TRANSIENT SURFACE RESPONSES OF A CRACKED HALF-PLANE

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ABSTRACT

The transient elastodynamic surface responses of an elastic half-plane are analyzed. The half-plane is subjected to an impact normal point force on the surface and contains an edge crack of an arbitrary length and orientation. The problem is considered as a superposition of the Lamb's problem and a superposition one. The Lamb's problem concerns an impact normal point force acting on the surface of a uncracked half-plane. While the superposition problem concerns a cracked half-plane subjected to the crack-face loading which are equal in magnitude but opposite in sign to the corresponding stress field in the Lamb's problem.

The solution to the Lamb's problem is well-known. On the other hand, owing to the existence of characteristic length in geometry, the analytic solution of the superposition problem is very difficulty, if it is not impossible. In this paper, a theory is proposed to analyze a reduced superposition problem instead. The crucial steps in the analysis are the application of integral transform, the Wiener-Hopf technique, coordinate transformation and the Cagniard-de Hoop method. Exact expressions are obtained for the resulting transient elastodynamic responses on the surface of the cracked half-plane as functions of time. The results are excepted to be exact within the time interval from the application of loading to the arrival time of waves other than P wave diffracted at the crack tip.

KEYWORDS:

Transient response, surface crack, Wiener-Hopf technique

INTRODUCTION

The transient surface responses of a cracked half-plane are very important in the application of nondestructive testing. Unfortunately, owing to the existence of geometric characteristic length in the problem, the analytic solution has been eluded for quite a long time. Freund (1974) analyzed a semi-infinite crack in unbounded medium with characteristic length in loading by using superposition of dislocations. Brock (1982) extended Fruend's result to the case of shear loading. Kuo and Cheng (1991) and Kuo and Chen (1992) used the Wiener-Hopf technique directly to analyze the case of Mode-III interface crack in a dissimilar anisotropic medium and the case of

Mode-I crack in a homogeneous unbounded medium, respectively. Kuo (1992) extended the results further to the case of Mode-III crack in the finite strip, which contained characteristic lengths not only in the loading but also in the geometry. Recently, Tsai and Ma (1992 and 1993) proposed a superposition procedure performing in the Laplace transformed domain to analyze semi-infinite cracks in a unbounded medium and in a half-plane, respectively. These problems contains characteristic length in loading (Tsai and Ma 1992), and/or in geometry (Tsai and Ma 1993). Nevertheless, exact transient analysis of problems with edge crack in a half-plane remains unattacked.

STATEMENTS OF PROBLEM

Consider a homogeneous, isotropic, linearly elastic half-plane containing an edge crack of arbitrary orientation. A global Cartesian coordinate (x_1, x_2) is defined in such a way that the half-plane is in the region of $x_2 \geq 0$, and the surface of the half-pane is coincided with the x_1 axis. The coordinate system, the applied force, the crack geometry and the half-plane are shown in Figure 1

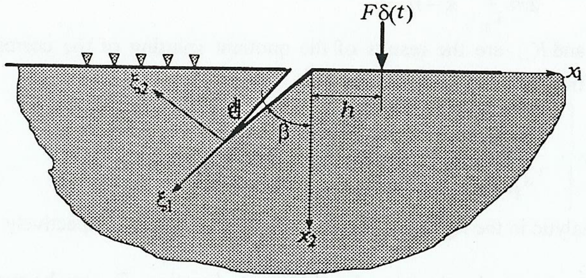

Figure 1: geometry of cracked half-plane and coordinate systems

The elastic half-plane is initially at rest, for $t < 0$. For time $t \geq 0$, an impact normal force of magnitude F acts on the free surface of the half-plane to the right of the crack mouth and at the point $(h,0)$. The boundary conditions on the surface of the half-plane are, along $x_2 = 0$,

$$\sigma_{22} = -F\,\delta(x_1 - h)\,\delta(t) \tag{1}$$

$$\sigma_{12} = 0 \tag{2}$$

where σ_{ij} are the components of the Cauchy stress tensor in the (x_1, x_2) coordinate, and δ is the Dirac delta function.

