VARIATIONAL APPROACH TO LIMIT LOAD EVALUATION IN FRAMEWORK OF THE IDEAL RIGID PLASTIC MODEL TAKING INTO ACCOUNT TENSILE CRACK FORMATION

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ABSTRACT

A new variational approach is developed which enable us to apply the ideal rigid plastic model replenished by a frscture criterion for evaluation of a limit load for structural elements taking into account presence and formation of tensile cracks. The approach is based on a new variational principle suggested by Alexandrov and Goldstein (1995). As an example of the approach application the plane strain necking and fracture of v-notched bar under tensile loading is considered.

KEYWORDS

Variational principle, rigid plastic model, crack, notch, limit load.

INTRODUCTION

The upper bound method (UBM) is the powerful tool, in particular, for modeling of the limit loads and states for structural elements when using the ideal rigid plastic model. The method is based on the variational principle of minimum of surface forces power on the real field of flow rates related to the limit state of a body when plastic zones occur along with existence of the rigid ones. First this variational principle had been proved for continuous fields of flow rates and then it had been extended for the case when the discontinuity of the shear component of the flow rate is admissible (Kachanov, 1956). Hence, the principle implies that a flow accompanied by formation of tensile cracks is impossible. At the same time tensile cracks are observing at both technological processes (Vilotic, 1987) and plastic fracture of structures. Hence, to model these phenomena a variational principle is required which could admit the flow rate fields with a discontinuity of the normal component the flow rate.

Such a variational principle will imply limit states with rigid and instantaneously occured plastic zones as well as tensile cracks of finite area.

The mentioned variational principle could be used as an adequate tool for modeling of conditions of structures plastic failure in assumption that fracture occuring is accompanied by instantaneous formation of cracks of finite area and each elementary act of the crack growth is accompanied by a finite increment of its area. The assumptions allow one to avoid the well known paradox that the energy flux to the crack edge in a plastic material is equal to zero and, hence, the Griffith criterion is not applicable for searching for crack growth conditions.

Note, that our assumptions are in agreement with criteria of the crack growth in elastoplastic bodies implying that a limit state is attaining simultaneously in finite region near the crack edge (see, e.g., Kfouri and Rice, 1978; Slepjan, 1981; Bui and Dang Van, 1987).

VARIATIONAL PRINCIPLE

Consider now the case when a tensile crack nucleation takes place at the stage of the plastic deformation of the body. The material is believed to be the ideal rigid plastic one. In accordance with the model described in the Introduction we will assume that the plastic collapse and finite crack formation occur instantly and simultaneously.

To evaluate the load of the crack formation and its geometric parameters we will use the special variational principle (Alexandrov and Goldstein, 1995) admiting fields of the flow rates with a discontinuity of the normal component of the flow rate.

Denote by σ_{ij} and ε_{ij} the components of the stress and strain rate tensors, respectively, and by u_i the components of the particle rates.

Let S_Δ be the surface in the body D, with the boundary $\partial D = S$, where the discontinuity of the normal component of the flow rate, $[u_i]_n$, can occur. Hence, $[u_i]_n \neq 0$ at the surface S_Δ. Assume that the limit stress is constant and equal to σ_f under the tensile fracture along the surface S_Δ.

Denote by t_{ni}^s, u_i^s the loads and flow rate components at the surface S. Let the boundary S of the body consists of the parts S_p, S_u and S_{pu} where the loads, flow-rates and their combinations are prescribed, respectively.

Introduce the kinematically admissible fields of the flow rates as the fields satisfying the boundary conditions for the flow rates and the condition of incompressibility everywhere in D except a small cylindrical region surrounding the surface S_Δ of the potential tensile crack. It can be proved (Alexandrov and Goldstein, 1995) the following

Variational principle: The functional

$$J = \int_D \tau_{ij}\varepsilon_{ij}\,d\omega + \sigma_f \int_{S_\Delta} \left|[u_i]_n\right| ds - \int_{S_p+S_{pu}^{\sigma}} t_{ni}u_i\,ds \tag{1}$$

has minimum at the real field of the flow rates among the all kinematically admissible fields of the flow rates. Here the last integral contains the terms depending only on the loads given at the surfaces S_p and S_{pu}.

