ON THE MODELLING OF INTERFACE CRACK YIELDING ZONES

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ABSTRACT

A crack of total length $2c$ lying on the interface between two half-planes of isotropic material with differing material moduli is considered. Applying Fourier integral transforms to Lame equations of plane elasticity problem and satisfying boundary conditions, a system of two singular integral equations with Cauchy kernels is obtained. The unknown functions are the shear stress over the fixed part of boundary and the derivative of normal displacement with respect to x over crack faces. Thin yield zones ahead of the crack continuation are assumed. The nonlinear system of singular integral equations is solved for this purpose and yield zone length is found for various values of ν and external load.

KEYWORDS

Interface crack, yielding zone, nonlinear singular integral equation

INTRODUCTION

Much attention has recently been focused on the interface crack problem which is of great importance in nonhomogeneous and composite material investigation. Particularly interface crack model (see bibliography in the paper of Rice (1988) and contact model by Comninou (1977, 1978) have been developed for elastic materials. Plastic zone at the tip of interface crack has been studied in the papers of Shih and Asaro (1988a, 1988b) and Zywicz and Parks (1989).

Plastic strip model by Leonov and Panasyuk (1959) and Dugdale (1960) for homogeneous materials is actively used now in fracture mechanics. A similar approach has been applied to the interface crack by Bastero and Atkinson (1988) where jump of tangential displacement has been taken into account.

In this paper an interface crack along the edge of a half-infinite plane is investigated. A thin yielding strip ahead of the crack tip is assumed and the corresponding mathematical model is formulated and analysed. The length of yield zone is found.

DERIVATION OF BASIC EQUATIONS

The plane deformation of an interface crack $|x| \leq c$ which is situated on the fixed boundary $y = 0$ of an elastic semi-plane $y \geq 0$ is considered. The lower half-plane $y < 0$ is absolutely rigid. The crack is loaded with uniform pressure of intensity P . Let E and ν be the Young's modulus and Poisson's ratio of the upper half-plane. The boundary conditions for $y = 0$ can be written as:

$$\sigma_{yy}(x,0) = -P, \qquad \sigma_{xy}(x,0) = 0, \qquad |x| < c, \tag{1}$$

$$u(x,0) = 0, \qquad v(x,0) = 0, \qquad |x| > c. \tag{2}$$

In order to obtain the solution of the problem the following unknown functions are introduced:

$$g_1(x) = \sigma_{xy}(x,0), \qquad g_2(x) = \frac{\partial v}{\partial x}(x,0). \tag{3}$$

The Fourier transforms are applied to the Lame plane elasticity equations. In consequence, the expressions for the derivative of normal displacement u and the normal stress σ_{yy} at $y = 0$ are obtained:

$$2\pi \frac{\partial u}{\partial x}(x,0) = \eta_{11} \int_{-\infty}^{+\infty} \frac{g_1(y)}{x-y} dy + \eta_{12} \int_{-\infty}^{+\infty} \frac{g_2(y)}{x-y} dy, \tag{4}$$

$$2\pi \sigma_{yy}(x,0) = \eta_{21} \int_{-\infty}^{+\infty} \frac{g_1(y)}{x-y} dy + \eta_{22} \int_{-\infty}^{+\infty} \frac{g_2(y)}{x-y} dy, \tag{5}$$

where

$$\eta_{11} = \frac{3-4\nu}{2\mu(1-\nu)}, \qquad \eta_{22} = -\frac{E}{1-\nu^2}, \qquad \eta_{12} = \eta_{21} = -\frac{1-2\nu}{1-\nu}, \tag{6}$$

μ is the shear modulus.
Satisfying the boundary conditions (1) and the derivatives with respect to x of the boundary conditions (2), we obtain the following system of singular integral equations:

$$\eta_{11} \int_c^h K(x,y) g_1(y) dy + \eta_{12} \int_0^c K(x,y) g_2(y) dy = 0, \quad |x| < c, \tag{7}$$

$$\eta_{21} \int_c^h K(x,y) g_1(y) dy + \eta_{22} \int_0^c K(x,y) g_2(y) dy = -2\pi P, \quad |x| > c, \tag{8}$$

where

$$K(x,y) = \frac{1}{x-y} - \frac{1}{x+y} \tag{9}$$

When we formulate the system of singular integral equations the symmetry of stress-strain field with respect to y is taken into consideration. The value h is used for the convenience of numerical analysis. The assumption

$$\sigma_{xy}(x,0) = 0, \quad v(x,0) = 0, \quad |x| > h. \tag{10}$$

is not a major one and according to St.Venant's principle will not influence the stress state near the crack, provided $h >> b$.
The system (7),(8) must be supplemented by the consistency conditions

$$\eta_{11} \int_c^h \ln\left|\frac{c-y}{c+y}\right| g_1(y) dy + \eta_{12} \int_0^c \ln\left|\frac{c-y}{c+y}\right| g_2(y) dy = 0, \tag{11}$$

