Strip Yielding in a Constrained Layer

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Abstract

This paper presents an investigation of the elastic-plastic deformation behaviour of a crack embedded in a layer which is sandwiched between two elastic adherends subjected to mode I loading. The stress intensity factors of cracks, contained in a layer, subjected to arbitrary surface tractions have been solved using integral transform technique, from which approximate solutions have been constructed. For layers with an elastic modulus lower than that of the adherends, the plastic yielding under the constraint of the elastic adherends exhibits two distinctly different regimes. A strip yielding solution has been obtained for the length of the plastic zone, which depends on the ratio of elastic modulus and crack length to layer thickness ratio. It has been found that the higher the modulus ratio, the sharper the transition between these two regimes. The agreement between the theoretical solution and finite element analysis is generally good.

Introduction

Layered structures which are consisted of two or more dissimilar adherends joined together with thin, ductile layers have been employed in a wide range of applications. Examples include adhesively bonded joints, a variety of composite materials, and ceramics joined by metal foils. For these structures the integrity of the layer and/or the interface between the layer and the substrate are often critical to the mechanical performance. Therefore understanding of the fracture behaviour of these material systems is not only of great importance to the technological advances in improving the quality of joining, but also essential to establishing appropriate design criteria for evaluating structural integrities.

Up to now in modelling the crack problems of layered materials, attention has been mainly focused on brittle layers [1] having low fracture toughness. The extent of the failure process is confined to a region much smaller than the relevant geometric length scales, such as layer thickness, thus the linear elastic fracture mechanics theories are applicable. However, for materials used in adhesive bonding or metal foil bonding, the layer is often made of very ductile materials to enhance the damage tolerance of a bonded structure. For instance, the structural adhesives [2] used in bonded joints are often toughened using additives such as rubber particles, which may undergo extensive plastic deformation prior to final fracture. In these cases, the plastic zone size at failure is generally greater or comparable to layer thickness [3]. Furthermore it has been found that in the case of metal foils [4,5] the fracture process is dictated by the nucleation and growth of voids at distances several foil thickness ahead of the crack tip, unlike for a crack in a bulk solid where voids nucleate and grow ahead of the crack tip within distances of the order of the crack tip opening. One important ramification of this difference in the fracture mechanisms is that crack tip opening displacement or the *J*-integral may fail to provide a unique fracture parameter for bonded structures, as both parameters characterise only the near tip deformation behaviour, hence unable to capture the essential features of fracture process occurring ahead of the crack tip at distance far greater than the crack opening.

Although elasto-plastic fracture mechanics for homogeneous materials is now highly advanced and mature, similar methods for constrained layers are virtually non-existent. In the case of fracture of homogeneous solids under large scale yielding conditions, the cohesive crack model by Dugdale [6] and Barrenblatt [7] provides a powerful tool for determining the plastic zone length and the crack tip opening displacement. The aim of this paper is to present a solution for the spread of plastic deformation in a constrained layer. The problem of a crack contained in a thin layer which is sandwiched between two elastic substrates of different modulus has been formulated using Fourier transform technique, which reduce the problem to a single integral equation. A simple, approximate solution has also been constructed based on the exact solution of the integral equation. The predicted plastic zone length was found to be in good agreement with finite element results.

Stress Intensity Factors for Cracks in a Constrained Layer

Let us first consider the case of elastic adhesive, as the plastic zone length can be determined by cancelling the stress intensity factor at the end of the plastic zone, similar to that proposed by Dugdale [6]. To this end, ideally a Green function for the stress intensity factor of a crack embedded in a constrained thin layer is required, which is the stress intensity factor caused by a pair of point forces acting at an arbitrary position on the crack faces. However, no solutions are presently available, except in the special case that the adhesive and the substrate have identical elastic properties. In the following, we will adopt a method similar to that used by Sih and Chen [8] to formulate the problem using Fourier transform technique. Assume that the crack surfaces are subjected to a traction $\sigma_{yy}(x,0) = -p(x)$, the Airy stress functions for the adhesive layer and the substrates can be expressed in terms of Fourier Transforms, and after some lengthy manipulations the problem can be reduced to a single unknown $A(s)$ that satisfies a pair of dual integral equations,