In order to describe the edge crack, a local coordinate system (ξ_1, ξ_2) is introduced in such a way that the crack lies in the line $\xi_2 = 0, -d < \xi_1 < 0$. Ahead of the crack tip the half-plane is perfectly welded. The global and local coordinates are related as

$$x_1 = -\xi_1 \sin\beta - \xi_2 \cos\beta - d\sin\beta \tag{3}$$

$$x_2 = \xi_1 \cos\beta - \xi_2 \sin\beta + d\cos\beta \tag{4}$$

where d and β are the length and the incline angel of the edge crack, respectively. Let τ_{ij} denote the component of the Cauchy stress tensor in the (ξ_1, ξ_2) coordinate. The conditions of zero traction on the crack faces are, along $\xi_2 = 0$ and $\xi_1 < 0$,

$$\tau_{22} = 0 \tag{5}$$

$$\tau_{12} = 0 \tag{6}$$

The components σ_{ij} and τ_{ij} are related by

$$\tau_{11} = \sigma_{11}\sin^2\beta + \sigma_{22}\cos^2\beta - \sigma_{12}\sin(2\beta) \tag{7}$$

$$\tau_{22} = \sigma_{11}\cos^2\beta + \sigma_{22}\sin^2\beta + \sigma_{12}\sin(2\beta) \tag{8}$$

$$\tau_{12} = (\sigma_{11} - \sigma_{22})\sin\beta\cdot\cos\beta - \sigma_{12}\cos(2\beta) \tag{9}$$

METHODS OF SOLUTION

In the Cartesian coordinate system, two dimensional in-plane wave motions of elastic solids are governed by two standard wave equations of displacement potentials ϕ and ψ (Achenbach 1973), as

$$\nabla^2\phi = s_p^2\ddot{\phi} \tag{10}$$

$$\nabla^2\psi = s_s^2\ddot{\psi} \tag{11}$$

where s_p and s_s are slowness of P and SV waves, respectively. The relationships between the relevant stress components and the displacement potentials can be found in (Achenbach 1973). In this article, the one-sided Laplace transform over the temporal variable t with kernel $\exp(-pt)$ will be applied to the problem. On the other hand, for the spatial variables, the two-sided Laplace transform over x_1 and ξ_1 with kernels $\exp(-pkx_1)$ and $\exp(-p\eta\xi_1)$, respectively, will be employed to the Lamb's problem and the superposition problems.

Lamb's Problem

The stresses in the Laplace transformed domain of the Lamb's problem, due to only the P-wave contribution, can be expressed as (Achenbach 1973)

$$\bar{\sigma}_{ij}^{L}(x_1, x_2, p) = \frac{p}{2\pi i}\int_{\Gamma_k} A_{ij}(k)\cdot e^{-p[\gamma_p x_2 - k(x_1 - h)]}\,dk \tag{12}$$

where the barred quantity and the superscript L stand for the one-sided Laplace transformed quantity and the solutions of Lamb's problem, respectively, Γ_k is the Bromwich contour of the one-sided Laplace transform, and

$$\begin{Bmatrix} A_{11} \\ A_{22} \\ A_{12} \end{Bmatrix} = \frac{(s_s^2 - 2\gamma_p^2)}{R(k)} \begin{Bmatrix} s_s^2 - 2\gamma_p^2 \\ s_s^2 - 2k^2 \\ -2k\gamma_p \end{Bmatrix} \tag{13}$$

The function $\gamma_\alpha = (s_\alpha^2 - k^2)^{1/2}$ for $\alpha = p, s$ are the vertical wavenumbers of P and SV waves, respectively, and $R(k)$ is the Rayleigh function which is defined as

$$R(k) = \left(s_s^2 - 2k^2\right)^2 + 4k^2\gamma_p\gamma_s \tag{14}$$

The stress field along the crack line in the local coordinate of $-d < \xi_1 < 0$, by the use of (7)-(9) and (12), is

$$\overline{\tau}_{ij}^{L}(\xi_1, 0, p) = \frac{p}{2\pi i}\int_{\Gamma_k} B_{ij}(k)\cdot e^{-p[q(\xi_1+d)+kh]}dk \tag{15}$$

where $q = \gamma_p \cos\beta + k\sin\beta$, and the relations between B_{ij} and A_{ij} are exactly the same as the ones between τ_{ij} and σ_{ij} in (7)-(9).

