On the Formation of Regular Crack Networks around a Circular Hole under Uniform Compression

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Abstract The energy approach is used to propose a model of arising of regular systems of cracks, emerging the surface of a circular cavity, being observed, for example, around oil and gas wells under uniform compression. The cracks are supposed to arise due to the accumulation of elastic compression energy in the system. The limit compression (the exhaustion of strength) being achieved, a network of cracks is formed in the most stressed layer adjacent to the interior of the body, thus utilizing the accumulated elastic energy of this layer. In this case, of all the possible grids the least energy-consuming one is formed, that is, the system with such a number n and length L of the cracks, that the energy needed for its creation is minimal. In the simplest scheme the number n of "petals"-wedges arising from the cracking turns out to be equal to 5 (which corresponds to $2n$ = 10 cracks) and (contrary to limit compression pressure magnitude) be independent on geometrical and physical-mechanical parameters of the problem.

Keywords Crack Systems, Oil and Gas Wells

1. Introduction

Around oil and gas wells under hydrostatic pressure [1, 2] (Fig. 1, 2) a pattern of regular cracks grids, emerging the surface of a circular cavity, is frequently observed. In the so-called «geoloosening» (directed layer unloading) method such systems are produced specially for increasing the rock fracturing and stimulation of oil input to the well [1, 2]. That is why it is important studying the structure of this crack networks, in particular, their number, length, etc.

Regular systems of cracks are found in many structures and natural objects (examples and references see, eg, [3]). They may be initially due to both non-mechanical factors, such as thermal stress, the effect of aggressive media, phase transforms (drying, freezing [4]), etc., as well as have purely mechanical nature. Note that the problem of regular cracking of bark in technically elementary but quite substantial statement was considered in [5] already in 1952. Point also the paper [6], in which the formation of regular ordered crack systems was used to illustrate the capabilities of the variational principle of cracks mechanics there proposed. In [3] for the near-surface thermal cracks the formation of "nested doll" systems was studied, where a regular system of equal cracks turns into two analogous crack systems, each with its own size but double period with respect to the original system. Moreover, it was shown that, in some cases, the crack may develop by jumps, and in others, for subsurface cracks it turns out more profitable not to extend into the material, but curl and form spalling. In [7] issues of arising of ordered crack systems and/or crack-like defects under compression were briefly touched upon. In [8] development of regular systems of surface cracks under the thermal shock, and in [9] – multiple cracking of brittle coatings upon loaded solids was studied. In [10-12] an experimental study was performed, and a model was proposed on formation and evolution of cracks echelon in the vicinity of a main longitudinal shear elastic-brittle crack. In [13-16] distribution of fragments of glass after break by size and time was experimentally studied and theoretically treated using methods of the theory of fractals.

Fig. 1 [4, p. 33]

Fig. 2 [4, p. 33]

2. Statement of the problem on regular cracks grid around oil and gas wells

In [17-20] the energy approach was used to propose a model of symmetric brittle crack formation in a thin plate (and a wedge) under bending by a point indenter. The main purpose of that idealized minimal model was to quantify the study of some possible mechanisms determining the number of the cracks arising. In this paper, similar approach is used to model the formation of a regular grid of cracks around a circular cavity under hydrostatic pressure [21-22]. The basic idea is that cracks are formed due to the accumulation in the system of elastic energy of compression (or rather, the shear energy). As the material strength is exhausted, in the most stressed layer, adjacent to the interior of the body, a regular network of cracks appears, which formation takes all the elastic energy of that layer. In this case, of all the possible grids the minimum-energy-consuming fracture scheme occurs, that is, a system with such a number n and length L of the cracks, that the energy needed for its creation, is minimal.

Unlike the case of plate bending here cracks go from the surface of the hole not perpendicular to it, but at an acute angle (though not necessarily at 45^0, ie in the direction of slip lines). In other words, they are not tensile cracks, though perhaps not quite shear ones. More precisely, in [2, p. 21, 37-38] "usual for rocks Coulomb-Mohr type criterion, according to which the failure on these planes occurs when the shear stress achieves some limit value $[\tau] = k - \sigma_n \mathrm{tg}\,\rho$, where k and ρ are cohesion modulus and internal friction angle of rock respectively, which are the strength characteristics of the rock" is taken as a failure criterion. Aimed to clarify the fundamental possibility of constructing a simple model of cracks ordered systems of such type and calculation of cracks number, as well as maximum simplification of the problem and using the features of its stress-strain state, we assume that the cracks propagate along the slip lines, and their formation is due to stored elastic energy, but not all, but only the shear energy, the energy of hydrostatic compression being not taken into account

So, the model is based on the following provisions:

I. Suppose we have an infinite plane with a circular cavity in a uniform compression, stress-strain state is plane.

II. As the material strength is exhausted, fracture occurs instantaneously with the formation of a symmetric system of shear cracks.

