DYNAMICAL SYSTEMS AND THE RATE-DEPENDENCE OF CRACKED MEDIUM

PÉTER B. BÉDA
Department of Mechanics Technical University of Budapest
P.O. Box 91. H-1502 Budapest, Hungary

ABSTRACT

The paper deals with rate-dependent and rate-independent smeared crack models. By considering a cracked medium as a dynamical system the case of the strain localization is studied as a static bifurcation. An important way of the loss of stability of a solution of dynamical systems is the static bifurcation. In such cases the uniqueness of the solution is lost at the loss of (Liapunov) stability. For a rate-independent crack model this phenomenon happens in a structurally unstable way. By adding a rate-dependent term the system of the basic equations regains structural stability.

KEYWORDS

Rate-dependent smeared crack model, strain localization, dynamical systems

INTRODUCTION

The numerical studies on the strain localization of smeared crack models show that in rate independent cases the results are essentially dependent on the finite element discretization, (Sluys, 1992), (Sluys & de Borst, 1992). This deficiency can be corrected by using several methods. One of them is to add rate-dependent terms, (Needleman, 1988). By considering a continuous material as a dynamical system (Wiggins, 1990) material instability is closely related to the Lyapunov stability (Rouche *et al.*, 1977) of some state of the material. There are two basic possibilities for a dynamical system of the loss of stability, the dynamic and static bifurcations. The first one means the onset of a self-sustained oscillation, the second means a change in the number of solutions. Rate-independence for continua causes nongeneric dynamic behavior (Béda, 1994), (Béda, 1996). This means that the loss of stability of a rate-independent continuum cannot be classified into the generic types.

The paper aims to treat smeared crack models as dynamical systems and study what happens in the case, when the numerical investigation finds mesh dependence. In the first part the basic equations of the cracked medium are derived by using a rate-independent smeared crack model. The next part shows, how the dynamical system concept can be used for cracked media. Then the nature of the nongeneric behavior is studied in the rate-independent case at strain localization. In the next part the effect of rate dependence is presented and at last a uniaxial problem is studied.

THE BASIC EQUATIONS OF A CRACKED MEDIUM

At first the basic equations of the cracked medium should be obtained. In case of small strain the kinematic equation is

$$\epsilon = \frac{1}{2}(\mathbf{u} \circ \nabla + \nabla \circ \mathbf{u}), \tag{1}$$

where ϵ is the strain tensor, $\mathbf{u}$ is the displacement vector and $\circ$ denotes diadic product. The equation of motion without body forces is

$$\rho \ddot{\mathbf{u}} = \mathbf{T}\nabla, \tag{2}$$

where ρ is the density and $\mathbf{T}$ denotes the symmetric Cauchy stress tensor. In the smeared crack models the strain rate $\dot{\epsilon}$ can be divided into an elastic and crack strain rate part

$$\dot{\epsilon} = \dot{\epsilon}_e + \dot{\epsilon}_{cr}. \tag{3}$$

By using matrix $\mathbf{D}_e$ of the elastic moduli and (3) the stress-strain relation is

$$\dot{\sigma} = \mathbf{D}_e(\dot{\epsilon} - \dot{\epsilon}_{cr}). \tag{4}$$

Let the crack model be a rate-independent one

$$\dot{\sigma} = \mathbf{D}_{cr}\dot{\epsilon}_{cr}. \tag{5}$$

Now like in (Sluys & de Borst, 1992) the constitutive equation can easily be obtained. From (4) and (5)

$$\dot{\epsilon}_{cr} = (\mathbf{D}_{cr} + \mathbf{D}_e)^{-1}\mathbf{D}_e,$$

then by subtituting into (5) the costitutive equation is

$$\dot{\sigma} = \mathbf{D}_{cr}(\mathbf{D}_{cr} + \mathbf{D}_e)^{-1}\mathbf{D}_e\dot{\epsilon},$$

or

$$\dot{\sigma} = \mathbf{D}\dot{\epsilon}. \tag{6}$$

The basic equations of the cracked medium are (1), (2) and (6). Let a Cartesian coordinate system with basis vectors $\mathbf{g}_j$, $(j = 1, 2, 3)$ be introduced,

$$\mathbf{u} = u_j\mathbf{g}_j.$$

By using the rate form of (1)

$$2\dot{\epsilon}_{ij} = v_{i,j} + v_{j,i}, \tag{7}$$

where v_i is the velocity and notation

$$v_{i,j} = \frac{\partial v_i}{\partial x_j}$$

is used for the velocity gradient.

