# A METHOD TO DETERMINE CONSTITUTIVE PROPERTIES OF THIN INTERFACE LAYERS LOADED IN SHEAR

K. S. ALFREDSSON and U. STIGH
*Department of Engineering Science, University of Skövde*
*S-541 28 Skövde, Sweden*

## ABSTRACT

A method to determine constitutive properties of thin interface layers loaded in shear is presented. To avoid instability a single-sided double-lap shear specimen is designed so that the stress distribution in the thin layer is non-uniform. The method is based on an exact inverse solution and is intended for determination of both hardening and softening behaviour of adhesives. The method is confined to monotonic loading.

## KEYWORDS

Thin layer, adhesive, test method, constitutive behaviour, strain softening, inverse solution

## INTRODUCTION

Adhesive bonding of structural components is a joining method which is attractive as an alternative or as a complement to more conventional methods like spot welding, riveting or bolting. The advantage of adhesive joints is that they introduce less stress concentration in the components to be joined than the conventional methods. In most applications the adhesive is much more compliant and has a much lower strength than the parts joined by the adhesive. Thus failure of adhesive joints often takes place by cohesive fracture in the adhesive or by adhesive fracture at the interface between the adhesive and the adherends. The strength of an adhesive joint depends, however, not only on the strength of the adhesive. The stress distribution in the adhesive will depend highly on the geometry of the joint. A lot of work, both theoretical and experimental, has been done regarding the strength of a number of specific joints, see the review in (Adams and Wake, 1984). The maximum strength is achieved if the joint is designed so that the stress distribution in the adhesive is uniform. A common experience is that the adhesive should be loaded in shear rather than in tension (peel) for the design to be successful. Results from tests performed for a specific joint geometry are, however, difficult to translate to a joint having a different geometry. The lack of reliable methods to dimension adhesive joints of arbitrary geometry is probably an important reason why adhesives have not been used up to their full potential. Kinloch (1987), for example, identifies a need to model more accurately the complete stress distributions in complex joint geometries. This requires a detailed knowledge of the constitutive behaviour of the adhesive. The issue of the present paper is a method for the experimental determination of constitutive relations for adhesives loaded in shear.

The most straightforward way to test adhesives is to use a test specimen made entirely of the adhesive material. To capture the softening behaviour for a ductile adhesive, however, a test specimen of small size need to be used in order to avoid local necking or cracking which will distort the measured data. It is therefore preferable to test adhesives in their natural form of thin films. When extracting constitutive relations from experiments on a thin film adhesive it is of utmost importance to know the stress distribution in the test specimen. As mentioned earlier the stress distribution will rely highly on the geometry of the specimen. One philosophy in achieving a known shear stress distribution is to use a test specimen for which the stress is uniform. The napkin ring test (Kinlcch, 1987) provides such a geometry. A test geometry which may be designed to give an approximately uniform stress distribution is the thick adherend test (Wernersson and Gustafsson, 1987). A disadvantage of having a uniform stress distribution is that the entire adhesive layer will soften and eventually fail simultaneously. This may cause instability which in turn cause difficulties in determining the softening part of the constitutive relation. A way of avoiding this is to use a test geometry where the stress distribution is non-uniform. In this paper a test specimen is proposed for the determination of constitutive relations for adhesives subjected to the fundamental deformation mode of pure shear. The geometry of the single-sided double-lap shear specimen is shown in Fig. 1a. The stress distribution of this specimen is known implicitly as will be shown later.

## GOVERNING EQUATIONS

In this section the equations governing the deformation of the shear specimen is presented. The assumptions made in the analysis are in the spirit of Volkersen's (1938) pioneer work. The bars are taken as linearly elastic and are assumed to be loaded uniaxially. Shear and bending deformation of the bars due to the transfer of adhesive shearing stresses at the interfaces is neglected. The adhesive is assumed to be loaded in pure shear. These assumptions are justified for the present geometry when the adhesive is weak compared to the bars. The constitutive relation for the adhesive is expressed in terms of the shear force, $q$, per unit length of the bars and the relative axial displacement, $u$, of the middle bar and the outer bars. A constitutive relation for an adhesive loaded in monotonically increasing shear reads

