VIBRATION BASED DETECTION OF LOCATION AND SIZE OF EDGE CRACK IN BEAMS ON MULTIPLE SUPPORTS

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ABSTRACT

In this paper a method has been presented to address an inverse eigenvalue problem to facilitate detection of location and size of an edge crack in an orientation normal to length in a beam on multiple supports from the measurement of natural frequencies. The crack is modelled by a rotational spring and the flexural vibration problem is formulated. This gives rise to a relationship involving the spring stiffness, location of crack and natural frequency. The applicability of the method is demonstrated by numerical experiments with a three span beam with end supports and a crack located either in the first or second span. The method is accurate up to 5% for a crack size more than 10% section depth and located away from the supports. The method does not require much iteration.

KEYWORDS

Nondestructive detection of crack size, vibration based detection of crack size, vibration of cracked beams, beams on multiple supports.

INTRODUCTION

A forward eigenvalue problem invloving determination of frequencies for specified crack location and size is straightforward. The corresponding inverse problem of determination of crack size and location from the knowledge of frequencies is rather difficult because of the lack of uniqueness and possibility of iteration. If these difficulties can be overcome the method can be cast into a nondestructive testing technique and it offers tremendous scope for its exploitation in practice. Such a method of detection offers some advantages. It can help to determine both location and size of a crack from the measurements made at a single, or at a few points, on the components.

The development of a crack in a component changes its vibration parameters, e.g. the structural parameters (i.e. mass, stiffness and flexibility) and the modal parameters (i.e. natural frequencies, modal damping values and mode shapes). The vibration based methods of crack detection utilize one or more of these parameters as the basis for crack detection. The methods based on structural parameters utilise mainly changes in stiffness matrix or flexibility matrix [Park *et al.* (1988), Mannan and Richardson (1988) and Pandey and Biswas (1994)]. There are some variants of the method based on modal parameters. The technique using changes in natural frequencies as the crack detection criterion have received a considerable attention. This is perhaps because the natural frequencies can be measured easily and monitoring is possible from any location on the component.

Adams *et al.* (1978) have demonstrated that the changes in natural frequencies under longitudinal vibration can be an important means to detect the crack location. They have represented the damage/crack by a linear spring and employed the receptance technique for analysis. For two dimensional, plate like, components, Cawley and Adams (1979) have employed finite element based approach and introduced the sensitivity analysis to facilitate the detection. Rizos *et al.* (1990) have proposed a method based on flexural vibration and represented the crack section by a rotational spring. The usefulness of the method for detection of both location and size is demonstrated for cantilever beams. The technique needs measurement only of amplitudes at any two locations of the beam.

Liang *et al.* (1991) have given a scheme, which has a lot of similarity with that of Rizos *et al.*'s, but it requires the measurement of three fundamental frequencies of the beam. Later Liang *et al.* [(1992a), (1992b)] have given a scheme where the inverse problem is addressed by deriving a function involving location and stiffness. This is done through the analysis of the forward problem and employing the perturbation method. This procedure offers promise even for multiple discrete cracks. For simple geometries, e.g., simply supported and cantilever beams, this function can be easily determined. For complicated cases, simulation package is used to derive the function [Liang *et al.* (1992b)]. All the methods presented in the literature have been mostly applied to single span beams with simple end conditions to illustrate their effectiveness. Attempts to examine beams on multiple supports, which can correspond to railway tracks, anchored pipe lines, etc., are lacking. In this paper a method, which may have relevance to such applications, has been developed, drawing on the analytical procedures of Liang *et al.* (1991) and Rizos *et al.* (1990). The effectiveness of the method is demonstrated with examples.

FORMULATION

The formulation presented in what follows is valid for any number of spans. For convenience, a continuous beam with 3 spans and an edge crack located in one of the spans is considered (Fig. 1). To model the transverse vibration the crack is represented by a rotational spring of stiffness K_t. The beam can be conveniently divided into 4 segments (AB, BC, CD and DE). The governing equation of transverse vibration of each span is of the form:

$$\frac{d^2}{dx^2}\left[EI\,\frac{d^2U}{dx^2}\right] + \omega^2\rho A U = 0 \qquad (1)$$

where U is dispalcement, ω is natural frequency of the vibration of the beam, E is the Young's modulus of elasticity, I is the second moment of full area of cross-section, ρ is the mass density, A is the cross-sectional area and x-axis is aligned with the axis of the beam. Introducing non-dimensional parameters $\lambda_i^4 = (\omega^2\rho A L_s^4)/EI$ and $\xi = (x - x_{ls})/L_s$ where x_{ls}= coordinate of the left support and L_s = length of the span, the equations for the four segments