Superposition Problem

The superposition problem considers a cracked half-plane subjected to crack face traction which are equal in magnitude but opposite in sign to (15). Owing to the existence of geometric characteristic length, the exact solution have not yet been found. In this article, a reduced problem will be considered instead first. In the reduced superposition problem, the half-plane is replaced by a unbounded solid, and the edge crack by a semi-infinite crack, while the new crack is subjected to exactly the same traction as for the original edge crack. Once the solution of the reduced superposition problem being found, the effect of the free surface of the half-plane in the original superposition problem will be taken into account exactly with the aid of the reflection coefficients of plane waves. The solution is then excepted to be exact up to the time when the first wave diffracted by the crack mouth at the free surface of the half-plane arrives.

Reduced Superposition Problem The reduced superposition problem concerns a problem of semi-infinite crack in a unbounded solid subjected to crack face traction of finite range. The problem contains only the characteristic length in loading but not in geometry. The solution can be found by the use of integral transform together with the direct application of Wiener-Hopf technique as described in Kuo and Chen (1992). Notice that an opposite sign to the expression in (15) is applied on the crack faces over $-d < \xi_1 < 0$, and it can be viewed as a superposition of surface tractions with forms of $\exp(-pq\xi_1)$ and described magnitudes over k. One-sided and two-sided Laplace transform over t and ξ_1 are employed to the problem. The problem is then solved by considering the symmetric and anti-symmetric ones with respect to the crack plane separately. The resulting displacement potentials in the transformed domain of (η, ξ_2, p) are

$$\begin{Bmatrix} \tilde{\overline{\phi}} \\ \tilde{\overline{\psi}} \end{Bmatrix} = -\frac{1}{2\pi i p^2}\int_{\Gamma_k} \begin{Bmatrix} B_{22}\overline{\Phi}_I^* D_{I-} + B_{12}\overline{\Phi}_{II}^* D_{II-} \\ B_{22}\overline{\Psi}_I^* D_{I-} + B_{12}\overline{\Psi}_{II}^* D_{II-} \end{Bmatrix} \cdot \exp\left[-p(\varsigma_p\xi_2 + kh)\right]dk \tag{16}$$

where tilde denotes quantity of the two-sided Laplace transform over ξ_1. The quantity ς_p is defined as $\varsigma_p = (s_p^2 - \eta^2)^{1/2}$, and

$$D_{i-}(q,\eta) = D_i(q,\eta)\exp(p\eta d) - D_{i+}(q,\eta) \tag{17}$$

$$\begin{Bmatrix} \overline{\Phi}_I^* \\ \overline{\Phi}_{II}^* \\ \overline{\Psi}_I^* \\ \overline{\Psi}_{II}^* \end{Bmatrix} = -\frac{1}{2\mu(s_s^2 - s_p^2)(\eta - s_R)\varsigma_p K_-(\eta)} \cdot \begin{Bmatrix} (s_s^2 - 2\eta^2)(s_p - \eta)^{1/2} \\ -2\eta\varsigma_p(s_s - \eta)^{1/2} \\ 2\eta\varsigma_p(s_p - \eta)^{1/2} \\ (s_s^2 - 2\eta^2)(s_s - \eta)^{1/2} \end{Bmatrix} \tag{18}$$

Functions D_i and D_{i+} in (17)-(18) are defined as

$$\begin{Bmatrix} D_I(q,\eta) \\ D_{II}(q,\eta) \end{Bmatrix} = \frac{1}{(\eta + s_R)(q+\eta)K_+(\eta)} \cdot \begin{Bmatrix} (s_p + \eta)^{1/2} \\ (s_s + \eta)^{1/2} \end{Bmatrix} \tag{19}$$

$$D_{i+}(q,\eta) = -\frac{1}{2\pi i}\int_{\Gamma_z} \frac{D_i(q,z)}{z - \eta}\cdot e^{pzd}\,dz \tag{20}$$

Functions K_+ and K_- are the results of the quotient splitting of the corresponding Rayleigh function in the transformed η domain as

$$K_\pm(\eta) = \exp\left\{-\frac{1}{\pi}\int_{s_p}^{s_s} \tan^{-1}\left[\frac{4z^2(z^2 - s_p^2)^{1/2}(s_s^2 - z^2)^{1/2}}{(s_s^2 - 2z^2)^2}\right]\frac{dz}{z \pm \eta}\right\} \tag{21}$$

and they are analytic in the right and left half complex η planes, respectively.