$$q = q(u) \tag{1}$$

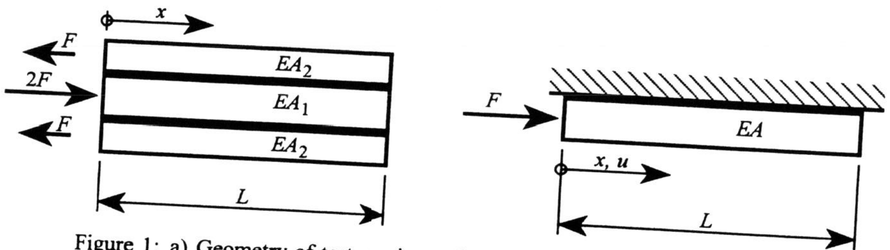


Figure 1: a) Geometry of test specimen (left); b) Equivalent system (right)



This relation is to be determined from measurements. It is assumed that the adhesive layer is homogeneous, that is the constitutive relation (1) is assumed to be valid along the entire adhesive layer. It is also assumed that $q(u) = ku$ for $|u| < M$, where $k$ and $M$ are positive numbers. Examples of typical adhesive constitutive relations are shown in Fig. 2a. For the system to be in equilibrium the following differential equation must be satisfied along the system (assuming the tensional stiffness $EA$ to be independent of $x$),

$$EA\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} - q(u) = 0\,, \qquad \frac{1}{EA} = \frac{2}{EA_1} + \frac{1}{EA_2} \tag{2a,b}$$

The boundary conditions are

$$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=0} = -\frac{F}{EA}\,, \qquad \left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=L} = 0 \tag{3a,b}$$

It is evident from the equations above that the original system is equivalent to a system consisting of a single bar with tensional stiffness $EA$ loaded by the force $F$ and connected to a rigid foundation via one of the adhesive layers of the original system. This equivalent system is shown in Fig. 1b. For a general constitutive relation $q(u)$ equations (1) and (2) together with the boundary conditions (3a,b) constitute a non-linear boundary value problem. In the following section an invers formula for the experimental determination of $q(u)$ is given. The development of the stress distribution including the issue of monotonic loading is also discussed in the following.

## INVERSE FORMULA FOR THE CONSTITUTIVE RELATION

Since, in general, the stress distribution in the adhesive layer of the test specimen will be non-uniform, it may not seem obvious how the constitutive relation $q(u)$ can be determined by performing measurements on the present test specimen. An inverse solution for the constitutive relation can, however, be derived by multiplying equation (2a) with $\mathrm{d}u/\mathrm{d}x$ and integrating along the system,

$$\int_0^L EA\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x = \int_0^L q(u)\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x \tag{4}$$

The integral on the left hand side may be evaluated directly which leads to

$$\left[\frac{EA}{2}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2\right]_0^L = \int_0^L q(u)\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x \tag{5}$$

By introducing $\delta = u(0)$ and $\delta_L = u(L)$ as the relative displacement at the left and right adhesive boundaries, respectively, and using the boundary conditions (3a,b), equation (5) may be written

$$\frac{F^2}{2EA} = \int_{\delta_L}^{\delta} q(u)\mathrm{d}u \tag{6}$$

The assumption of a long specimen ($L \to \infty$) implies that $\delta_L = 0$. Thus, for a long specimen equation (6) reads

$$\frac{F^2}{2EA} = \int_0^{\delta} q(u)\mathrm{d}u \tag{7}$$

If the constitutive law is known this equation provides a relation between the force $F$ and the relative displacement $\delta$ at the point of load application. In general numerical integration is needed, but for simple mathematical models of $q(u)$ explicit relations may be obtained. It should be noted that due to the physical requirement $q(u) \geq 0$, equation (7) implies that the displacement $\delta$ will increase monotonically with $F$ (and vice verse for a prescribed displacement $\delta$). For an adhesive having a linear constitutive relation $q(u) = ku$ a linear relation between $F$ and $\delta$ is of course obtained,

