij} u_{i,1} - W\ \delta_{1j}) q_{,j} dA \tag{3}$$

In the interaction energy integral method, the auxiliary fields, including displacements (u^{aux}), strains (ε^{aux}), and stresses (σ^{aux}) are used. These auxiliary fields need to be suitably defined in order to evaluate T-stress and TSIFs in nonhomogeneous materials. Here we adopt analytical fields originally developed for homogeneous materials and the "incompatible formulation" is chosen in this paper [14].

As shown in Fig.1, there is a point force *F* applied to the crack tip in an infinite plane. The auxiliary displacement fields and the auxiliary stress fields for T-stress are chosen as follows [11, 12],

$$\begin{aligned} u_1^{aux} &= -\frac{F(\kappa_{tip}+1)}{8\pi\mu_{tip}} \ln\frac{r}{d} - \frac{F}{4\pi\mu_{tip}} \sin^2\omega \\ u_2^{aux} &= -\frac{F(\kappa_{tip}-1)}{8\pi\mu_{tip}} \omega + \frac{F}{4\pi\mu_{tip}} \sin\omega\cos\omega \end{aligned} \tag{4}$$

$$\begin{aligned} \sigma_{11}^{aux} &= -\frac{F}{\pi r}\cos^3\omega \\ \sigma_{22}^{aux} &= -\frac{F}{\pi r}\cos\omega\sin^2\omega \\ \sigma_{12}^{aux} &= -\frac{F}{\pi r}\cos^2\omega\sin\omega \end{aligned} \tag{5}$$

where *d* is the coordinate of a fixed point on the x_1 axis, μ_{tip} is the shear modulus evaluated at the crack-tip, and

$$\kappa_{tip} = \begin{cases} \dfrac{3-\nu_{tip}}{1+\nu_{tip}} & \text{(plane stress)} \\ 3-4\nu_{tip} & \text{(plane strain)} \end{cases} \tag{6}$$

And the auxiliary strain fields $\varepsilon_{ij}^{aux} = S_{ijkl}(\mathbf{x})\sigma_{kl}^{aux}$, which is incompatible with the auxiliary displacement fields [15]. Since S_{ijkl} is the compliance tensor of the actual materials, not the compliance tensor of the crack tip, it can be got that: $\varepsilon_{ij}^{aux} \neq (u_{i,j} + u_{j,i})/2$.

Superposition of the actual and auxiliary fields leads to a new equilibrium state.

$$J^{1+2} = \int_A ((\sigma_{ij}+\sigma_{ij}^{aux})(u_{i,1}+u_{i,1}^{aux}) - \frac{1}{2}(\sigma_{ik}+\sigma_{ik}^{aux})(\varepsilon_{ik}^{m}+\varepsilon_{ik}^{aux})\delta_{1j})_{,j}\, q dA$$
$$+\int_A ((\sigma_{ij}+\sigma_{ij}^{aux})(u_{i,1}+u_{i,1}^{aux}) - \frac{1}{2}(\sigma_{ik}+\sigma_{ik}^{aux})(\varepsilon_{ik}^{m}+\varepsilon_{ik}^{aux})\delta_{1j}) q_{,j} dA \quad (7)$$

The interactional part is called interaction energy integral [14]. Here, it can be derived as

$$I = \int_A (\sigma_{ij}^{aux}u_{i,1} + \sigma_{ij}u_{i,1}^{aux} - \sigma_{ik}^{aux}\varepsilon_{ik}^{m}\delta_{1j}) q_{,j} dA$$
$$+\int_A (\sigma_{ij}^{aux}u_{i,j1} + \sigma_{ij}u_{i,j1}^{aux} - \sigma_{ij,1}^{aux}\varepsilon_{ij}^{m} - \sigma_{ij}^{aux}\varepsilon_{ij,1}^{m}) q dA \quad (8)$$

