DYNAMIC CRACK PROPAGATION ANALYSIS BY A BOUNDARY INTEGRAL EQUATION METHOD

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ABSTRACT

Presented is a time-domain boundary element method for the simulation of dynamic crack propagation in finite and infinite elastic domains. A fracture criterion is applied to determine the direction and speed of crack advance without any a priori assumptions regarding the crack path. General loading conditions are possible because crack closure is taken into account by solving the related contact problem. The governing system of boundary integral equations in time-domain is solved numerically by a collocation method in conjunction with a time-stepping scheme. To check the accuracy of the solution procedure a straight crack propagating at constant speed in a finite body is investigated first and results are compared to those obtained by other numerical techniques. Examples of curvilinear crack growth at variable speed showing the influence of stress waves and crack closure on the computed crack paths illustrate the versatility of the method.

KEYWORDS

Dynamic crack propagation, time-domain boundary element method, impact loading, crack curving, stress waves

INTRODUCTION

Fast running cracks observed in experiments or in real structures generally show a curved trajectory resulting from loading conditions, specimen geometry and stress waves travelling through the cracked body (Knauss & Ravi-Chandar, 1985; Ramulu & Kobayashi, 1985). A realistic model for the simulation of dynamic crack propagation therefore requires the temporal and spatial evolution of a crack not to be prescribed but to be controlled by a physically meaningful fracture criterion. For the mathematical formulation and solution of the problem that means that the crack trajectory itself must be determined from the analysis. Discretization with respect to space and time is complicated by the fact that at any time the future crack path is not known. Within the framework of linear elastodynamics the problem description may be reduced solely to the body's boundary and to the crack by means of boundary integral equations (BIEs). Their discretization by boundary elements which allow for an easy representation of curved and moving boundaries (e.g. crack paths) is an appropriate numerical tool. Nevertheless, most approaches up to now are strongly restricted – for example to prescribed crack trajectories – either from the kind of BIE applied or from the chosen discretization. The aim of the present paper is to overcome these deficiencies by a new approach applicable to rather general dynamic crack propagation problems. It makes use of a non-classical derivation of BIEs proposed by Zhang (1991) and a simple modelling of crack growth similar to that of Koller *et al.* (1992) who treated the 'free' propagation of a mode-III crack in an unbounded domain. The present method has been successfully applied to two-dimensional dynamic crack propagation in an unbounded domain under a variety of mixed-mode loading conditions (Gross & Seelig, 1995; Seelig & Gross, 1996). An extension to the more realistic situation of finite bodies where interesting effects like the influence of stress waves reflected from boundaries can be studied, will be given here.

BOUNDARY VALUE PROBLEM AND BOUNDARY INTEGRAL EQUATIONS

The problem under consideration is an isotropic linear elastic body B containing a crack $\Gamma(t)$ which may grow with time (Fig. 1). The exterior boundary ∂B_{ex} of the body is subject to some time dependent loading and the crack is a surface of discontinuity with respect to the displacement $u_i(\mathbf{x},t)$. Initially the material is stress free and at rest $u_i(\mathbf{x},t=0)=0$, $\dot{u}_i(\mathbf{x},t=0)=0$.

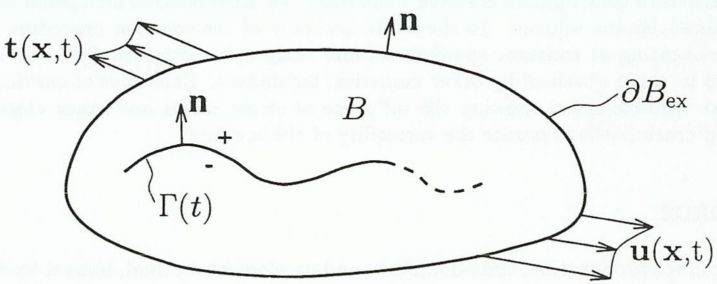


Fig. 1 : Cracked body, $\partial B = \partial B_{\mathrm{ex}} \cup \Gamma(t)$

Formulating the initial boundary value problem by boundary integral equations (BIEs) we have to deal with the following boundary data: the displacements $u_i(\mathbf{x},t)$ und tractions $t_i(\mathbf{x},t)=\sigma_{ij}(\mathbf{x},t)\,n_j(\mathbf{x})$ on the exterior boundary ($\mathbf{x}\in\partial B_{\mathrm{ex}}$) and the jumps in displacement $\Delta u_i(\mathbf{x},t) := u_i^+(\mathbf{x},t) - u_i^-(\mathbf{x},t)$ (see Fig. 1) on the crack ($\mathbf{x}\in\Gamma(t)$). Here, σ_{ij} denotes the stress tensor and n_i is the unit normal vector as indicated in Fig. 1. The crack faces are assumed to be traction-free or subject to some contact stress in case of crack closure. Thus, the traction vector $t_i^\Gamma(\mathbf{x},t)$ on the crack is always continuous. The initial boundary value problem can be expressed by the following system of coupled time-domain boundary integral equations:

$$\begin{aligned} c_{kl}(\mathbf{x})\,u_k(\mathbf{x},t) = \int_0^t \Big[\oint_{\partial B_{\mathrm{ex}}} \{ & u_{il}^G(\mathbf{x},\mathbf{y},t-\tau)\,t_i(\mathbf{y},\tau) \; - \\ & - \sigma_{ijl}^G(\mathbf{x},\mathbf{y},t-\tau)\,n_j(\mathbf{y})\,u_i(\mathbf{y},\tau) \}\, dA(\mathbf{y}) \; + \\ + \int_{\Gamma(\tau)} & \sigma_{ijl}^G(\mathbf{x},\mathbf{y},t-\tau)\,n_j(\mathbf{y})\,\Delta u_i(\mathbf{y},\tau)\,dA(\mathbf{y}) \;\Big]\, d\tau \;, \qquad \mathbf{x}\in\partial B_{\mathrm{ex}} \end{aligned} \tag{1}$$

$$\begin{aligned} t_p^\Gamma(\mathbf{x},t) = C_{pqlk}\,n_q(\mathbf{x}) \int_0^t \Big[\int_{\partial B_{\mathrm{ex}}} \{ & \sigma_{ijl}^G(\mathbf{x},\mathbf{y},t-\tau)\,(n_k(\mathbf{y})\,\partial_j - n_j(\mathbf{y})\,\partial_k)\,u_i(\mathbf{y},\tau) \; + \\ & + \rho\,u_{il}^G(\mathbf{x},\mathbf{y},t-\tau)\,\ddot{u}_i(\mathbf{y},\tau)\,n_k(\mathbf{y}) \; - \\ & - \frac{\partial u_{il}^G}{\partial y_k}(\mathbf{x},\mathbf{y},t-\tau)\,t_i(\mathbf{y},\tau) \}\, dA(\mathbf{y}) \; - \\ - \oint_{\Gamma(\tau)} \{ & \sigma_{ijl}^G(\mathbf{x},\mathbf{y},t-\tau)\,(n_k(\mathbf{y})\,\partial_j - n_j(\mathbf{y})\,\partial_k)\,\Delta u_i(\mathbf{y},\tau) \; + \\ & + \rho\,u_{il}^G(\mathbf{x},\mathbf{y},t-\tau)\,\Delta\ddot{u}_i(\mathbf{y},\tau)\,n_k(\mathbf{y}) \}\, dA(\mathbf{y}) \;\Big]\, d\tau \;, \qquad \mathbf{x}\in\Gamma(t) \end{aligned} \tag{2}$$

A superscript G denotes the fundamental solutions (elastodynamic Green's functions) for displacement and stress, C_{pqlk} is the elasticity tensor, ρ the mass density and superscript dots indicate derivatives with respect to time. In points of a smooth boundary $c_{kl}(\mathbf{x})\,u_k(\mathbf{x},t)$ reduces to $u_l(\mathbf{x},t)$. The derivation of BIEs from integral identities ('conservation integrals') is well known from literature ('direct method'). BIE (1) – called the 'displacement equation' because of its left hand side – follows from the classical Betti-Rayleigh reciprocal theorem (Eringen & Suhubi, 1975). The 'traction equation' (2) is obtained using an alternative integral identity of linear elastodynamics proposed and successfully applied to stationary cracks in unbounded domains by Zhang (1991). In contrast to traction BIEs derived from the Betti-Rayleigh theorem eqn (2) is non-hypersingular which simplifies its numerical treatment. All integrals exist at least in the Cauchy principal value sense.
In case of frictionless crack face contact the traction vector on the crack is $t_p^\Gamma(\mathbf{x},t) = p(\mathbf{x},t)\,n_p(\mathbf{x})$ where the contact pressure p has to be determined from the constraint of vanishing material penetration. A penalty method is applied to solve this contact problem approximately. Therefore the contact pressure is assumed to be proportional to the 'forbidden' material penetration