PLANE STRAIN NECKING AND FRACTURE OF V-NOTCHED TENSILE BARS

Plane and axisymmetric notched bars subjected to a tensile load produce a high stress triaxiality level near the net section. Very often this high stress triaxiality results in a crack at the center of specimen (Nadai, 1950). However, this depends on the hydrostatic pressure (Bridgman, 1964) and material properties. Ductile materials usually exhibit a cup-cone type of tensile fracture at atmospheric pressure. At such a pressure the notch sharpness can change a fracture mechanism.

The experiments carried out by Joubert and Valentin (1976) have shown that a ductile material can exhibit either ductile or brittle rupture. The influence of material properties on the fracture behavior of notched wide plates have been studied by Ikeda (1976). Experimental results presented in this work showed the difference in the macroscopic condition of ductile fracture between the high strength steel plate and the aluminum alloy 5083-0. In order to describe the behavior of tensile bars the pioneer stress and strain analysis of Bridgman (1964) has been developed by Clausing (1969) and Argon et al. (1975) and Earl and Brown (1976). Finite element technique has been adopted by Argon et al. (1975) and Glinka et al. (1988) and others. Majima and Furukawa (1990) have determined the development of the stress and strain distributions with deformation applying a numerical method to the hardness test results. All of these studies have dealt with the behavior of specimen deforming without fracture. In order to introduce a fracture process into considerations a damage model is usually applied (see, e.g., Zavaliangos and Anand, 1993). This leads to complicated equations and their solution sometimes show that the damage parameter influences the specimen behavior just before rupture only.

Unlike, calculations based on the proposed model can be carried out using the simple and well-known techniques such as the limit analysis and slip-line theory. On the other hand, they take into account the material properties, indeed, σ_f value is a material characteristic adjusted by the stress triaxiality level since the fracture criterion depends on the stress state. In particular, the slip-line solutions to a plane notched bar under tension have been found by Neimark (1968), and Richmond (1969), and Gu (1987), and others. In the subsequent analysis here Richmond's (1969) solution will be used. It is important that it is the analytical solution to the nonsteady problem with two space variables. This allows one to take into account large strains and changes in the sharpness of a notch and, thus, to predict the rupture moment and final reduction of area at rupture.

The configuration chosen for the analysis is shown in Fig.1a and consists of the plane strain deformation of a doubly notched plate of minimum width 2T and notch angle 2θ. In this figure the region ABCDE is the current plastic zone proposed by Richmond (1969). Since a nonsteady process is considered the plastic zone changes in time such that θ and T are functions of time. The free surface DE rotates but remains linear. In order to calculate the load required to deform this specimen it is only necessary to know σ_y stress on the surface AE. This stress is constant and equal to (Richmond, 1969)

$$\sigma_y = k(2+\pi-2\theta) \tag{2}$$

k being the shear resistance, a material constant.

A kinematically admissible velocity field admiting a crack of width 2a is shown in Fig.1b. The plastic region A' B' C' D' E' is simply the part of Richmond's original solution, however, the tangent velocity is discontinuous at the surface A' B' C' D'. One can conclude from the well known property of slip-line fields that σ_y is a constant on the surface A' E' and it is defined by expression (2). Let us denote $\sigma_y=\sigma_o$ at this surface. By assumption, σ_y is also a constant at the surface AA', $\sigma_y=\sigma_f$. Therefore, by incorporating the fact that y-axis is an axis of symmetry and $E'D'$ is a free surface, the load may be calculated from the following expression