According to the relationship of the displacement and strain in elastic mechanics, one may write

$$\sigma_{ij}^{aux}u_{i,j1} = \frac{1}{2}\sigma_{ij}^{aux}(u_{i,j1}+u_{j,i1}) = \sigma_{ij}^{aux}\varepsilon_{ij,1}^{t} = \sigma_{ij}^{aux}(\varepsilon_{ij,1}^{m}+\varepsilon_{ij,1}^{th}) \quad (9)$$

The second integral can be obtained as

$$\int_A (\sigma_{ij}^{aux}u_{i,j1} + \sigma_{ij}u_{i,j1}^{aux} - \sigma_{ij,1}^{aux}\varepsilon_{ij}^{m} - \sigma_{ij}^{aux}\varepsilon_{ij,1}^{m}) q dA = \int_A (\sigma_{ij}u_{i,j1}^{aux} - \sigma_{ij,1}^{aux}\varepsilon_{ij}^{m} + \sigma_{ij}^{aux}\varepsilon_{ij,1}^{th}) q dA \quad (10)$$

If the extra strain field $\varepsilon_{ij}^{aux0} = S_{ijkl}^{tip}\sigma_{kl}^{aux}$ is introduced, we can get the formulation $\varepsilon_{ij}^{aux0} = \frac{1}{2}(u_{i,j}^{aux}+u_{j,i}^{aux})$, Here S_{ijkl}^{tip} is a compliance tensor at the crack tip [15]. Thus, the interaction energy integral can be written as

$$I = \int_A (\sigma_{ij}^{aux}u_{i,1} + \sigma_{ij}u_{i,1}^{aux} - \sigma_{ik}^{aux}\varepsilon_{ik}^{m}\delta_{1j}) q_{,j} dA$$
$$+\int_A (\sigma_{ij}(S_{ijkl}^{tip} - S_{ijkl}(x))\sigma_{kl,1}^{aux}) q dA + I^{th} \quad (11)$$

where

$$I^{th} = \int_A (\sigma_{ij}^{aux}\varepsilon_{ij,1}^{th}) q dA = \int_A \sigma_{ii}^{aux}[\alpha_{,1}(\theta-\theta_0) + \alpha\theta_{,1}] q dA \quad (12)$$

In the above derivation process, the domain of the integral is chosen arbitrarily around the crack-tip, so the interaction energy integral for thermal fracture problems is domain-independent.

3. Evaluation of the T-stress from the interaction energy integral

The contour integral around the crack tip can be written as

$$I = \lim_{\Gamma_0\to 0}\oint_{\Gamma_0} (\sigma_{ik}\varepsilon_{ik}^{aux}\delta_{1j} - \sigma_{ij}u_{i,1}^{aux} - \sigma_{ij}^{aux}u_{i,1}) n_j d\Gamma \quad (13)$$

Here we can consider only the stress parallel to the crack direction, i.e.

$$\sigma_{ij} = T\delta_{1i}\delta_{1j} \quad (14)$$

Where T denotes the T-stress. The force f is in equilibrium can be expressed as

$$f = -\lim_{\Gamma_0\to 0}\oint_{\Gamma_0} \sigma_{ij}^{aux} n_j d\Gamma \quad (15)$$

We can also obtain the following relationship from the kinematic equations

$$u_{i,1} = \varepsilon_{11}^{t}\delta_{i1} = (\varepsilon_{11}^{m}+\varepsilon_{11}^{th})\delta_{i1} \quad (16)$$

It can be rewritten as

$$u_{i,1} = (\frac{\sigma_{11}}{E} + \alpha\Delta\theta C^{*})\delta_{i1} \quad (17)$$

where $C^{*}=1$ for generalized plane stress and $C^{*}=1+\nu(x)$ for plane strain. Substituting Eq. (14) and Eq.(17) into the contour integral, one obtains

$$I = -\lim_{\Gamma_0\to 0}\oint_{\Gamma_0} (\frac{T}{E} + \alpha\Delta\theta C^{*})\sigma_{ij}^{aux} n_j d\Gamma = (\frac{T}{E_{tip}^{*}} + \alpha_{tip}\Delta\theta_{tip}C_{tip}^{*})\cdot f \quad (18)$$