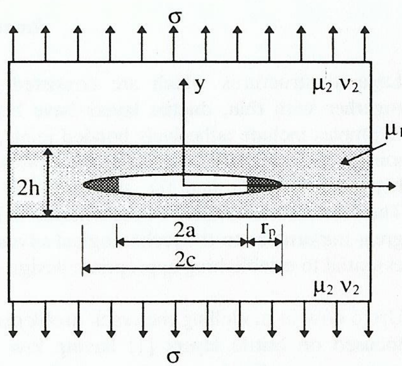

Fig.1 A crack in a constrained layer

$$\int_0^\infty sF_1(s)A(s)\cos(sx)ds = \frac{\pi}{2}p(x) \qquad (x < c) \tag{1a}$$

$$\int_0^\infty A(s)\cos(sx)ds = 0 \qquad (x \geq c) \tag{1b}$$

where function $F_1(s)$ is given by the following expression, correcting a typographical error in Ref. [8],

$$F_1(s) = \frac{a_1 + [a_2(sh)^2 + a_3]e^{-2sh} + a_4e^{-4sh}}{a_1 - a_2(sh)e^{-2sh} - a_4e^{-4sh}} \tag{2}$$

where h denotes the half the layer thickness, and a_1, a_2, a_3, and a_4 are dependent on shear modulus ratio, μ_1/μ_2, and the Poisson's ratios, ν_1 and ν_2 of the adhesive and substrate, respectively; expressions can be found in Ref.[8]. The dual integral equations can be further reduced to a single Fredholm integral equation after making the following representation,

$$A(s) = \int_0^c \phi(t)J_0(st)dt \tag{3}$$

and hence

$$\phi(t) + t\int_0^c \phi(x)K(t,x)dx = t\int_0^t \frac{p(x)}{\sqrt{t^2 - x^2}}dx \tag{4}$$

where

$$K(t,x) = \int_0^\infty s[F_1(s) - 1]J_0(st)J_0(sx)ds \tag{5}$$

The stress intensity factor is given by

$$K = \frac{2\phi(c)}{\sqrt{\pi c}} \tag{6}$$

In the case of uniform traction, $p(x) = \sigma$, the right-hand side of the integral equation (4) becomes $\pi\sigma t/2$, and it is easy to verify that the stress intensity factor for a central crack in a single material, $K = \sigma\sqrt{\pi c}$, is recovered, noting $F_1(s) = 1.0$.

The integral equation (4) can be solved using Gauss-Legendre method. A convergence study showed that an accuracy of 0.1% in stress intensity factor could be achieved for N=60. The numerical results of the integral equation are shown in Fig.2, which agree with those reported in Refs. [8,9] using a slightly different solution method. It should be pointed, however, that a fundamental error occurred in Ref.[8] regarding the limiting case of $h/a \to 0$. The asymptotic solution

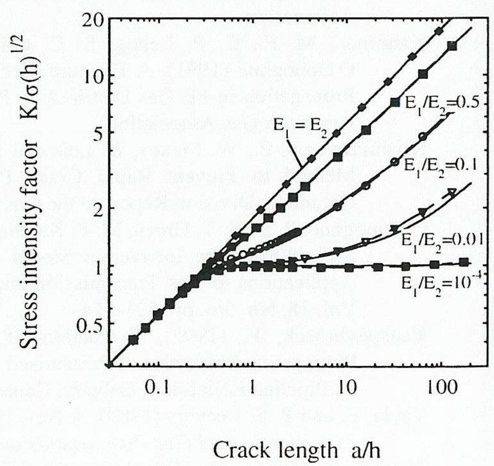

Fig.2 Stress intensity factors of a crack embedded in a constrained layer. Symbols: numerical solution of the integral equation. Solid lines: approximate solution.

of the dual integral equations (1) in this case cannot be simplified by setting $F_1(s)$ equal to a constant, thus reducing the problem to a single material crack with the applied stress σ being replaces by $\mu_1(1-\nu_2)/\mu_2(1-\nu_1)\sigma$, as suggested by Sih and Chen [8]. This can be easily demonstrated by letting $F_1(s)$ equal to a constant and then solve the integral equation (4). The correct stress intensity factor for this limiting case is given by [13], derived using *J*-integral theory.