III. Cracks occur along the slip lines, forming a regular pattern like Fig. 1-3.

IV. One can neglect the irreversible (inelastic, heat, etc.) losses (plate behaves quasi-brittle), and possible dynamics (waves).

V. Energy balance equation expresses the equality of the energy of formation of new surfaces (cracks) to the elastic energy of shear, released from the ring (layer), which was cut through with those cracks.

VI. The minimum-energy-consuming fracture scheme occurs, i.e., the scheme with such a number n and length L of the cracks, that the energy needed for its creation, is minimal.

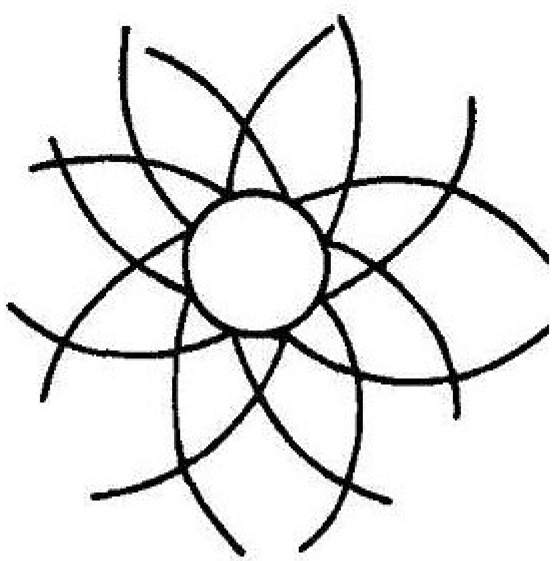

Fig. 3 [24, p. 326]

The proposed scheme can essentially be considered as a version of that classical approach by Griffith [23], adopted for cases where the problem symmetry requires the hypothesis on occurrence of one crack to be replaced with the assumption of arising a symmetrical system with n cracks.

3. Crack creation energy

To formalize the above speculations we write according to the condition «V» the basic equation of energy balance

$$W = W_e \tag{1}$$

where W, W_e are the energy of formation of new surfaces (shear cracks) and elastic energy gone to it of the layer, weakened (destroyed) by the cracks net arised, respectively.
Somewhat roughenning the real situation, we assume that the fracture pattern is axially periodic

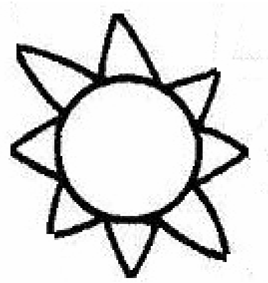

Fig. 4

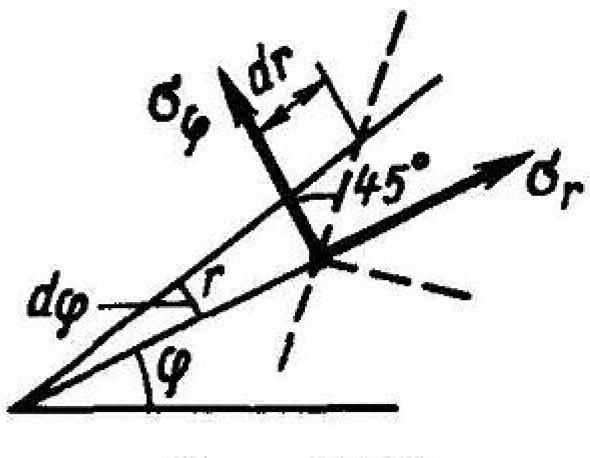

Fig. 5 [24, p. 326]