$$\int_{-c}^{c} g_2(y) dy = 0. \tag{12}$$

The unknown functions $g_i(x)$ are assumed to have integrable singularities at the points c and d as follows:

$$g_1(x) = \frac{g_1^*}{\sqrt{h-x}(x-c)^\alpha}, \qquad g_2(x) = \frac{g_2^*}{\sqrt{x}(c-x)^\alpha}, \tag{13}$$

where $g_i(x) \in H$, H is a class of functions which satisfies the Holder condition.
The method of determining α has been discussed by Muskhelishvili (1953). Using this method we find the order of singularity $\alpha = 0.5 + i\beta$. Thus, the singularity oscillates near the crack tip. The stresses $\sigma_{yy}(x,0)$ and $\sigma_{xy}(x,0)$ therefore tend to infinity, and the crack faces overlap.

YIELD STRIP MODEL

To avoid material overlap, assume that thin yield zones ahead of the crack tips. We model these zones by crack continuations where displacement arise jumps take place. Namely, normal stress acts on the part $c < |x| < b$ of the crack continuation which is equal to yield stress $\sigma_s = 2\tau_s$, and shear stress τ_s acts on the section $b < |x| < d$. Therefore, the boundary conditions can now be written as

$$\sigma_{yy}(x,0) = -P, \quad \sigma_{xy}(x,0) = 0, \quad |x| \le c \tag{14}$$

$$\sigma_{yy}(x,0) = \sigma_s, \quad \sigma_{xy}(x,0) = 0, \quad c < |x| < b, \tag{15}$$

$$\sigma_{xy}(x,0) = \mp\tau_s, \quad v(x,0) = 0, \quad b < |x| < d, \tag{16}$$

$$u(x,0) = 0, \quad v(x,0) = 0, \quad d \le |x| \le h. \tag{17}$$

Satisfying these conditions by means of (4),(5), we obtain the following system of singular integral equations

$$\eta_{11} \int_d^h K(x,y) g_1(y) dy + \eta_{12} \int_0^b K(x,y) g_2(y) dy = -\eta_{11}\tau_s \ln\left|\frac{d^2-x^2}{b^2-x^2}\right|, \quad d \le x \le h, \tag{18}$$

$$\eta_{21} \int_d^h K(x,y) g_1(y) dy + \eta_{22} \int_0^b K(x,y) g_2(y) dy =$$

$$= -\eta_{21}\tau_s \ln\left|\frac{d^2-x^2}{b^2-x^2}\right| + \begin{cases} -2\pi P, & x < c, \\ 2\pi\sigma_s, & c < x < b, \end{cases} \quad 0 \le x < b. \tag{19}$$

with the consistency conditions

$$\eta_{11}\int\limits_d^h \ln\left|\frac{d-y}{d+y}\right| g_1(y)dy+\eta_{12}\int\limits_0^b \ln\left|\frac{d-y}{d+y}\right| g_2(y)dy =$$

$$= \eta_{11}\tau_s((2d)\ln(2d)-(d-b)\ln(d-b)-(b+d)\ln(b+d)), \qquad (20)$$

$$g_2^*(0)=0. \qquad (21)$$

For arbitrary positions of points b and d, the stresses have square root singularity in their vicinity. Corresponding stress intensity factors can be introduced as follows:

$$K_1(b)=\lim_{x\to b+0}\sqrt{[2(x-b)]}\sigma_{yy}(x,0), \qquad K_2(d)=\lim_{x\to d+0}\sqrt{[2(x-d)]}\sigma_{xy}(x,0). \qquad (22)$$

A numerical method which takes into account the singularities of unknown functions is used (Loboda (1993) and Loboda and Sheveleva (1995)). This method can be used for any positions of the points b and d, but their real locations are found from the conditions

$$K_1(b)=0, \qquad K_2(d)=0. \qquad (23)$$

The system of singular integral equations (18),(19) is reduced to a system of linear algebraic equations which was solved numerically.
Numerical results have been obtained for various values of ν, P, and $E=10^{+6}$. The yield zone lengths $\lambda_1=\frac{b-c}{c}$ and $\lambda_2=\frac{d-b}{c}$, displacements $u(b,0)$ and $v(c,0)$ for various values of $\frac{P}{\tau_s}$ are shown in Tables 1 and 2.