While for small strain $\mathbf{T}$ and σ are the same tensors, the equation of motion in rate form is

$$\rho\ddot{v}_i = \dot{\sigma}_{ij,j}. \tag{8}$$

To get a convenient form for the following part, the basic equations should be given on the velocity field. Such form can be obtained from (8) with (6) and (7)

$$\rho\ddot{v}_i = D_{ijkl}v_{k,lj}, \tag{9}$$

because $D_{ijkl} = D_{ijlk}$.

DYNAMICAL SYSTEMS

In abstract form (9) reads

$$\frac{d^2 v}{dt^2} = Fv. \tag{10}$$

Here $v = (v_1, v_2, v_3)$ is a vector of the coordinates of the velocity field satisfying the boundary conditions and F is a linear differential operator defined by the left hand side of (9). Equation (10) defines an infinite dimensional dynamical system.

The stability of a state of the continuum means the Liapunov stability of a solution $v^0(t)$ of (10). That is, a state represented by $v^0(t)$ is stable, when the perturbed velocity field $v^0(t) + v'(t)$ remains sufficiently close to the unperturbed one. Such definitions are also used in solid mechanics (Hill, 1962), (Eringen, 1975), (Nguyen, 1992). The stability investigation of the solution $v^0(t)$ starts with a transformation into a local form at that solution by substituting

$$v(t) = v^0(t) + v'(t)$$

into (10),

$$\frac{d^2(v^0 + v')}{dt^2} = F(v^0 + v'). \tag{11}$$

While v^0 is a solution of (10) and F is a linear operator, the first terms of each parts in (11) are equal, thus the equation of motion (11) of the perturbation $v'(t)$ has the same form as (10). Then (11) should be transformed into a system of first order equations by introducing new variables

$$y_1 = v_1, \quad \ldots, \quad y_3 = v_3, \quad y_4 = \dot{v}_1, \quad \ldots, \quad y_6 = \dot{v}_3,$$

and vectors

$$y_\varphi, \quad (\varphi = 1, \ldots, 3), \quad y_\psi, \quad (\psi = 4, \ldots, 6).$$

The transformed equations are

$$\dot{y}_\varphi = y_\psi \tag{12}$$

$$\dot{y}_\psi = Fy_\varphi. \tag{13}$$

Now the stability properties are determined by the eigenvalues of the linear operator $\hat{F}$ defined by the right hand sides of (12) and (13),

$$\hat{F}(y_\varphi, y_\psi) = (y_\psi, Fy_\varphi).$$

By using Liapunov's indirect method (Chetayev, 1961) the state v^0 is asymptotically stable, when the real parts of all eigenvalues of $\hat{F}$ are negative. In case of zero real parts, the state is stable. For linear systems this implies stability but nonlinearities can ruin it. The characteristic equation of $\hat{F}$ reads

$$\begin{aligned} \lambda y_\varphi &= y_\psi, \\ \lambda y_\psi &= F y_\varphi. \end{aligned} \tag{14}$$

By substituting the first equations of (14) into the second the system contains only the squares of λ,

$$\lambda^2 y_\varphi = F y_\varphi.$$

Thus equations like (9) can not give strict results for stability, because the set of eigenvalues consists of pairs $\pm\sqrt{\beta}$. When $\beta > 0$ there is a positive real part, consequently the state is unstable. When $\beta < 0$ the real part of the eigenvalues is zero. Such kind of behavior in ungeneric for dynamical systems, because typically it should have eigenvalues with nonzero real parts. In this sense (9) is called structurally unstable, (Arnold, 1983).