(Fig. 4) and is built of n «petals", each seen from the center of the cavity at an angle $2\varphi_n$ and formed by a couple of cracks emanating from the surface of the cavity at an angle of 45^0 and directed to each other

$$2\varphi_n = 2\pi/n,\ \varphi_n = \pi/n \tag{2}$$

The energy of formation of new surfaces (shear cracks)

$$W = \gamma L = \gamma \cdot 2n\Lambda \tag{3}$$

where γ is corresponding effective (specific) surface fracture energy (or more precisely, the specific energy of cracking, since there are two surfaces but one crack), L – total length of all the cracks, n – number of petals-sectors formed ($2n$ cracks), Λ – the length of a crack.
Taking that fracture occurs along the slip lines and the cracks grid forms a regular pattern like Fig. 4, calculate the total length of cracks

$$L = 2n\Lambda = n \cdot 2 \cdot \int_{\varphi=0}^{\varphi=\frac{\pi}{n}} dS \tag{4}$$

where n is the number of pieces (wedges), cut out by the cracks, φ – polar coordinate, dS – differential of arc length along the crack.
As is known, the direction of the maximum shear stresses divides the angles between the principal axes of the stress tensor in two [25, p. 265]. Circular cavity considered is a special case of a tube with an infinite outer radius. The principal stresses in the cross section of the tube are directed radially and circumferentially, the slip lines being inclined to these directions at an angle 45^0 (Fig.

5). From this figure we see that

$$dR = \pm R d\varphi \tag{5}$$

where R and φ are polar coordinates. This equation gives two orthogonal families of slip lines

$$R(\varphi) = Ce^{\pm\varphi}; \qquad R|_{\varphi=0} = R^* = C, \qquad R(\varphi) = R^* e^{\pm\varphi} \tag{6}$$

where C is the integration constant, R^* – the radius of the hole. Taking R^* and

$$W^* = 2\sqrt{2}\,\pi\gamma R^*, \tag{7}$$

as the length and energy reference scales, respectively, pass on to dimensionless variables r, λ, l, w by the formulas

$$r = \frac{R}{R^*}, \quad \lambda = \frac{\Lambda}{R^*}, \quad l = \frac{L}{R^*}, \quad w = \frac{W}{W^*} = \frac{W}{2\sqrt{2}\pi\gamma R^*} \tag{8}$$

$$R = R^* r, \qquad \Lambda = R^* \lambda, \qquad L = R^* l, \qquad W = W^* w \tag{9}$$

Then (6) takes the form $r(\varphi) = e^{\pm\varphi}$, and choosing to be definite one of the branches (families) with the sign +, we obtain the dimensionless radial coordinate of the "petal"-wedge end r_n

$$r_n = r(\varphi_n) = |(2)| = r(\pi/n) = e^{\pi/n}, \qquad n = \pi/\ln r_n \tag{10}$$

Writing now the expression for the differential of arc length in polar coordinates, expressing φ in R using (5) and integrating over the entire crack length, we find the length of one crack Λ, λ and the total dimensionless length of all the cracks l

$$\Lambda = \int_{\varphi=0}^{\varphi=\frac{\pi}{n}} dS = \int_{\varphi=0}^{\varphi=\frac{\pi}{n}} \sqrt{(Rd\varphi)^2 + (dR)^2} = |(5)| = \int_{R^*}^{R_n} \sqrt{2}dR = R^* \int_{1}^{r_n} \sqrt{2}dr = \sqrt{2}R^*(r_n - 1) \tag{11}$$

$$\lambda = \sqrt{2}(r_n - 1)$$

$$l = 2n\lambda = 2\sqrt{2}n(r_n - 1) = |(10)| = 2\sqrt{2}\pi\frac{r_n - 1}{\ln r_n} \tag{12}$$

For the dimensionless cracking energy with (8), (3), (11) we have

$$w = \frac{W}{W^*} = |(3),(8)| = \frac{2n\gamma\Lambda}{2\sqrt{2}\pi\gamma R^*} = |(11)| = \frac{2n\gamma\sqrt{2}R^*(r_n - 1)}{2\sqrt{2}\pi\gamma R^*} = |(10)| = \frac{r_n - 1}{\ln r_n} \qquad 13)$$

A plot of the dimensionless cracking energy w on r_n is represented by the upper curve (almost straight) in Fig. 6. It is the function defined for $r_n \geq 1$, monotonically increasing, concave upwards and equal to 1 for $r_n = 1$.