$$F = \sqrt{kEA}\,\delta \tag{8}$$

Figure 2b shows the force displacement relation for the adhesive types shown in Fig. 2a. If equation (7) is differentiated with respect to the relative displacement $\delta$, the following inverse formula for the constitutive relation is obtained,

$$q(\delta) = \frac{F}{EA}\frac{\mathrm{d}F}{\mathrm{d}\delta} \tag{9}$$

This is the key equation of the present paper. The constitutive relation may by use of this equation theoretically be obtained from measurements of the force $F$ and the relative displacement $\delta$. An inverse solution, of the same form as the one given here, has previously been presented for a DCB-specimen (Olsson and Stigh, 1989) where the adhesive is loaded in peel.

The physical meaning of equation (9) is not obvious, but equations (6) and (7) may be explained in terms of energy release rates and use of the $J$-integral (Eshelby, 1951 and Rice, 1968),

$$J = \int_\Gamma \left( W\mathrm{d}y - T_i \frac{\partial u_i}{\partial x}\mathrm{d}s \right) \tag{10}$$

where $\Gamma$ is a curve surrounding the crack tip and taken in a counter-clockwise sense, $W$ is the strain energy density, $T_i$ is the traction vector defined according to the outward normal along $\Gamma$ and $u_i$ is the displacement vector. It should be noted that the $J$-integral in equation (10) is expressed in terms of the energy release rate per unit width. However, in the following $J$ is evaluated as the total energy release rate. For this purpose half of the symmetric shear specimen

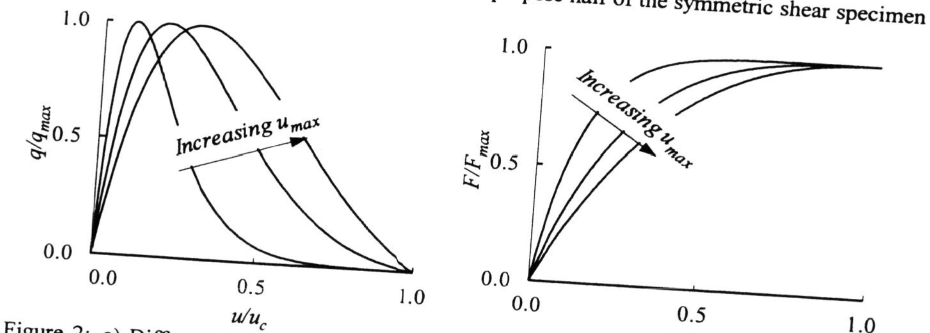


Figure 2: a) Different types of adhesive constitutive relations, $q = \beta e^{-\alpha u}\sin(\pi u/u_c)$, used for demonstration (left); b) Force vs. displacement for adhesives according to the left figure (right). Location of peak value of $q$ is denoted $u_{max}$.



containing one adhesive layer is studied, see Fig. 3. Evaluation of the $J$-integral along the two curves $\Gamma_A$ and $\Gamma_B$ yields the following values,

$$J_A = \frac{F^2}{2EA} + \int_0^{\delta_L} q(u)\mathrm{d}u\,, \qquad J_B = \int_0^{\delta} q(u)\mathrm{d}u \tag{11a,b}$$

Path independence of $J$ implies that $J_A = J_B$, which is equivalent to equation (6). Here $J_A$ may be interpreted as the elastic energy released by the system for a unit increase of the "crack length" $a$ in Fig. 3. Further, $J_B$ is the energy consumed by the adhesive layer for a unit increase of $a$. This is the physical explanation of equation (6) from which equation (7) and subsequently the inverse formula (9) are deduced.

In the derivations performed above it has been assumed that the constitutive relation is valid both at loading and unloading. Thus the adhesive is assumed to be non-linearly elastic in the general case. It is not plausible that an adhesive will behave the same in unloading as in loading. Due to damage and plasticity unloading follows a different path. The present equations will, however, be valid also for a real material as long as the entire adhesive layer is loaded monotonically. In the next section, where the stress distribution in the layer is studied in detail, it is shown that for a monotonically increasing load $F$ (or displacement $\delta$) the entire adhesive layer will be loaded monotonically.