$$p(\mathbf{x},t) = \frac{c_p}{2}\,(\Delta u_{\mathrm{n}}(\mathbf{x},t) - |\Delta u_{\mathrm{n}}(\mathbf{x},t)|) \quad , \qquad \Delta u_{\mathrm{n}}(\mathbf{x},t) := \Delta u_i(\mathbf{x},t)\,n_i(\mathbf{x}) \tag{3}$$

with a penalty stiffness c_p much bigger ($\times 10^3$) than the elastic stiffness of the body. Obviously p vanishes when the crack is open, otherwise eqn (2) becomes nonlinear. Taking into account some frictional stress depending on the contact pressure would produce no difficulty.

NUMERICAL SOLUTION PROCEDURE

In the following the analysis is restricted to two-dimensional problems reducing ∂B_{ex} and $\Gamma(t)$ to a closed and an open curve, respectively. Only one-sided propagation of the crack is investigated (Fig. 1). Both boundary curves are approximated by polygons consisting of elements of constant length Δy_{ex} on ∂B_{ex} and Δy_{Γ} on $\Gamma(t)$. Equidistant time steps $t_m = m\Delta t$ and $\tau_n = n\Delta t$ $(n = 0, ..., m)$ are chosen to discretize the current time t and the past τ appearing in the temporal convolution. The number of elements E_{ex} on the exterior boundary is fixed whereas the number of crack elements $E_{\Gamma}(m)$ increases with time when the crack grows. Due to the different element lengths on ∂B_{ex} and $\Gamma(t_m)$ a finer mesh may be used on the crack. But for a good temporal resolution the smallest element size should be chosen such that $\min \Delta y \geq c_L \Delta t$ where c_L is the longitudinal wave speed. On the other hand, too small values of Δt may cause instabilities in the solution at large times. A piecewise constant spatial approximation is chosen for all boundary data except at the crack tips where square root shape functions serve to model the proper asymptotic behaviour of the displacement jumps: $\Delta u_i(r,t) \sim \sqrt{r}$ (r : distance from crack tip). The temporal variation within each time step is taken linear for displacements and displacement jumps and piecewise constant for the tractions.
Inserting these spline approximations into BIEs (1) and (2) and applying a collocation method leads to a system of algebraic equations for the unknown coefficients in the boundary data approximation. In each time step the unknown coefficients are determined by those computed in previous time steps and the given boundary data. The system of algebraic equations to be solved in each time step is nonlinear in case of crack face contact and then can be treated by a Newton iteration.

FRACTURE CRITERION AND MODELLING OF CRACK ADVANCE

The dynamic stress intensity factors at a running crack tip are given by the computed displacement jumps Δu_n, Δu_t normal and tangential to the crack and universal functions of the crack tip speed $\dot{a}$ (see e.g. Freund, 1990) :

$$K_{I,II}(t;\dot{a}) = k_{I,II}(\dot{a}) \lim_{r\to 0} \frac{\Delta u_{n,t}(r,t)}{\sqrt{r}} \quad . \tag{4}$$

Due to the $\sqrt{r}$ - crack tip shape functions they are determined directly by the coefficients of the displacement jump approximation on the crack tip elements.
Via the stress field at the moving crack tip the SIFs enter the fracture criterion of Erdogan & Sih (1963) applied here. It is widely accepted in application to brittle fracture and states that crack advance will take place in the direction φ_0 of maximum circumferential stress σ_φ when this stress reaches the same critical value as in pure mode-I

$$\max \sigma_\varphi(\varphi;\dot{a},K_I,K_{II}) - \sigma_\varphi(\varphi=0;\dot{a},K_{Ic}) \begin{cases} =0 & \text{for} \quad \dot{a}>0 \\ <0 & \text{for} \quad \dot{a}=0 \end{cases} \quad . \tag{5}$$

The angle φ is measured from the tangent at the crack tip. For a propagating crack the value of $\dot{a}$ is such that the equality in (5) holds. The critical stress is represented by the dynamic fracture toughness $K_{Ic} = K_{I_D}(\dot{a})$ which as a function of crack tip speed has to be determined experimentally.
Crack growth is modelled by adding a new element of constant length to the moving crack

tip whenever condition (5) is violated. By virtue of the restrictions on crack tip speed and space-time discretization ($\dot{a} < c_L < \Delta y/\Delta t$) this can take place only after several time steps have passed, say at discrete times $t_{m_{k-1}}, t_{m_k}, \ldots$. Therefore the crack tip speed can be determined only as the average value over the whole interval between two such instants of discrete crack advance:

$$\dot{a}(t) \approx \frac{\Delta y}{t_{m_k} - t_{m_{k-1}}} = \frac{\Delta y}{\Delta t\,(m_k - m_{k-1})} \qquad \text{for} \quad t \in [t_{m_{k-1}}, t_{m_k}] \quad . \tag{6}$$