$$L = 2\sigma_f a + 2\sigma_o(T-a) \tag{3}$$

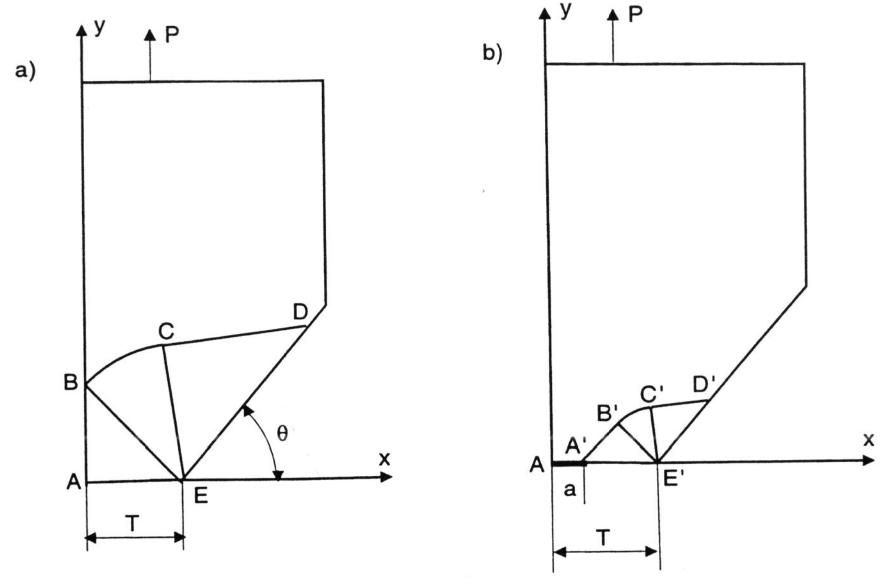

Fig.1. (a) One quadrant of notched bar and plastic zone
(b) Kinematically admissible velocity field with crack

In order to apply the variational principle proposed it is necessary to consider L as a function of a and to find its minimum value. From (3) it is clear that L has no minimum at $0 < a < T$, therefore, a minimum is reached at one of the end points, a=0 or a=T. This means that the specimen of such type deforms with no crack up to a point of separation. Separation can occur either by plastic flow or appearence of a crack of finite area which is equal to the current area of the minimum section. The type of separation depends on the material properties and stress state. From (3) it follows that a transition point is defined by the equation

$$\sigma_f = \sigma_o \tag{4}$$

If the material obeys the von Mises yield criterion then $k = \sigma_Y / \sqrt{3}, \sigma_Y$ being the yield stress in tension. In this case substituting (2) into (4) we obtain that

$$\frac{\sigma_Y}{\sigma_f} = \frac{\sqrt{3}}{2+\pi-2\theta} \tag{5}$$

to find the transition point. If $\sigma_Y \le \sqrt{3}\sigma_f / (2+\pi)$ then the separation by plastic flow occurs. If $\sigma_Y \ge \sqrt{3}\sigma_f / (2+\pi-2\theta_o)$ then the crack of width 2T appears at the initial moment and no plastic flow occurs (θ_o being the initial value of θ). If $\sqrt{3}\sigma_f / (2+\pi-2\theta_o) > \sigma_Y > \sqrt{3}\sigma_f / (2+\pi)$ then at the beginning the plastic flow occurs which leads to changes in the value of θ. When θ reaches a magnitude θ_1 the separation takes place. The value θ_1 may be determined from (5) at a

given ratio σ_y/σ_f

$$\theta_1 = 1 + \frac{\pi}{2} - \frac{\sqrt{3}}{2}\frac{\sigma_f}{\sigma_Y} \tag{6}$$