## STRESS DISTRIBUTION IN THE ADHESIVE LAYER

Even though the stress distribution in the adhesive layer is governed by a non-linear boundary value problem the relation between the applied load $F$ and the relative displacement $\delta$ can be determined quite easily, see equation (7). In fact, the stress distribution in the layer can be determined in a similar manner if the constitutive relation is known. The procedure of multiplying the governing equation (2a) with $\mathrm{d}u/\mathrm{d}x$ and integrating along the system, used in the previous section, is used here as well. Instead of integrating along the entire system, however, the integration is performed from an arbitrary coordinate $x$ to the right end of the specimen. By use of the boundary condition (3b) the resulting equation for a long specimen ($\delta_L = 0$) is

$$\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 = \frac{2\,H(u)}{EA}\,, \qquad H(u) \equiv \int_0^{u(x)} q(\tilde{u})\mathrm{d}\tilde{u} \tag{12a,b}$$

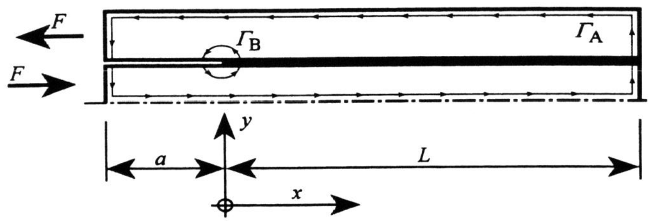


Figure 3: Paths for evaluation of $J$-integrals. Upper half of shear specimen is studied.

where $H \geq 0$ represents the strain energy per unit length of the adhesive at the position $x$. Taking the square root of equation (12a) leads to

$$\frac{du}{dx} = -\sqrt{\frac{2\,H(u)}{EA}} \tag{13}$$

where the sign is given by physical reasons explained later in this text. This equation expresses a local relation between the gradient of the relative displacement and the strain energy density of the adhesive. The equation may be separated and integrated from the point of load application to an arbitrary point $x$. This leads to the following implicit relation between the relative displacements $u(x)$ and $\delta$,

$$x = \sqrt{\frac{EA}{2}} \int_{u(x)}^{\delta} \frac{d\tilde{u}}{\sqrt{H(\tilde{u})}} \tag{14}$$

For a linear elastic adhesive as in equation (8) this equation gives an exponentially decaying displacement field,

$$u(x) = \delta\, e^{-\sqrt{\frac{k}{EA}}x} \tag{15}$$

as is to be expected. This shows that the negative sign in equation (13) is correct for small loads and that both $u$ and $q$ are positive along the entire system for a linear elastic adhesive. In order to show that the negative sign in equation (13) is correct for a non-linear adhesive it is first assumed that $u > 0$ for all $x$. This is obviously true at the onset of non-linear behaviour. From $u > 0$ it follows that $q \geq 0$. Furthermore from equation (2a) $d^2u/dx^2 \geq 0$, that is $du/dx$ is increasing with $x$. From the boundary conditions (3a,b) it follows that $du/dx$ is non-positive and increasing with x. Thus, for $u > 0$ the negative sign in equation (13) is correct for a non-linear constitutive relation. Evaluation of equation (14) for two different values of $\delta$ ($\delta_a < \delta_b$) yields two different values of the relative displacement ($u_a$ and $u_b$) for a given position $x$. After some manipulations this gives

$$\int_{u_a}^{u_b} \frac{d\tilde{u}}{\sqrt{H(\tilde{u})}} = \int_{\delta_a}^{\delta_b} \frac{d\tilde{u}}{\sqrt{H(\tilde{u})}} \tag{16}$$

This equation shows that an increase of the prescribed displacement $\delta$ leads to an increase of the displacement $u(x)$ at every point of the system ($u_a < u_b$). This implies further that, when the linear relation is no longer valid, the relative displacement $u(x)$ is still positive everywhere. According to the previous explanation this means that the sign chosen in equation (13) is correct also for a non-linear constitutive relation. Accordingly the entire adhesive layer is loaded monotonically.