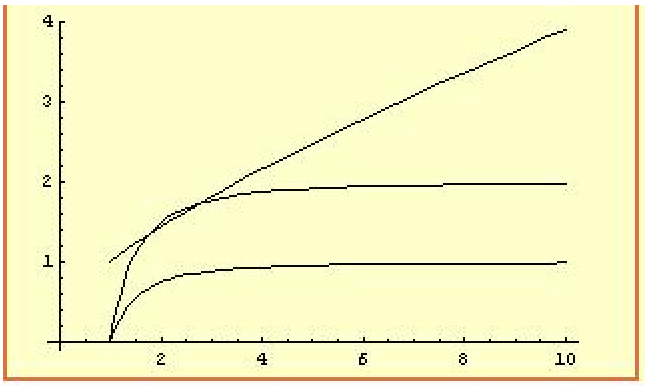

Fig. 6.

4. The elastic energy

Now we write the expression for the right hand side of (1) – elastic energy W_e, stored in the near-well layer, cut through with the cracks, which goes to the formation of cracks

$$W_e = W_f = \int_1^{R_n} W_{fs} 2\pi R dR$$

where W_f is shear energy, W_{fs} (s = specific) – specific shear energy, and the integration is over the ring $R \in [1, R_n]$. According to [25, p. 284, formula (7.28)], the expression for the specific shear energy can be written as

$$\begin{aligned} W_{fs} &= \frac{(1+\mu)}{6E}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]= \\ &= \frac{(1+\mu)}{6E}\left[(\sigma_1-\sigma_2)^2+{\sigma_2}^2+{\sigma_1}^2\right] \end{aligned} \tag{14}$$

where μ is Poisson's ratio, E – Young's modulus, $\sigma_{1,2,3}$ – principal stresses.
Substituting for the right side of (14) the well-known Lame solution for the very thick cylinder with inner radius R^*, being under internal pressure P [25, p. 338-339, formula (9.21)]

$$\sigma_{1,2} = \pm P\left(\frac{R^*}{R}\right)^2 = \pm\frac{P}{r^2} \tag{15}$$

we obtain for the specific W_{fs} and entire W_f shear energy in the ring (layer) $r \in [1, r_n]$

$$\begin{aligned} W_{fs} &= \frac{(1+\mu)}{6E}\left(2^2+1+1\right)\frac{P^2}{r^4} = \frac{(1+\mu)}{E}\frac{P^2}{r^4} \\ W_f(r) &= \int_1^{R_n}\frac{(1+\mu)}{E}\frac{P^2}{r^4}2\pi R dR = \left|R = R^* r\right| = \pi(1+\mu)ER^{*2}\left(\frac{P}{E}\right)^2\left(1-\frac{1}{r_n^2}\right) \end{aligned} \tag{16}$$

Going from P and W_f with the aid of scales of pressure E and energy

$$W^{**} \equiv \pi(1+\mu)ER^{*2} \tag{17}$$

respectively, to dimensionless p and w_f by formulas

$$p = \frac{P}{E}, \quad w_f = \frac{W_f}{W^{**}}, \quad P = pE, \quad W_f = W^{**} w_f \tag{18}$$

we obtain for the dimensionless energy w_f

$$w_f(p, r_n) = p^2\left(1 - \frac{1}{r_n^2}\right) \tag{19}$$

The formula (19) determines for $r_n \geq 1$ monotonically increasing function, concave upwards, equal to 0 for $r_n = 1$ and getting onto a horizontal asymptote at $r_n \to \infty$. The graphs of this function for two values of p are shown in Fig. 6 by two lower curves

.

5. Calculation of the index *n*

For small internal pressures p shear energy plot $w_f(r_n)$ lies below the cracking energy plot $w(r_n)$. The pressure arising, the in-tube stored elastic energy increases monotonically, the plot $w_f(r_n)$ goes higher and at some moment it touches the graph $w(r_n)$ for some r_n. The moment of contact will be the first moment when the elastic energy be equal to the energy required for the formation of an appropriate system of cracks (see similar arguments in Mohr theory [25, § 61, p. 300-306]). Touching specifies two conditions (equality of functions and equality of their derivatives) to determine two unknowns: the pressure and the thickness of the elastic layer, which gives its elastic energy for cracking.
From the condition of equality of functions, by substituting into (1) the expressions (7), (13), (17), (19), we obtain

$$W = W^* w = W_f = W^{**} w_f \tag{20}$$

$$\begin{gathered} 2\sqrt{2}\pi\gamma R^* \frac{r_n - 1}{\ln r_n} = \pi(1+\mu)Ep^2 R^{*2}\left(1 - \frac{1}{r_n^2}\right), \\ \frac{r_n - 1}{\ln r_n} = \frac{(1+\mu)ER^*}{2\sqrt{2}\gamma} p^2 \left(1 - \frac{1}{r_n^2}\right) \end{gathered} \tag{21}$$

or, setting the dimensionless constant

$$A = \frac{(1+\mu)ER^*}{2\sqrt{2}\gamma} \tag{22}$$

$$W = W_e: \qquad \frac{r_n - 1}{\ln r_n} = Ap^2\left(1 - \frac{1}{r_n^2}\right), \qquad Ap^2(r_n + 1)\ln r_n = r_n^{\,2} \tag{23}$$