Knowing θ_1 the corresponding minimum width of plate can be defined from (Richmond, 1969)

$$T_1 = T_o\left(\frac{1-\cos\theta_1}{1-\cos\theta_o}\right)^2 \tag{7}$$

T_o being the minimum width of plate at initial moment. The neck profile is given by (Richmond, 1969)

$$\frac{x}{T_o} = \frac{T}{T_o}(1+2\cos\theta)$$
$$\frac{y}{T_o} = \frac{c}{T_o} + 2\frac{T}{T_o}\sin\theta \tag{8}$$

with $\theta_o \le \theta \le \theta_1$ and

$$\frac{T}{T_o} = \left(\frac{1-\cos\theta}{1-\cos\theta_o}\right)^2$$

$$\frac{c}{T_o} = \left(1-\cos\theta_o\right)^{-2}\left[3(\theta-\theta_o)+4(\sin\theta_o-\sin\theta)+\sin\theta\cos\theta-\sin\theta_o\cos\theta_o\right]$$

From the above considerations it follows that the stress σ_y on the surface AE' is exactly equal to the net fracture stress, $\sigma_{f,net}$, at the separation moment. Ikeda (1976) has experimentally found the variation of $\sigma_{f,net}$ and σ_Y with the temperature for high strength steel and aluminum alloy 5083-0. For this aluminum alloy it has been found that the ratio $\sigma_{f,net}/\sigma_Y \approx 1.33$ when the temperature changes from -200°C to 40°C. Using this result and Eq. (5) one can obtain $\theta_1 \approx 1.42$. Thus, the separation by plastic flow is not possible and the separation at the initial moment takes place if $\theta_o < 1.42$. For the high strength steel investigated by Ikeda (1976) the ratio $\sigma_{f,net}/\sigma_Y$ depends on the temperature. Fig.2 shows the variation of θ_1 and T_1 with the temperature calculated by means of (6) and (7). The configurations of specimen after rupture obtained by means of (8) are given in Fig.3 for different ratios σ_f/σ_y.

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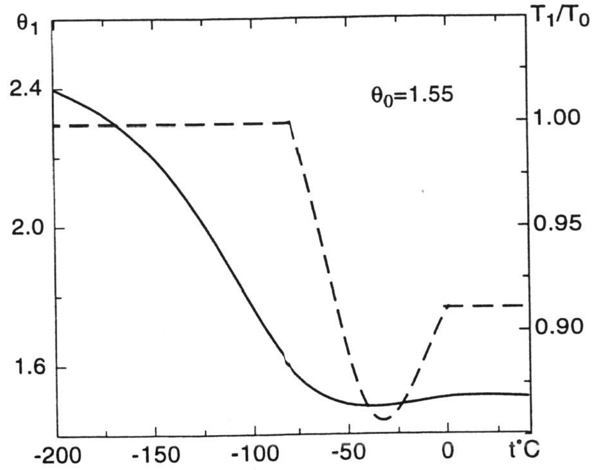

Fig.2. Variation of θ_1 and T_1 with temperature for steel specimen

(——— θ_1, – – – T_1)

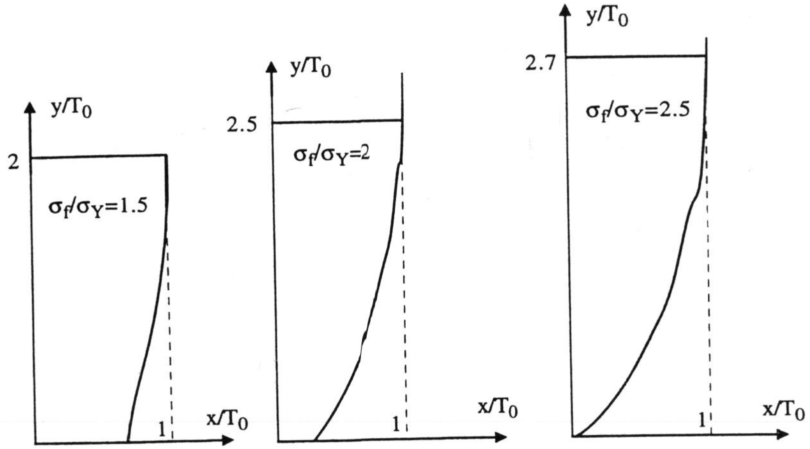

Fig.3. Configurations of specimen (one quadrant) after rupture

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