For the same reason the dynamic SIFs (4) are averaged from the last instant of discrete crack advance on (here t_{m_k}) before entering the fracture criterion. After the onset of crack growth $(\dot{a} > 0)$

$$K(t_m) = \frac{1}{(t_m - t_{m_k})} \int_{t_{m_k}}^{t_m} K(t)\,dt \tag{7}$$

can be regarded to be the SIF at time t_m consistent with the discretization. From (6) the crack tip speed can be determined only for a time interval between two known instants of discrete crack growth. The evaluation of the fracture criterion at any current time therefore requires an update of the crack tip speed and must be performed iteratively.
The discrete modelling of crack growth chosen here is similar to that proposed by Koller *et al.* (1992). It is a rather rough approximation, but in contrast to methods using 'moving singular elements' (see e.g. Gallego & Dominguez, 1992) it requires no remeshing.

RESULTS

All physical quantities are normalized appropriately and illustrations showing the problem under consideration are true to scale.

Crack Growth at Constant Speed

The accuracy of the presented method in application to crack propagation in finite bodies is checked by solving a problem which has been treated before using different numerical techniques. A rectangular plate containing initially a central crack of length $2a_o$

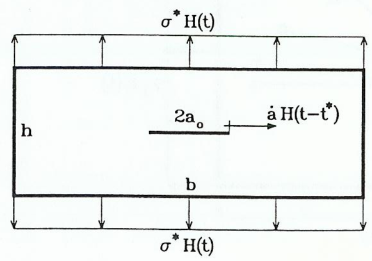


Fig. 2 : Center-cracked plate

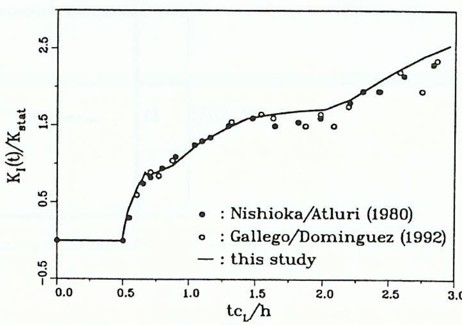


Fig. 3 : Dynamic mode-I SIF

($b : h : 2a_o = 2.6 : 1 : 0.6$) is uniformly loaded by a traction of Heaviside-function time dependence (Fig. 2). At time t^* the crack starts growing at constant speed $\dot{a} = 0.29\,c_T$ in its original direction. There is no fracture criterion involved here. Figure 3 shows the normalized mode-I SIF versus dimensionless time in comparison with numerical results by Nishioka & Atluri (1980) and Gallego & Dominguez (1992) who applied a finite element method and a boundary element substructure technique, respectively. The results are in good agreement.

Crack Growth Controlled by a Fracture Criterion

For the dynamic fracture toughness the relation shown in Fig. 4 which is typical for steel and other moderately brittle materials is used in the following examples. The steep increase of $K_{I_D}(\dot{a})$ at about 40% of the shear wave velocity c_T results from micromechanisms not considered here in detail and acts like a barrier to the crack tip speed.

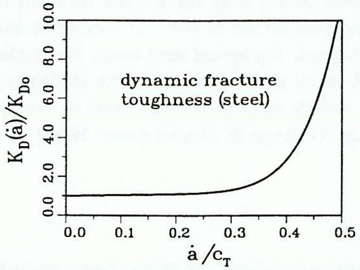


Fig. 4 : Dynamic fracture toughness

Stress Wave Induced Crack Curving. One reason for crack curving observed in brittle solids is the presence of stress waves travelling through a cracked body, being reflected at the boundaries and altering the stress state at the moving crack tip at different times (Knauss & Ravi-Chandar, 1985; Ramulu & Kobayashi, 1985).