Table 1
Yield zone lengths and displacements at the crack tip for $\nu=0,03$

$\frac{P}{\tau_s}$	λ_1	λ_2	$u(b,0)$	$v(c,0)$
0,20	$2,9\cdot10^{-3}$	$2,0\cdot10^{-2}$	$-0,1473\cdot10^{-6}$	$0,1079\cdot10^{-7}$
0,25	$6,1\cdot10^{-3}$	$3,0\cdot10^{-2}$	$-0,1719\cdot10^{-6}$	$0,1585\cdot10^{-7}$
0,33	$1,0\cdot10^{-2}$	$4,6\cdot10^{-2}$	$-0,1989\cdot10^{-6}$	$0,2081\cdot10^{-7}$
0,40	$5,7\cdot10^{-2}$	$5,5\cdot10^{-2}$	$-0,2092\cdot10^{-6}$	$0,2520\cdot10^{-7}$
0,50	$2,4\cdot10^{-2}$	$7,7\cdot10^{-2}$	$-0,2274\cdot10^{-6}$	$0,3230\cdot10^{-7}$
0,66	$3,9\cdot10^{-2}$	$1,0\cdot10^{-1}$	$-0,2442\cdot10^{-6}$	$0,4170\cdot10^{-7}$
1,00	$8,4\cdot10^{-2}$	$1,6\cdot10^{-1}$	$-0,2625\cdot10^{-6}$	$0,6355\cdot10^{-7}$

Table 2
Yield zone lengths and displacements at the crack tip for $\nu=0,3$

$\frac{P}{\tau_s}$	λ_1	λ_2	$u(b,0)$	$v(c,0)$
0,20	$4,6\cdot10^{-3}$	$8,6\cdot10^{-3}$	$-0,5660\cdot10^{-7}$	$0,1395\cdot10^{-7}$
0,25	$1,0\cdot10^{-2}$	$1,3\cdot10^{-2}$	$-0,6449\cdot10^{-7}$	$0,2016\cdot10^{-7}$
0,33	$1,7\cdot10^{-2}$	$1,9\cdot10^{-2}$	$-0,7230\cdot10^{-7}$	$0,2605\cdot10^{-7}$
0,40	$2,4\cdot10^{-2}$	$2,3\cdot10^{-2}$	$-0,7448\cdot10^{-7}$	$0,3155\cdot10^{-7}$
0,50	$3,9\cdot10^{-2}$	$3,1\cdot10^{-2}$	$-0,7944\cdot10^{-7}$	$0,4070\cdot10^{-7}$
0,66	$6,5\cdot10^{-2}$	$4,5\cdot10^{-2}$	$-0,8484\cdot10^{-7}$	$0,5346\cdot10^{-7}$
1,00	$1,3\cdot10^{-1}$	$7,7\cdot10^{-2}$	$-0,9273\cdot10^{-7}$	$0,7861\cdot10^{-7}$

It is interesting to note that for $\nu=0,5$ the values $\eta_{12}=\eta_{21}=0$ and from the system (18), (19) we obtain two independent equations with respect to $g_1(y)$ and $g_2(y)$. First one have the limited solution at the point d when $b=d$ or $\tau_s=0$ while the second one attains in this case the following form

$$\eta_{22}\int\limits_0^b K(x,y)g_2(y)dy=\begin{cases}-2\pi P, & x<c,\\ 2\pi\sigma_s, & c<x<b,\end{cases} \qquad 0\le x<b \qquad (24)$$

with additional condition (21). This equation is similar to the equation used for modelling of plastic strips at the crack tips in a homogeneous material and its exact solution can be found. From this solution and condition $K_1(b)=0$ the following analytical expressions for λ_1 and $v(c,0)$ were found.

$$b=c\sec\left(\frac{\pi P}{2(\sigma_s+P)}\right), \qquad (25)$$

$$v(c,0)=-\frac{4c(\sigma_s+P)(1-\nu^2)}{\pi E}\ln\left(\frac{c}{b}\right) \qquad (26)$$

Their values calculated for the same E and P/τ_s ratious as in the Tables 1 and 2 are shown in the Table 3.

Table 3
Yield zone lengths and displacement $v(c,0)$ at the crack tip for $\nu=0,5$

$\frac{P}{\tau_s}$	λ_1	$v(c,0)$
0,20	$1,0\cdot10^{-2}$	$0,2149\cdot10^{-7}$
0,25	$1,5\cdot10^{-2}$	$0,3289\cdot10^{-7}$
0,33	$2,5\cdot10^{-2}$	$0,5552\cdot10^{-7}$
0,40	$3,5\cdot10^{-2}$	$0,7945\cdot10^{-7}$
0,50	$5,1\cdot10^{-2}$	$0,1198\cdot10^{-6}$
0,66	$8,1\cdot10^{-2}$	$0,1980\cdot10^{-6}$
1,00	$1,6\cdot10^{-1}$	$0,4121\cdot10^{-6}$

DISCUSSION

It leads from the Tables 1-3 that increasing of p/τ_s calls of all calculated parameters increasing. Enereasing of Poison's ratio ν leads to λ_1, $v(c,0)$ increasing and λ_2, $u(c,0)$ decreasing. Particularly for uncompressed material the zones of shear stresses (b,d) dissapeare and considered model is equivalent in mathematical sence to strip slip model for homogeneous material and obtained results in this case are in good agreement with exact analytical solution.
It worth to note that due to considered interface crack model an oscillating singularity was elaminated and due to correct determination of the points b and d positions the nonsingular crack state was found. The values of $u(b,0)$ and $v(c,0)$ can be used as fracture parameters for considered composite material. To predict the crack propogation they should be compared with critical values which are to be found experimentally.

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