The results shown in Fig.2 indicated two distinctly different regimes, denoted here as the 'short crack' and the 'long crack' regimes. For crack lengths far shorter than the height of the layer, the crack field is not perturbed by the constraint imparted by the substrates. However, in the 'long crack' limit, viz crack length much greater than layer thickness, the stress intensity factor is strongly influenced by the ratio between adhesive and substrate moduli. Two important scaling relations have been identified by the analytical formulation, as indicated in Fig.2, which are respectively the elastic modulus ratio (μ_1/μ_2) and crack length to layer thickness ratio (c/h). Since no closed-form solutions of the integral equation are possible, it is desired to construct an approximate solution to provide an accurate interpolation over the full range of the two key scaling parameters mentioned above.

Apart from the short crack limit, the two branches at the long crack limit that ought to be recovered as special cases are (i) $K=\sigma\sqrt{h}$ as $\mu_1/\mu_2\to 0$ [10,11] and (ii) $K=\sigma\sqrt{\pi c}\sqrt{\mu_1/\mu_2}$ as $h/c\to 0$, see e.g. Refs.[12,13]. One closed-form solution has been recently derived [13],

$$K = Y_1(h/c, E_1/E_2)\sigma\sqrt{\pi c} \tag{7}$$

where

$$Y_1 = \left[\frac{E_1}{E_2}+\frac{h}{\pi c}\left(1-\frac{E_1}{E_2}\right)\right]^{1/2} \quad (c \ge h) \tag{8}$$

where E_1 and E_2 are the Young's moduli of the adhesive layer and the substrate. Here the factor Y reflects the changes in the stress intensity factor due to the perturbation of elastic modulus within a strip of height of h. While the correction factor given by equation (8) has been shown to be in good agreement with numerical solutions of the problem in the range $c/h>1.0$, it is easy to see that Y_1 does not recover unity as $c/h\to 0$, the short crack limit. To circumvent this problem, we will combine it with a simple interpolation formulas [13] obtained for the limiting case of $E_1/E_2\to 0$, viz, a thin layer clamped at $y=\pm h$,

$$Y_2 = \left(1-\nu_1^2\right)^{1/2}\left(\frac{h}{\pi c}\tanh\frac{\pi c}{h}\right)^{1/2} \quad (E_a/E_s=0) \tag{9}$$

which recovers both the short crack limit (except the factor $\left(1-\nu_1^2\right)^{1/2}$, which reflects the difference between clamped boundary versus shear free boundary) and a special long crack limit ($c>>h$) with $\mu_2\to\infty$, a crack contained in a thin strip which is clamped at the edges [10,11].

Combining equations (8) and (9) would yield a new geometry factor for a crack in a constrained layer,

$$Y(h/c, E_1/E_2) = \left[\beta\frac{E_1}{E_2}+\frac{h}{\pi c}\tanh\frac{\pi c}{h}\left(1-\beta\frac{E_1}{E_2}\right)\right]^{1/2} \tag{10}$$

where an additional parameter, β, has been included to improve the correlation with the exact solution of the integral equation. The comparison between the numerical solutions and the above formulas is shown in Fig.2, where β is given by

$$\beta = 1+\left[1-(E_1/E_2)^{1/3}\right]e^{-Ac/h} \quad (A=0.048) \tag{11}$$

where A was obtained by curve-fitting the numerical results of the integral equation. It is evident from Fig.2 that this new formulas provides good approximation over the entire range, including the short crack and the long crack limits with various modulus ratios.