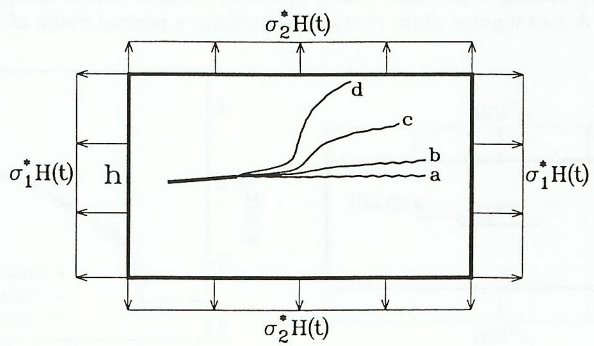
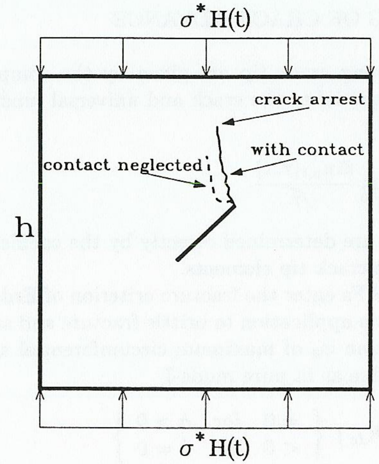


Fig. 5 : Crack curving due to reflected stress waves;
$a: \sigma_1^* = 0,\quad b: \sigma_1^* = \sigma_2^*,\quad c: \sigma_1^* = 1.5\sigma_2^*,\quad d: \sigma_1^* = 2\sigma_2^*$

Figure 5 shows computed crack paths in a rectangular plate of height h with a slightly inclined initial crack (5°, thick line) for different ratios of biaxial loading. Because of the fracture criterion the crack always tends to propagate under mode-I conditions. So, for uniaxial loading of amplitude σ_2^* (curve a) it remains parallel to the horizontal boundaries. Of course, additional tractions of amplitude σ_1^* applied on the vertical boundaries lead to slightly different angles of crack initiation. But two strong changes in crack propagation direction (clearly to be seen at curves c and d) take place at later times $t \approx 1.5\,h/c_L$ and $t \approx 2.5\,h/c_L$. At these times waves of amplitude σ_2^* generated at the upper and lower boundary at $t = 0$ and reflected ones and twice, respectively, on the opposite sides reach the middle region of the plate, i.e. the crack. The sudden changes of the state of stress and hence the preferred direction of crack advance at the two indicated times can easily be explained from a simplified one-dimensional consideration of σ_2^*-wave propagation in an uncracked plate. Each of the two waves coming from the upper and lower boundary carries a jump in stress of σ_2^*. Thus, the stress in 2-direction in the middle of the plate drops from $2\,\sigma_2^*$ to zero at their arrival after the first reflection and jumps back to $2\,\sigma_2^*$ after the second reflection.

Influence of Crack Face Contact. The pressure loaded square plate with an inclined initial crack (45°) shown in Fig. 6 serves to illustrate the influence of crack closure on the computed crack path. When correctly modelled – i.e. by solving the related contact problem – crack initiation takes place under pure mode-II conditions with an kinking angle of approximately 70°. The wing pattern of the resulting crack path is well known. Neglection of crack face contact leads to crack initiation with a physically meaningless negative mode-I SIF and results in a totally different crack path (dashed line).



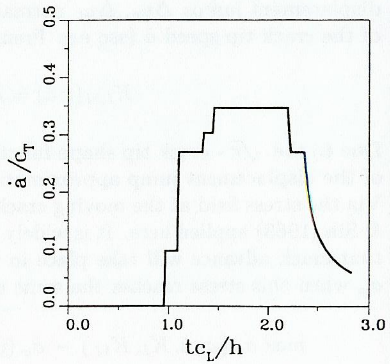


Fig. 6 : Influence of crack face contact

Fig. 7 : Crack tip speed

The temporal evolution of the fracture process up to crack arrest can be studied approximately from Fig. 7 where the computed crack tip speed versus time is shown.

CONCLUSIONS

A numerical method has been presented which enables the simulation of rather general dynamic crack propagation problems in finite and infinite elastic domains. Although very simple interpolation functions and only a rough approximation of crack growth have been chosen the solution procedure proved to be of sufficient accuracy when compared to other techniques. By consequently exploiting the advantages of the boundary element method it is a reasonable tool to investigate complex phenomena of dynamic fracture on a macroscopic level.

ACKNOWLEDGEMENTS

This work was supported by the Deutsche Forschungsgemeinschaft (DFG) through Grant No. Gr 596/20-1.

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