Then we can get

$$T = \frac{I}{f} \cdot E_{tip}^{*} - \alpha_{tip} \Delta\theta_{tip} C_{tip}^{*} E_{tip}^{*} \tag{19}$$

Where

$$E_{tip}^{*} = \begin{cases} E_{tip} & \text{(plane stress)} \\ \dfrac{E_{tip}}{1-\nu_{tip}^{2}} & \text{(plane strain)} \end{cases} \qquad C_{tip}^{*} = \begin{cases} 1 & \text{(plane stress)} \\ 1+\nu_{tip} & \text{(plane strain)} \end{cases} \tag{20}$$

So, the T-stress can be evaluated easily if we can obtain the interaction energy integral I. As Yu et al. [15] proved, the equivalent domain integral is not referred to in the above analysis and as a result, both the domain size and the material properties in the integral domain are not limited for the equivalent domain integral.

4. Numerical examples and discussions

To test the validity of the method developed in the above section, two crack problems in nonhomogeneous materials are considered.

Example 1: An edge crack in a homogeneous plate.

An edge crack of length "*a*" is located in a homogeneous plate which subjected to steady-state thermal loading. The temperature boundary of the example is assumed to be $\theta_1 = \theta_0 = 0°C$ and $\theta_2 = 1°C$. This problem have been studied by Sladek and Sladek [9] and KC and Kim [11]. The following data are used for the numerical analysis:

$$L/W = 4,\ a/W = 0.1 \sim 0.8,\ \nu = 0.3,\ E(x) = 1.0\times10^{5},\ \alpha(x) = 1.67\times10^{-5},\ \lambda = 1$$

Tables 1 present the mode-I TSIFs and the T-stress for various crack lengths. The T-stress obtained is in good agreement with those reported in Sladek and Sladek [9].

Table 1 Comparison of the mode-I TSIF and T-stress in homogeneous materials under thermal loading (Example 1)

a/W	Sladek and Sladek's results		Present results	
	K_1	T-stress	K_1	T-stress
0.1	0.6454	-0.4317	0.6432	-0.4198
0.2	0.776	-0.2179	0.7756	-0.2196
0.3	0.7951	-0.0314	0.7953	-0.0322
0.4	0.7527	0.1463	0.753	0.1489
0.5	0.6705	0.3258	0.6708	0.3295
0.6	0.5601	0.5075	0.5605	0.5142
0.7	0.4288	0.698	0.4291	0.7064
0.8	0.2825	0.896	0.2828	0.9095

Example 2: An edge crack in a FGMs plate.

In this example, an edge crack problem in a FGMs plate is studied. The material properties, Young's modulus and thermal expansion coefficient are exponential functions of x, while Poisson's ratio is constant. The temperature boundary of the example is assumed to be $\theta_1 = \theta_2 = 0.5\theta_0$ and $\theta_0 = 10°C$.

The following data are used in the FEM analyses:

$$L/W = 8,\ a/W = 0.5,\ E(x) = E_1 \times e^{\frac{\delta x}{W}},\ \alpha(x) = \alpha_1 \times e^{\frac{\gamma x}{W}},\ \lambda(x) = \lambda_1 \times e^{\frac{\beta x}{W}},\ \delta = \ln(\frac{E_2}{E_1}),$$

$$\gamma = \ln(\frac{\alpha_2}{\alpha_1}),\ \beta = \ln(\frac{\lambda_2}{\lambda_1}),$$

$$E_1 = 1.0,\ E_2 = 5,\ \alpha_1 = 0.01,\ \alpha_2 = 0.02,\ \lambda_1 = 1,\ \lambda_2 = 10,\ \nu = 0.3$$

Different element numbers $N_W \times N_L$ (81×648, 101×808 and 121×968) for two different temperature conditions are chosen. The results are shown in Table 2. It can be seen that the present results agree well with the solutions provided by KC and Kim [11]. From these two examples, the convergence and the accuracy of the present method are verified.