A Strip Yield Model

From the forgoing analysis the stress intensity factor of a crack embedded in a constrained layer can be much lower than in a single material, due to the 'shielding' of stiff substrates. In the case of small scale yielding, the plastic zone length can be easily evaluated using standard fracture mechanics. The much lower stress intensity factor in the case of low modulus layer means that the plastic zone length is also much smaller than that would exist in a single material subjected to the same remote stresses. However, it has been found [13] that outside the singular field the y-stress ahead the tip of a crack in a constrained layer is much higher than that exists in a single material subjected to the same stress intensity factor. A plateau in the *y*-stress has also been observed from finite element analysis [13]. The spread of plastic yielding can thus be rather 'abrupt' as this plateau region is crossed over. Consequently the plastic zone could grow to an extent much greater than the layer thickness, or the singular field [3,13]. In the following we will call this behaviour as large scale yielding.

The length of the plastic zone under large scale yielding conditions can be determined by superimposing the solutions of two elastic problems, as advocated by Dugdale [6]. This is done by cancelling the stress intensity factor at the tip of a fictitious crack, which is subjected to the remote applied load and a cohesive stress over a region near the crack tip. The length over which the cohesive stress is required to cancel the stress intensity factor at the end of this fictitious crack is the plastic zone length. This can be expressed as

$$K + K_{\sigma_{ys}} = 0 \tag{12}$$

where K and $K_{\sigma_{ys}}$ are the stress intensity factors corresponding to a crack subjected to uniform stress and a cohesive stress. In the present case, plastic yielding is assumed to be contained in the adhesive layer while the substrates are assumed to remain elastic. For a partially loaded crack that is subjected to a traction $p(x)=-\sigma_{ys}H(x-a)$, where H is the Heaviside function, the right-hand side of the integral equation (4) becomes

$$t\int_0^t \frac{p(x)}{\sqrt{t^2-x^2}}dx = \begin{cases} 0 & t<a \\ -t\sigma_{ys}\cos^{-1}\dfrac{a}{t} & a<t\le c \end{cases} \tag{13}$$

For a given crack of length a, the plastic zone length ($r_p = c-a$) can be obtained by searching for the value of c which would satisfy equation (12). This would require an iteration procedure which involves solving the integral equation at each iteration. Alternatively, if we can construct an approximate solution for a partially loaded crack, we can establish a simple, closed-form solution of the plastic zone length. To this end, it is proposed to combine the correction factor (10) with the well known stress intensity factor of partially loaded crack in a single material,

$$K_{\sigma_{ys}} = \left[\frac{2}{\pi}\sigma_{ys}\sqrt{\pi c}\cos^{-1}\frac{a}{c}\right] Y(h/r_p, \mu_1/\mu_2) \qquad (14)$$

where the first term in the bracket represents the solution of a partially loaded crack in a single material. It should be noted that the crack length c in the correction factor Y has now been replaced by the plastic zone length, $r_p = c - a$, as the cohesive stress is only applied over this length.