The development of the stress distribution is remarkably simple. By use of equation (14) it may be shown that the distance $\Delta x$ between two points where the relative displacement is $u_a$ and $u_b$, respectively, is independent of the prescribed displacement $\delta$ and given by

$$\Delta x = x_a - x_b = \sqrt{\frac{EA}{2}} \int_{u_a}^{u_b} \frac{d\tilde{u}}{\sqrt{H(\tilde{u})}} \tag{17}$$



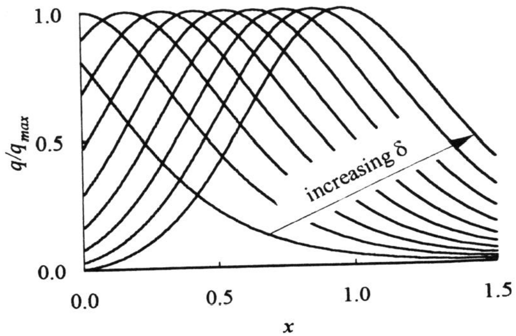


Figure 4: Development of stress distribution for $u_{max}/u_c = 0.2$.

This means that the displacement field and the stress distribution will translate undistorted in the positive $x$-direction as $\delta$ is increased. The translational velocity of the distributions is found from equation (14) as

$$\dot{x} = \sqrt{\frac{EA}{2\,H(\delta)}}\;\dot{\delta} \tag{18}$$

where a dot denotes time-differentiation. Figure 4 shows the development of the stress distribution for one of the adhesive types shown in Fig. 2a. The figure shows the stress distribution for situations at constant increments of the prescribed displacement $\delta$. This corresponds to a decreasing translational velocity $\dot{x}$, since $H(\delta)$ is an increasing function. The rightmost curve in Fig. 4 depicts the situation when the adhesive at $x = 0$ is unable to carry any further load. For a further increase of $\delta$ the "crack length" $a$ (see Fig. 3) will increase with a speed proportional to $\dot{\delta}$. The stress distribution will remain the same as in Fig. 4 but it will translate together with the "crack tip".

## DISCUSSION

In a physical realization of the method proposed in this paper it is essential to assure that the assumptions made in the analysis are fulfilled. Beside the requirements that the bars behave elastically and that the specimen is sufficiently long it is also important that the design ensures that symmetry prevails and that bending of the bars is not introduced at the points of load application. The procedure will require that displacements of small magnitude are determined with good enough accuracy so that the measured relation between the applied force $F$ and the displacement $\delta$ can be differentiated with sufficient accuracy. It should also be noted that in its present form the method is not suited for determining unloading characteristics of adhesives. The main advantage of the method is that instability is avoided because of the non-uniform stress distribution. This is believed to be essential when the softening behaviour of an adhesive is determined.

## ACKNOWLEDGEMENTS

This work was financed in part by the Swedish Ministry for Education and Science. It is part of a project supported by Saab Military Aircraft and Volvo.


## REFERENCES


Adams, R. D. and Wake, W. C. (1984). *Structural Adhesive Joints in Engineering*. Elsevier Applied Science Publishers, London, UK.

Eshelby, J. D. (1951). The force on an elastic singularity. *Phil. Trans. Roy. Soc. London* **A244**, 87-112.

Kinloch, A. J. (1987). *Adhesion and Adhesives*. Chapman and Hall, London, UK.

Olsson, P. and Stigh U. (1989). On the determination of the constitutive properties of thin interface layers - an exact inverse solution. *Int. J. Fracture* **41**, R71-R76.

Rice, J. R. (1968). A path independent integral and the approximate analysis of strain concentration by notches and cracks. *J. Appl. Mech.* 379-386.

Volkersen, O. (1938). Die Nietkraftverteilung in zugbeanspruchten Nietverbindungen mit konstanten Laschen-querschnitten. *Luftfahrtforschung*, Vol. 15, 41-47.

Wernersson, H. and Gustafsson, P. J. (1987). The complete stress-slip curve of wood adhesives in pure shear. In *Mechanical behaviour of adhesive joints* (Edited by G. Verchery and A. H. Cardon), pp 139-150. Edition Pluralis, Paris.