are as follows:

$$\frac{d^4U_i}{d\xi^4} + \lambda_i^4\,U_i = 0 \qquad i = 1, 2, 3 \text{ and } 4 \qquad (2)$$

where $\lambda_1^4 = \lambda_2^4 = \omega^2\rho A L_1^4/EI$, $\lambda_3^4 = \omega^2\rho A L_2^4/EI$ and $\lambda_4^4 = \omega^2\rho A L_3^4/EI$. $\xi = x/L_1$ for both the segments AB and BC (i.e. origin $\xi = 0$ is at the left support of the span), and U_i stands for the dispalcement of the segment i.

The solutions to Eqn. (2) is given by

$$U_i = A_{i1}\cos\lambda_i\xi + A_{i2}\cosh\lambda_i\xi + A_{i3}\sin\lambda_i\xi + A_{i4}\sinh\lambda_i\xi \qquad (3)$$

where A_{ij}, i, j = 1, 2, 3 and 4, are arbitrary constants to be determined from the boundary conditions, continuity and compatibility conditions. The boundary conditions at the two end supports are

$$U_1|_{\xi=0} = 0\,,\quad U_1''|_{\xi=0} = 0\,,\quad U_4|_{\xi=1} = 0\,,\quad U_4''|_{\xi=1} = 0 \qquad (4)$$

At intermediate supports 2 and 3 the continuity conditions are

$$\begin{aligned} &U_2|_{\xi=1} = 0\,,\quad U_3|_{\xi=0} = 0\,,\quad U_2'|_{\xi=1} = U_3'|_{\xi=0}\,,\quad U_2''|_{\xi=1} = U_3''|_{\xi=0} \\ &U_3|_{\xi=1} = 0\,,\quad U_4|_{\xi=0} = 0\,,\quad U_3'|_{\xi=1} = U_4'|_{\xi=0}\,,\quad U_3''|_{\xi=1} = U_4''|_{\xi=0} \end{aligned} \qquad (5)$$

The continuity of displacement, moment and shear forces at the crack location (say, $\xi = \beta$) and jump condition in the slope can be written in the following form :

$$U_1 = U_2\,,\quad U_1'' = U_2''\,,\quad U_1''' = U_2'''\,,\quad U_2' = U_1' + \frac{\lambda_1}{K}U_1'' \qquad (6)$$

where $K = \dfrac{K_t L_1}{EI}$ is the non-dimensional stiffness of the rotational spring representing the crack.

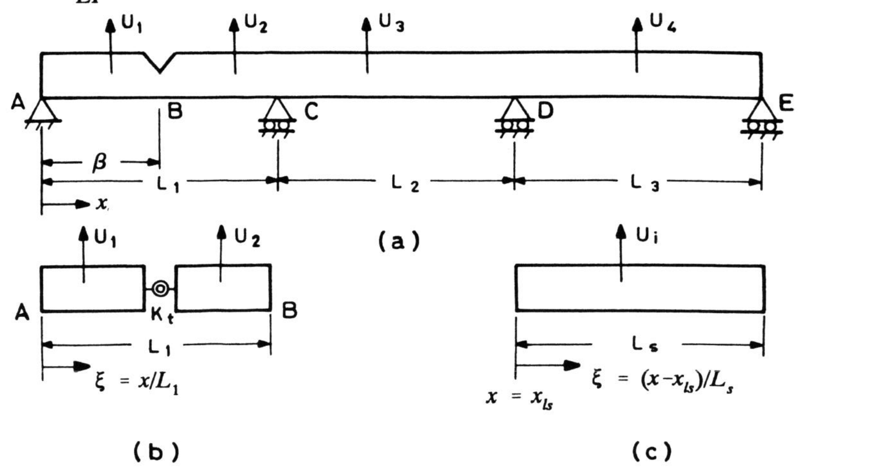

Fig. 1. (a) Three span continuous beam. (b) Representation of crack by rotational spring. (c) Local coordinate for a span.