RATE-DEPENDENT CRACK MODEL

In the followings the rate-dependent crack model is used to end up with a stucturally stable setting of the basic equations. This model is taken from the numerical studies of (Sluys, 1992) and (Sluys & de Borst, 1992), where a rate-dependent smeared crack model is used to omit the mesh dependence of the results of finite element analysis. The model can be obtained from (5) by adding a rate-dependent term in form

$$\dot{\sigma} = \mathbf{D}_{cr}\dot{\epsilon}_{cr} + \mathbf{M}\ddot{\epsilon}_{cr}, \tag{15}$$

where $\mathbf{M}$ is the matrix of the rate-sensitivity parameters. From (4)

$$\dot{\epsilon}_{cr} = \left(\dot{\epsilon} - \mathbf{D}_e^{-1}\dot{\sigma}\right).$$

Then by substituting into (15) the constitutive equation in form

$$\dot{\sigma} = \mathbf{D}_{cr}\left(\dot{\epsilon} - \mathbf{D}_e^{-1}\dot{\sigma}\right) + \mathbf{M}\left(\ddot{\epsilon} - \mathbf{D}_e^{-1}\ddot{\sigma}\right) \tag{16}$$

is obtained. Now the basic equations of the cracked medium are (1), (2) and (16). To define a dynamical system like in the previous part the basic equations should be written on the velocity field. Such form can be obtained from equations (1), (2) and (16)

$$\mathbf{M}\mathbf{D}_e^{-1}\dddot{\mathbf{v}} + \left(\mathbf{I} + \mathbf{D}_{cr}\mathbf{D}_e^{-1}\right)\ddot{\mathbf{v}} - \frac{1}{\rho}\mathbf{M}\left(\dot{\mathbf{v}}\circ\nabla + \nabla\circ\dot{\mathbf{v}}\right)\nabla - \frac{1}{\rho}\mathbf{D}_{cr}\left(\mathbf{v}\circ\nabla + \nabla\circ\mathbf{v}\right)\nabla = 0. \tag{17}$$

By introducing the new variables like in the previous part

$$\mathbf{y}_1 = \mathbf{v}, \mathbf{y}_2 = \dot{\mathbf{v}}, \mathbf{y}_3 = \ddot{\mathbf{v}}$$

(17) can be transformed into a system of differential equations

$$\begin{aligned} \dot{\mathbf{y}}_1 &= \mathbf{y}_2 \\ \dot{\mathbf{y}}_2 &= \mathbf{y}_3 \\ \dot{\mathbf{y}}_3 &= -\mathbf{D}_e\mathbf{M}^{-1}\left(\mathbf{I} + \mathbf{D}_{cr}\mathbf{D}_e^{-1}\right)\mathbf{y}_3 + \tfrac{1}{\rho}\mathbf{M}\left(\mathbf{y}_2\circ\nabla + \nabla\circ\mathbf{y}_2\right)\nabla\cdot \\ &\quad + \tfrac{1}{\rho}\mathbf{D}_{cr}\left(\mathbf{y}_1\circ\nabla + \nabla\circ\mathbf{y}_1\right)\nabla \end{aligned} \tag{18}$$

After proper rearrangements the eigenvalue equation is

$$\left(\lambda^3 + \mathbf{D}_e\mathbf{M}^{-1}\left(\mathbf{I} + \mathbf{D}_{cr}\mathbf{D}_e^{-1}\right)\lambda^2\right)\mathbf{y}_1 = \frac{1}{\rho}\mathbf{D}_e\left(\lambda\mathbf{I} + \mathbf{M}^{-1}\mathbf{D}_{cr}\right)\left(\mathbf{y}_1\circ\nabla + \nabla\circ\mathbf{y}_1\right)\nabla. \tag{19}$$

Now a λ satisfying (19) can have nonzero real part, thus the rate-dependent smeared crack model is structurally stable.

A condition for the (Liapunov) stability of a state of the cracked medium can also be formulated. A state of this medium is asymptotically stable, when all λ satisfying (19) has negative real parts. When there is a zero λ, the system is on the stability boundary. Then from (19) the condition of strain localization is

$$\frac{1}{\rho}\mathbf{D}_e\mathbf{M}^{-1}\mathbf{D}_{cr}\left(\mathbf{v}\circ\nabla + \nabla\circ\mathbf{v}\right)\nabla = 0$$

for all velocities $\mathbf{v}$ satisfying the boundary conditions.