Table 2 Comparison of the mode-I TSIF and T-stress in FGMs for different thermal loading (Example 2)

Loading condition	Element numbers	Present results		KC and Kim's results		
		K_1/K_0	T-stress	K_1/K_0	T-stress	3D T-stress
$T_1 = 0.5T_0$	81×648	0.0023	0.00589	0.0229	0.0067	0.006
	101×808	0.0023	0.00592			
	121×968	0.0023	0.00594			
$T_1 = 0.05T_0$	81×648	0.00438	0.01118	0.00437	0.0126	0.0115
	101×808	0.00438	0.01124			
	121×968	0.00438	0.01129			

5. Conclusions

In this paper, a modified interaction energy integral for thermal loading condition is derived for the T-stress computations. The interaction energy integral is proved to be domain-independent for thermal conditions. It can be found that the numerical results are in good agreement with those in published papers. The present method is effective to analyze the thermal fracture problems of nonhomogeneous materials.

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References

[1] J.R. Rice. Limitations to the small scale yielding approximation for crack tip plasticity. J Mech Phys Solids. 22 (1974) 17-26.

[2] Z.Z. Du, J.W. Hancock. The Effect of Nonsingular Stresses on Crack-Tip Constraint. J Mech

Phys Solids. 39 (1991) 555-67.
[3] S.G. Larsson, Carlson AJ. Influnece of non-singular stress terms and specimen geometry on small scale yielding at crack tips in elastic-plastic materials. . J Mech Phys Solids. 21 (1973) 263-77.
[4] N.P. O'DOWD, C. Shih, R. Dodds. The role of geometry and crack growth on constraint and implications for ductile/brittle fracture. American Society for Testing and Materials. 2 (1995) 134-59.
[5] J.H. Kim, G.H. Paulino. T-stress, mixed-mode stress intensity factors, and crack initiation angles in functionally graded materials: a unified approach using the interaction integral method. Comput Method Appl M. 192 (2003) 1463-94.
[6] C.S. Chen, R. Krause, R.G. Pettit, L. Banks-Sills, Ingraffea AR. Numerical assessment of T-stress computation using a p -version finite element method. Int J Fracture. 107 (2001) 177-99.
[7] A. Sutradhar, G.H. Paulino. Symmetric Galerkin boundary element computation of T-stress and stress intensity factors for mixed-mode cracks by the interaction integral method. Eng Anal Bound Elem. 28 (2004) 1335-50.
[8] G.H. Paulino, J.H. Kim. A new approach to compute T-stress in functionally graded materials by means of the interaction integral method. Eng Fract Mech. 71 (2004) 1907-50.
[9] J. Sladek, V. Sladek. Evaluation of T-stresses and stress intensity factors in stationary thermoelasticity by the conservation integral method. Int J Fracture. 86 (1997) 199-219.
[10] S. Dag. Mixed-mode fracture analysis of functionally graded materials under thermal stresses: A new approach using J(k)-integral. J Therm Stresses. 30 (2007) 269-96.
[11] A. KC, J.H. Kim. Interaction integrals for thermal fracture of functionally graded materials. Eng Fract Mech. 75 (2008) 2542-65.
[12] J.H. Kim, A. KC. A generalized interaction integral method for the evaluation of the T-stress in orthotropic functionally graded materials under thermal loading. J Appl Mech-T Asme. 75 (2008).
[13] J.R. Rice. A path independent integral and the approximate analysis of strain concentration by notches and cracks. J Appl Mech-T Asme. 35 (1968) 379-86.
[14] H.J. Yu, L.Z. Wu, L.C. Guo, S.Y. Du, Q.L. He. Investigation of mixed-mode stress intensity factors for nonhomogeneous materials using an interaction integral method. Int J Solids Struct. 46 (2009) 3710-24.
[15] H.J. Yu, L.Z. Wu, L.C. Guo, H. Li, S.Y. Du. T-stress evaluations for nonhomogeneous materials using an interaction integral method. Int J Numer Meth Eng. 90 (2012) 1393-413.