From the above 16 conditions (given by Eqns. 4 to 6) the following system of equations is obtained.

$$[\mathbf{D}]\,\{\mathbf{A}\} = \{\mathbf{0}\} \tag{7}$$

where $\{\mathbf{A}\}$ is the vector of the arbitrary constants A_{ij} and the matrix $[\mathbf{D}]$ is as follows.

$$[\mathbf{D}] = \begin{bmatrix} [\mathbf{B_L}] & [\mathbf{0}] & [\mathbf{0}] & [\mathbf{B_R}] \\ [\mathbf{0}] & [\mathbf{S_2}] & [\mathbf{F}] & [\mathbf{0}] \\ [\mathbf{0}] & [\mathbf{0}] & [\mathbf{S_3}] & [\mathbf{F}] \\ [\mathbf{C_L}] & [\mathbf{C_R}] & [\mathbf{0}] & [\mathbf{0}] \end{bmatrix} \tag{8}$$

The non-zero submatrices of $[\mathbf{D}]$ are given by

$$[\mathbf{B_L}] = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\quad [\mathbf{B_R}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \cos\lambda_4 & \cosh\lambda_4 & \sin\lambda_4 & \sinh\lambda_4 \\ -\cos\lambda_4 & \cosh\lambda_4 & -\sin\lambda_4 & \sinh\lambda_4 \end{bmatrix},\quad [\mathbf{F}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 \end{bmatrix},$$

$$[\mathbf{S}_i] = \begin{bmatrix} \cos\lambda_i & \cosh\lambda_i & \sin\lambda_i & \sinh\lambda_i \\ 0 & 0 & 0 & 0 \\ -\sin\lambda_i & \sinh\lambda_i & \cos\lambda_i & \cosh\lambda_i \\ -\cos\lambda_i & \cosh\lambda_i & -\sin\lambda_i & \sinh\lambda_i \end{bmatrix}, \qquad i = 2 \text{ and } 3,$$

$$[\mathbf{C_L}] = \begin{bmatrix} \cos\alpha & \cosh\alpha & \sin\alpha & \sinh\alpha \\ -\cos\alpha & \cosh\alpha & -\sin\alpha & \sinh\alpha \\ \sin\alpha & \sinh\alpha & -\cos\alpha & \cosh\alpha \\ (-\frac{K}{\lambda_1}\sin\alpha & (\frac{K}{\lambda_1}\sinh\alpha & (\frac{K}{\lambda_1}\cos\alpha & (\frac{K}{\lambda_1}\cosh\alpha \\ -\cos\alpha) & \cosh\alpha) & -\sin\alpha) & \sinh\alpha) \end{bmatrix},$$

$$[\mathbf{C_R}] = \begin{bmatrix} -\cos\alpha & -\cosh\alpha & -\sin\alpha & -\sinh\alpha \\ \cos\alpha & -\cosh\alpha & \sin\alpha & -\sinh\alpha \\ -\sin\alpha & -\sinh\alpha & \cos\alpha & -\cosh\alpha \\ \frac{K}{\lambda_1}\sin\alpha & -\frac{K}{\lambda_1}\sinh\alpha & -\frac{K}{\lambda_1}\cos\alpha & -\frac{K}{\lambda_1}\cosh\alpha \end{bmatrix}, \quad \text{and} \quad \alpha = \lambda_1\beta.$$

For a crack in any other span the Eqns. (7) and (8) have the same form, only the relative positions of the various submatrices in $[\mathbf{D}]$ change.

The characteristics equation obtained from Eqn. (7) can be rewritten in the form

$$\frac{K}{\lambda_1}|\Delta_1| + |\Delta_2| = 0 \qquad \text{or,} \qquad K = -\lambda_1\frac{|\Delta_2|}{|\Delta_1|} \tag{9}$$

where $|\Delta_1|$ and $|\Delta_2|$ have the same form as $|\mathbf{D}|$ except for the differences in the last row. The last rows for $|\Delta_1|$ and $|\Delta_2|$ are respectively as follows:

$$\lfloor -\sin\alpha \quad \sinh\alpha \quad \cos\alpha \quad \cosh\alpha \quad \sin\alpha \quad -\sinh\alpha \quad -\cos\alpha \quad -\cosh\alpha \quad 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \rfloor$$

$$\lfloor -\cos\alpha \quad \cosh\alpha \quad -\sin\alpha \quad \sinh\alpha \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \rfloor$$

METHODOLOGY FOR CRACK DETECTION

For detection it is necessary to measure the first three natural frequencies of the beam. Assuming the crack in a particular span, the variation of K with crack location β in the span is obtained using Eqn. (9) for each of the three fundamental modes of vibration. Since there is only one crack, the position where the three curves intersect, gives the crack location [Liang *et al.* (1991)]. The crack size is then computed from the standard relation between the stiffness K and the crack size [Ostachowicz and Krawkczuk (1991)]

$$K = \frac{bh^2 L_s}{72\pi I(a/h)^2 f(a/h)} \tag{10}$$

$$f(a/h) = 0.6384 - 1.035\,(a/h) + 3.7201\,(a/h)^2 - 5.1773\,(a/h)^3 - 7.553\,(a/h)^4 - 7.332\,(a/h)^5 + 2.4909\,(a/h)^6 \tag{11}$$

where b is thickness and h is depth of the beam. While calculating K for a particular β and vibration mode it is important that a proper value of modulus of elasticity E is employed. The modulus of elasticity E must correspond to the uncracked natural frequency in the same mode.