ONE DIMENSIONAL CASE

In this part the stability of a rod of length L is studied. Then instead of the vector $\mathbf{y}_1$ a scalar v is used. Firstly, the case of the rate-independent crack model is considered. Then instead of the matrices in (4) and (5) scalar material parameters are used, the Young modulus E and $h = D_{cr11}$. Introducing notation $c_e^2 = \frac{E}{\rho}$ the one dimensional form of the rate-independent constitutive equation is

$$\dot{\sigma} = c_e^2\frac{h}{E+h},$$

and of (8) is

$$\ddot{v} = c_e^2\frac{h}{E+h}\frac{\partial^2 v}{\partial x^2}.$$

The eigenvalue equation reads

$$\lambda^2 v = c_e^2\frac{h}{E+h}\frac{\partial^2 v}{\partial x^2}. \tag{20}$$

In the case of homogeneous boundary conditions

$$v = e^{i\alpha_k x}, \quad \text{where} \quad \alpha_k = \frac{k\pi}{L} \quad (k = 1, \ldots)$$

should be substituted into (20) and for the eigenvalues

$$\lambda_k^2 = -\alpha_k^2 c_e^2\frac{h}{E+h} \tag{21}$$

is obtained. Fig.1 shows the eigenvalues from (21) on the complex plane.

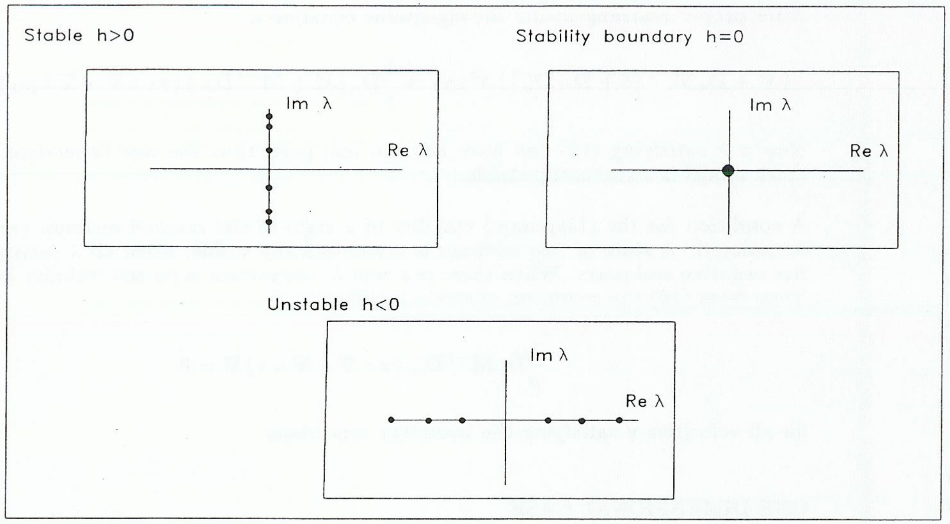


Fig.1 The eigenvalues in the case of rate-independent crack model

In the case of stability all λ_k of (21) are imaginary numbers, thus $h > 0$. The loss of stability happens at $h = 0$. Then all the eigenvalues are zero, which means an additional degeneracy, because in a typical loss of stability only one (real) eigenvalue or one pair of conjugate complex eigenvalues cross the imaginary axis.

For a rate-dependent crack model the results of the fourth part can be applied. Then equation (19) reads

$$\lambda^2 \left(\frac{E+h}{m} + \lambda\right) v = c_e^2 \left(\frac{h}{m} + \lambda\right) \frac{\partial^2 v}{\partial x^2}, \tag{22}$$

where $m = M_{11}$. In the case of homogeneous boundary conditions from (22),

$$\lambda^3 + \frac{E+h}{m}\lambda^2 + \alpha_k^2 c_e^2 \lambda + \frac{h}{m}\alpha_k^2 c_e^2 = 0. \tag{23}$$

From the Routh-Hurwitz criterion (Meirovitch, 1986) concludes that when

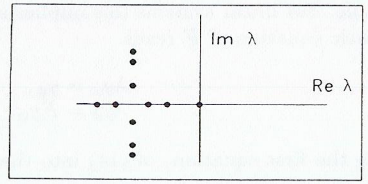
$$\frac{E+h}{m} > 0, \quad \alpha_k^2 c_e^2 > 0, \quad \frac{h}{m} > 0$$

all the real parts of the solutions λ of (23) are negative.