By deforming the integration contour Γ_z of (20), the function D_{i+} can be expressed as a sum of two closed form terms, which are the contribution from the poles at $\eta = -s_R$ and $\eta = -q$, and a branch line integral. Hence the expression of D_{i-} in (17) can be written as

$$D_{i-} = D_i e^{p\eta d} + E_{iq}e^{-pqd} + E_{iR}e^{-ps_R d} + \int_{-s_p}^{-\infty} E_{iz}e^{pzd}dz \tag{22}$$

Notice that D_i, E_{iq}, E_{iR} and E_{iz} are all independent on the parameter p. Substituting (22) into (16), taking the inverse two-sided Laplace transform over η, and using the relations between the global and local coordinates (3)-(4), the typical terms in $\overline{\phi}$ (and $\overline{\psi}$) will be as

$$\overline{\phi} = \frac{1}{4\pi^2 p}\int_{\Gamma_k}\int_{\Gamma_\eta} G_1(k,\eta)e^{-p[H(k,\eta)+wd]}d\eta\,dk \tag{23}$$

$$\overline{\phi} = \frac{1}{4\pi^2 p}\iiint G_2(k,\eta)e^{-p[H(k,\eta)+zd]}d\eta\,dz\,dk \tag{24}$$

where the function H in the exponent of (23)-(24) is defined as

$$H = kh + (\eta\sin\beta - \varsigma_p\cos\beta)x_1 - (\varsigma_p\sin\beta + \eta\cos\beta)x_2 \tag{25}$$

and w, according to (22), could be $-\eta$, q, or s_R. It is of interesting to mention that (23)-(24)

can be interpreted as the superposition of plane waves with horizontal wavenumber $(\eta \sin\beta - \varsigma_p \cos\beta)$ over η. Notice that the vertical wavenumber is $(-\varsigma_p \sin\beta - \eta\cos\beta)$, which is the square root of the difference between s_p^2 and the square of the horizontal wavenumber as one would expect.

Free Surface Effect By expressing the diffracted waves as superposition of plane waves as (23)-(24), the reflection of the diffracted waves at the free surface of the half-plane can easily be constructed. Each plane P wave will be reflected to yield both P and SV waves. In the application of nondestructive testing, the surface responses are of most interest. The resulting surface displacements can be obtained from the displacement potentials. The typical terms resulting from contribution of (23)-(24) are as

$$\overline{u}_j = \frac{1}{4\pi^2 p} \int_{\Gamma_k} \int_{\Gamma_\eta} G_1(k,\eta)\, S_{jp}(\eta \sin\beta - \varsigma_p \cos\beta) \cdot e^{-p[H(k,\eta)+wd]} d\eta\, dk \quad (26)$$

$$\overline{u}_j = \frac{1}{4\pi^2 p} \iiint G_2(k,\eta)\, S_{jp}(\eta \sin\beta - \varsigma_p \cos\beta) \cdot e^{-p[H(k,\eta)+zd]} d\eta\, dz\, dk \quad (27)$$

where the function S_{jp} is the surface receiver function of the displacement components in the generalized ray theory, and is defined as

$$S_{1p}(\alpha) = \left(s_p^2 - \alpha^2\right)^{1/2}\left(1 - R^{pp}\right) + \alpha R^{ps} \quad (28)$$

$$S_{2p}(\alpha) = \alpha\left(1 + R^{pp}\right) + \left(s_s^2 - \alpha^2\right)^{1/2} R^{ps} \quad (29)$$

and R^{pp} and R^{ps} are reflection coefficients of plane waves at the free surface as

$$R^{pp} = \left[\left(s_s^2 - 2\alpha^2\right)^2 - 4\alpha^2\left(s_p^2 - \alpha^2\right)^{1/2}\left(s_s^2 - \alpha^2\right)^{1/2}\right] \Big/ \Delta \quad (30)$$

$$R^{ps} = 4\alpha\left(s_p^2 - \alpha^2\right)^{1/2}\left(s_s^2 - 2\alpha^2\right)^2 \Big/ \Delta \quad (31)$$

$$\Delta = \left(s_s^2 - 2\alpha^2\right)^2 + 4\alpha^2\left(s_p^2 - \alpha^2\right)^{1/2}\left(s_s^2 - \alpha^2\right)^{1/2} \quad (32)$$