CASE STUDY

To demonstrate the effectiveness of the method a case study is presented. In the absence of experimental data, the first three natural frequencies are obtained by finite element method. A continuous steel beam with three equal spans with the following material properties is studied: $E = 2.1\text{x}10^{11}$ N/m^2, Poisson's ratio $\nu = 0.3$ and density $\rho = 7860$ kg/m^3. Natural frequencies for both the cracked and uncracked geometries are computed with the help of a finite element package. The beam is discretised by mostly 8-noded isoparametric elements. Around the crack tip 12 quarter point singularity elements have been used. The various cases considered and corresponding natural frequencies are given in Table 1 and Table 2. The variation of K with β is obtained from Eqn. (9). Typical plots of K vs β are shown in Fig. 2.

To eliminate any graphical and subjective error, the three intersection points (β_1, K_1), (β_2, K_2) and (β_3, K_3) corresponding to the three sets of frequencies, e.g., (ω_1, ω_2), (ω_2, ω_3) and (ω_3, ω_1), are determined. The centroid of the triangle formed by (β_1, K_1), (β_2, K_2) and (β_3, K_3) is taken

Table 1 Comparison of predicted and actual location and size for crack in first span.

Crack details location β	Crack details size (a/h)	Natural frequencies (rad/s) ω_1	ω_2	ω_3	Predicted location β	location % error	stiffness K	size (a/h)	size % error
uncracked		1466.46	1872.83	2719.13					
0.1	0.1	1466.26	1872.39	2718.72					
0.1	0.2	1465.54	1870.86	2717.34	0.1247	24.68	77.25	0.1578	-21.09
0.1	0.3	1464.23	1868.09	2714.84	0.1209	20.89	30.27	0.2514	-16.20
0.1	0.4	1461.97	1863.42	2710.57	0.1121	12.13	13.17	0.3670	-8.26
0.2	0.1	1465.58	1871.00	2717.57	0.2030	1.49	196.47	0.0974	-2.55
0.2	0.2	1462.95	1865.69	2713.10	0.2096	4.82	52.56	0.1918	-4.10
0.2	0.3	1458.09	1856.36	2705.29	0.2087	4.33	22.24	0.2906	-3.14
0.2	0.4	1449.57	1841.26	2692.74	0.2060	3.02	11.05	0.3953	-1.16
0.5	0.1	1463.83	1868.95	2717.53	0.4994	-0.11	185.10	0.1005	0.51
0.5	0.2	1456.22	1858.49	2713.28	0.5009	0.18	48.05	0.2006	0.30
0.5	0.3	1442.37	1841.56	2706.43	0.5011	0.22	20.59	0.3010	0.34
0.5	0.4	1418.81	1817.75	2696.77	0.5009	0.18	10.49	0.4040	1.00
0.6	0.1	1464.10	1870.23	2718.74	0.6013	0.22	185.27	0.1005	0.46
0.6	0.2	1457.32	1863.13	2717.62	0.6003	0.05	48.09	0.2005	0.26
0.6	0.3	1445.20	1851.53	2715.76	0.6001	0.01	20.59	0.3010	0.35
0.6	0.4	1425.27	1835.02	2713.01	0.5997	-0.05	10.51	0.4038	0.95
0.8	0.1	1465.59	1872.83	2717.83					
0.8	0.2	1463.06	1872.70	2714.07	0.8012	0.15	49.37	0.1979	-1.05
0.8	0.3	1458.70	1872.50	2707.60	0.8003	0.04	20.99	0.2984	-0.52
0.8	0.4	1451.86	1872.14	2697.42	0.7994	-0.08	10.60	0.4022	0.56

as the common intersection point (β, K). The crack size is then computed using Eqn. (10). The computed crack locations and sizes are presented in Tables 1 and 2.