Tangency condition we obtain by differentiating (23)

$$\frac{dW}{dr} = \frac{dW_e}{dr}: \quad \frac{\ln r_n - \dfrac{r_n - 1}{r_n}}{\ln^2 r_n} = Ap^2 \frac{2}{r_n^{\,3}}, \qquad Ap^2\left(\ln r_n + \frac{r_n + 1}{r_n}\right) = 2r_n \tag{24}$$

Dividing (23_1) to (24_1), we obtain governing equations for r_n

$$\frac{\dfrac{r_n - 1}{\ln r_n}}{\dfrac{\ln r_n - \dfrac{r_n - 1}{r_n}}{\ln^2 r_n}} = \frac{\left(1 - \dfrac{1}{r_n^{\,2}}\right)}{\dfrac{2}{r_n^{\,3}}} \tag{25}$$

$$\ln r_n = \frac{r_n + 1}{r_n + 2} = 1 - \frac{1}{r_n + 2} \tag{26}$$

Equation (26) for r_n is transcendental and needs numerical solution, but it can be seen that its root is close to e, therefore, representing r_n in (26) as $r_n = e(1 + \varepsilon)$, expanding in powers of ε and holding the first order, one can obtain simple approximate estimates for ε, r_n and n

$$\varepsilon \approx -0.25,$$

$$n = \frac{\pi}{\ln r_n} = \left| r_n = e(1+\varepsilon) \right| = \frac{\pi}{\ln[e(1+\varepsilon)]} \approx \frac{\pi}{1+\varepsilon} \approx \frac{3.14}{0.75} \approx 4.2$$

It is seen, that the value for n obtained is a fractional number. For the model assumed it means that the solution will be one of the two integers closest to the fractional value found (ie, either 4 or 5), which gives a lower value for the elastic energy. Those integer solutions n appear with changing $W_e(p)$ on increasing p, which, as is clear from Fig. 6, leads to the splitting of the solution found above into two solutions (corresponding to one and the same p), one of which, as can be seen from the graphs in Fig. 6, crawls down ($r_{n1}\downarrow$), and the other – up ($r_{n2}\uparrow$). By (13) $w(r_n) = (r_n - 1)/\ln r_n$ is a monotonically increasing function (↑), ie $r_{n1} < r_{n2} \Rightarrow w(r_{n1}) < w(r_{n2})$; vice versa, $n(r_n)$ decreases monotonically (by (10): $n = \pi / \ln r_n \downarrow$). The lower elastic energy corresponds to the lower value of r_n and, accordingly, to the larger value of n. Consequently, $n_{\min} = 5$. Here, with the growth of p the total stored elastic energy in the body increases, but cracking consumes less energy due to the fact that though the number n of cracks increases, but because of the reduction in r_n their total length l in (12) becomes less.

6. CONCLUSIONS

The energy approach is used to propose a model of arising of regular systems of cracks, emerging the surface of a circular cavity, being observed, for example, around oil and gas wells under uniform compression. The cracks are supposed to arise due to the accumulation of elastic compression energy in the system. The limit compression (the exhaustion of strength) being achieved, a network of cracks is formed in the most stressed layer adjacent to the interior of the body, thus utilizing the accumulated elastic energy of this layer. In this case, of all the possible grids the least energy-consuming one is formed, that is, the system with such a number n and length L of the cracks, that the energy needed for its creation is minimal. In the simplest scheme the number n of "petals"-wedges arising from the cracking turns out to be equal to 5 (which corresponds to $2n = 10$ cracks) and (contrary to limit compression pressure magnitude) be independent on geometrical and physical-mechanical parameters of the problem. The approach presented can be obviously formalized in the form of a corresponding variational principle of E.M. Morozov type [5, p. 11-24], provided that the core of the functional there introduced, would be modified appropriately and will be proportional not to the maximum normal stress or the maximum linear strain (similar to the first and second strength theories [5, p. 12]), but to the value, corresponding to the failure criterion here adopted.

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