Cagniard-de Hoop Inversion

Investigate the one-sided Laplace transform inversion by Cagniard-de Hoop method with care, we conclude that the arrival time of (27) is corresponding to the P-wave traveling along the free surface from the source point to the crack mouth, then along the crack line to the crack tip, finally from the crack tip directly to the receiver point. The arrival time of (26) with $w = s_R$ corresponds to the P-wave from source point to the crack mouth, then Rayleigh wave from crack mouth to the crack tip, finally P-wave again directly from the crack tip to the receiver. The arrival time of (26) with $w = \eta$ corresponds to the P-wave along the free surface from source point to the crack mouth, then from the crack mouth along the free surface again to the receiver. This term is supposed to be canceled out with the contribution of Lamb's problem, if we use the original superposition problem rather than the reduced one.

The only term of interest is (26) with $w = q$, which corresponds to P-wave directly from source point to the crack tip, and then directly from crack tip to the receiver.

NUMERICAL RESULTS

To have some confidence on the proposed theory by considering the reduced superposition problem together with the free surface effect instead of the original superposition problem, the results are compared numerically to the finite difference computation first. For convenience of comparison the source time function is taken as half-sine shape with duration 5 μ s, which is about the real source in the experiment of dropping steel ball on a specimen. The crack length is 2cm, and the force is acting at $x_1 = 2$ cm. Receivers are on the surface of the half-plane and at $x_1 = -1$, -2, -4cm for the case of $\beta = 0^0$ and $\beta = 30^0$.

Time history of transient vertical surface displacements for the cases of $\beta = 0°$ and 30° are shown in Figures 2a and 2b, respectively. The solid lines are the finite difference results, while the dotted lines are the analytic solution from the contribution of (26) with $w = q$. The satisfactory agreements are observed. Notice that this analytic solution is only valid from the initial time up to the arrival of any wave other than the diffracted P-wave from the crack tip. Hence the dotted lines terminate at the time of the second wave arrives.

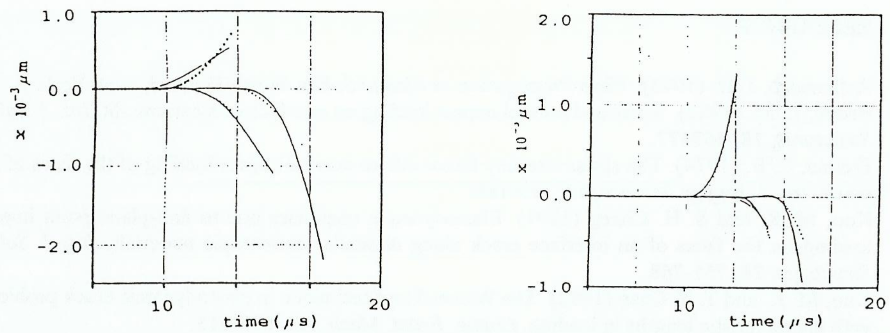

Figure 2. Vertical displacement of the surface versus time, at $x_1 = -1$, -2, -4cm, for $\beta = 0^0$ and $\beta = 30^0$. The solid line is the finite difference result, and the dotted line is the present analytic result.

CONCLUSION

In this paper a theory is proposed to construct the analytic solution of the transient elastodynamic surface displacement for a cracked half-plane due to an impact normal point force. The basic idea is a superposition of the Lamb's problem and an appropriate superposition problem. In the theory the half-plane with an edge crack in the original superposition problem is replaced by a reduced problem, which concerns an infinite plane with a semi-infinite crack, to construct the initial diffracted wave field from the crack tip. The crucial steps in the analysis of the diffracted waves are the application of integral transform together with the Wiener-Hopf technique, the coordinate transform and the Cagniard-de Hoop method. The surface response on the free surface of the half-plane due to the diffracted wave field are then constructed with the aid of the surface receiver function in the generalized ray theory.

Only the first term in the solution is inverted from the integral transformed domain, hence the solution is excepted to be exact only from the initial time up to the arrival of the second diffracted wave from the crack tip to the surface of the half-plane. The transient vertical displacements are computed numerically for cases of vertical and inclined edge crack, and are compared to the finite difference results. Satisfactory agreements are observed.

This result is also very useful to identify the first arrival time of signal from the experimental data. That arrival time is a key factor of locating the crack tip in the nondestructive testing application.

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