The loss of stability happens at $h = 0$. Then there is a $\lambda = 0$ solution of (23), thus this is a static bifurcation or localization. The other eigenvalues are

$$\lambda_{2,3,k} = -\frac{E}{2m} \pm \sqrt{\frac{E^2}{4m^2} - \alpha_k^2 c_e^2} \qquad (k = 1, \ldots). \tag{23}$$

The real parts of all eigenvalues $\lambda_{2,3,k}$ are negative. The eigenvalues are shown in Fig.2.



Stability boundary h=0

Fig.2 The eigenvalues in the case of rate-dependent crack model

There is a change in the type of λ in (23) when the expression under the root changes sign. Introducing notation

$$\alpha_* = \frac{E}{2mc_e}$$

the types of the roots can be given. When $\alpha \leq \alpha_*$ the eigenvalues are real, when $\alpha > \alpha_*$ they are complex numbers. While the solution of (22) is a combination of functions

$$e^{\lambda t} e^{i\alpha x},$$

when α is small, there is an oscillatory behavior and when α is large, there is no oscillation. From α_* an internal length can be defined

$$l_* = \frac{2mc_e\pi}{E},$$

which depends only on the material parameters. The results show that there is no oscillation with length $l > l_*$.

CONCLUSION

When the cracked medium is described with a rate-independent model, the resulting dynamical system is unproper in dynamical sense. A kind of degeneracy can also be found at the mesh dependence of the numerical investigations described in literature, (Sluys, 1992), (Sluys & de Borst, 1992). This behavior disappears by adding rate-dependence both in the numerical studies of the literature and in the present analysis based on dynamical systems theory.

In the uniaxial case also an additional degeneracy is found at the rate-independent crack model, because all eigenvalues coincide at the loss of stability. By using rate-dependent model the equations are nondegenerate. Moreover, similarly to (Sluys, 1992), the dynamical systems theory also results a kind of internal length having almost the same value.

ACKNOWLEDGEMENT

This work was supported by the Hungarian Development and Reseach Fund (OTKA F0175331).

REFERENCES

Arnold, V.I. (1983). *Geometrical Methods in the Theory of Ordinary Differential Equations*. Springer, New York.

Béda, P.B. (1994). Localization as instability of dynamical systems. In:*Material Instabilities* (R.C. Batra, H. Zbib, eds.) ASME Press, New York, pp. 279-284.

Béda, P.B. (1996). Material instability in dynamical systems. *European Journal of Mechanics*, (under publ.)

Chetayev, N.G. (1961). *The Stability of Motion*, Pergamon Press, New York.

Hill, R. (1962). Accelerations waves in solids. *Journal of the Mech. and Phys. of Solids*, **10**. 1-16.

Eringen, C. (1975). *Continuum Physics, Vol. II*. Academic Press, New York.

Meirovitch, L. (1986). *Elements of Vibration Analysis*, McGraw-Hill, New York

Needleman, A. (1988). Material rate-dependence and mesh sensitivity on localization problems. *Comp. Meth. Appl. Mech. Engng* **67**, 69-86.

Nguyen, Q.S. (1992). *Bifurcation and Stability of Dissipative Systems*, Springer, New York.

Rouche, N., P. Habets and M. Laloy (1977). *Stability Theory by Liapunov's Direct Method*, Springer, New York.

Sluys, L.J. (1992). Wave propagation, localization and dispersion in softening solids. Dissertation, Delft University of Technology, Delft.

Sluys, L.J. and de Borst, R. (1992). Wave propagation and localization in a rate-dependent cracked medium - model formulation and one-dimensional examples. *Int. J. Solids Structures*, **29**, 2945-2958.

Wiggins, S. (1990). *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, Springer, New York.