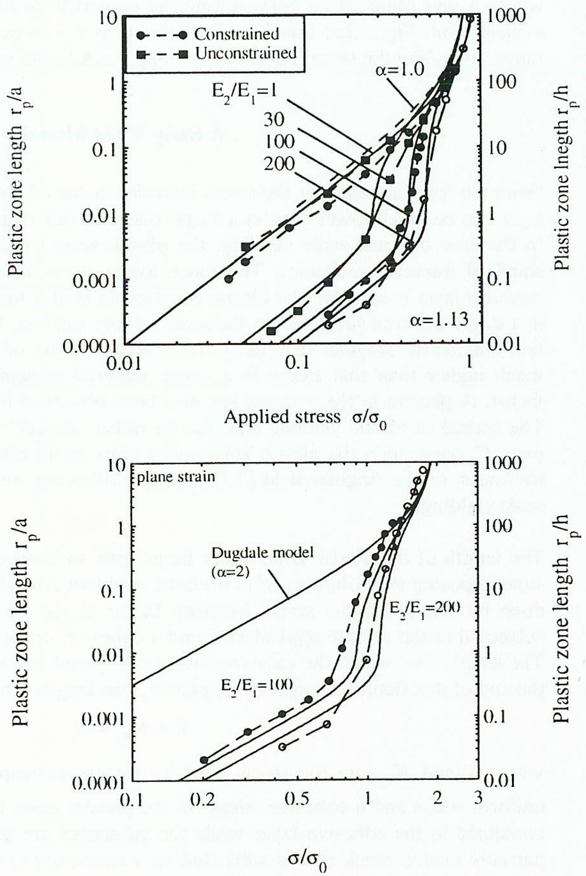

Fig.3 Lengths of plastic zone at various stress levels under (a) plane stress and (b) plane strain conditions. Symbols: finite element results. Solid lines: theory.

The solutions of the plastic zone length for the plane stress and plane strain conditions are shown in Fig.3, together with finite element results to be discussed later. The cohesive stress σ_{ys} is a factored yield stress to reflect the influence of the triaxial stress state on the plastic yielding of a constrained layer [13], i.e.,$\sigma_{ys} = \alpha\sigma_0$. Here σ_0 denotes the uniaxial yield stress of the layer material and α refers to the plastic constraint factor. Under plane stress and plane strain conditions, it has been found that [13] the plastic constraint factor,

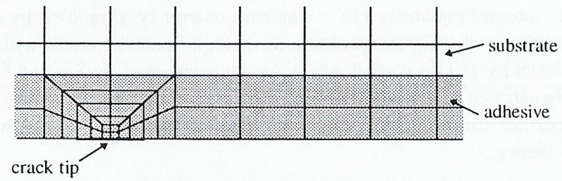

Fig.4 Finite element mesh near crack tip

$$\alpha = \begin{cases} 3/\sqrt{7} & \text{plane stress} \\ 2.0 & \text{palne strain} \end{cases} \qquad (\nu_1 = 0.33) \qquad (15)$$

which are used respectively in computing the theoretical predictions. The finite element model of the problem was developed using a general purpose code (PAFEC) [14], and the mesh near the crack tip region is shown in Fig.4. The adhesive was assumed to be elastic-perfectly plastic and associated plasticity theories were used with the von Mises yield criterion.

With the case of single material crack as a base reference, the problem of a crack embedded in a thin layer which is sandwiched between two elastic substrates can be divided into two categories: the elastic modulus of the layer (E_1) being less than or greater than that of the substrate (E_2). In the extreme of the latter category, such as two pieces of ceramic plates bonded with a metal foil, the elastic deformation of the foil is almost negligible, thus the layer representing an incompressible solid. When compressed between two elastic substrates, the metal foil would behave rather like a rigid-plastic material, developing a high hydrostatic stress ahead of the crack tip, which can increase indefinitely with the applied load [4,5]. In the present case, however, as the adhesive layer has a modulus much lower than those of the substrates, the hydrostatic stress arising from the constrained yielding is far less pronounced than in the case of metal foil. In fact, the maximum hydrostatic stress has been found not to exceed three times of the uniaxial yield stress of the material, even under plane strain condition, before the entire adhesive layer yields. Nevertheless, the analysis presented here should also be applicable to the latter category. Further work is in progress to develop a strip yield model which incorporates the effect of hydrostatic stresses.

Conclusions

The spread of plastic yielding of a crack embedded in a constrained layer has been analysed using integral transform technique. An approximate solution has also been obtained, which is shown to be in close agreement with the finite element results. The transition from small scale yielding and to large scale yielding normally occurs at a plastic zone length to layer thickness much less than unity. The smaller the ratio between the Young's moduli of the adhesive and substrate, the sharper the transition from small scale yielding to large scale yielding.

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