DISCUSSIONS AND CONCLUSIONS

The effectiveness of the method is predominantly dependent on the changes in frequency in response to a change in crack length. Greater this change higher will be the effectiveness. For components like slender beams, the method will have greater suitability. Though the problem of edge crack has been addressed here, if a similar modelling is possible for an internal crack, the method can be routinely extended to an internal crack.

The method is able to predict both the location and size reasonably accurately for the cases considered. The error increases when the crack is close to a support. For such a location (e.g.

Table 2 Comparison of predicted and actual location and size for crack in second span.

Crack details location β	Crack details size (a/h)	Natural frequencies (rad/s) ω_1	ω_2	ω_3	Predicted location β	location % error	stiffness K	size (a/h)	size % error
uncracked		1466.46	1872.82	2719.13					
0.1	0.1	1466.26	1870.73	2717.18	0.0984	-1.59	200.81	0.0963	-3.65
0.1	0.2	1465.55	1864.82	2711.35	0.0999	-0.06	49.59	0.1975	-1.27
0.1	0.3	1464.30	1854.78	2701.68	0.1002	0.18	21.04	0.2981	-0.64
0.1	0.4	1462.24	1839.43	2687.44	0.1002	0.18	10.67	0.4013	0.31
0.3	0.1	1464.76	1872.18	2717.26	0.2999	-0.04	188.44	0.0996	-0.42
0.3	0.2	1459.93	1870.38	2711.98	0.3002	0.07	48.52	0.1996	-0.19
0.3	0.3	1451.49	1867.35	2703.01	0.3004	0.12	20.77	0.2999	-0.05
0.3	0.4	1438.01	1862.78	2689.31	0.3005	0.18	10.58	0.4027	0.66
0.4	0.1	1464.11	1872.66	2714.14	0.4017	0.43	187.50	0.0998	-0.16
0.4	0.2	1457.43	1872.18	2700.54	0.3999	-0.01	48.34	0.2000	-0.00
0.4	0.3	1445.76	1871.37	2677.80	0.3998	-0.04	20.68	0.3004	0.15
0.4	0.4	1427.20	1870.14	2644.13	0.3998	-0.05	10.54	0.4033	0.83

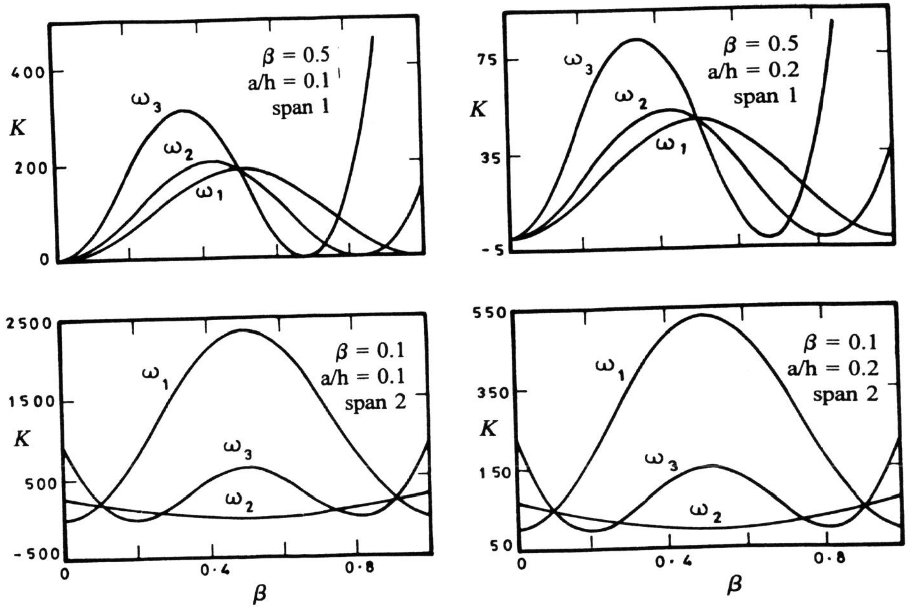

Fig. 2. Typical plots of K vs β for crack in first or second spans.

$\beta = 0.1$ and $a/h = 0.1$, or $\beta = 0.8$ and $a/h = 0.1$) the change in frequency is too small to provide any basis for a crack detection. The error again depends on whether the crack is close to any end or intermediate supports (Table 1). Barring these, for the first span, the error in location is less than 1%, except for the case of $\beta = 0.2$ where it is about 5%. The corresponding figure for crack size determination is less than 4.5%; in many cases it is less than 1%. For cracks located in the second span, even a 10% crack can be predicted when it is close to any end support (e.g., β=0.1). The error in location or size is less than in the previous case.

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