USING MORI-TANAKA METHOD FOR EFFECTIVE MODULI OF CRACKED THERMOPIEZOELECTRIC MATERIALS*

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ABSTRACT

A Mori-Tanaka effective modulus theory is developed for microcrack-weakened thermopiezo-electric solids. The theory is capable of determination of effective thermal conductivity, effective electroelastic moduli, effective thermal expansion and pyroelectric coefficients. The microcrack induced material constants are derived by way of a recently developed explicit solutions of thermal-, electric- and elastic- fields for a crack in an infinite piezoelectric solid. Numerical results are presented graphically in comparison with those by self-consistent method(SCM) and generalized self-consistent method(GSCM).

KEYWORDS

Piezoelectricity, effective modulus, Mori-Tanaka method, microcrack, thermal analysis

INTRODUCTION

Many brittle materials, such as concrete, piezoceramic, have a large number of pre-existing microcracks. The determination of their effective material properties has been the focus of considerable research. Current research into the development of effective methods to predict the effective material properties has mostly concentrated on the Taylor's method(Fanella and Krajcinovic, 1988), the self-consistent method(Budiansky and O'Connell, 1976), the differential scheme(Hashin, 1988), the Mori-Tanaka method(Mori and Tanaka, 1973; Benveniste, 1986), Off all the methods, the Mori-Tanaka method is one of the most versatile methods developed in recent years and, has several advantages over the others(Ferrari, 1991). Firstly, the method was found to be in remarkable agreement with experimental data. Secondly its predictions are

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comprised between the Hashin-Shtrikman bounds at all fiber concentration levels. The third is d to the work of Benveniste(1987). He showed that this method accounts for the interaction of t inclusions, under specific circumstances, and that it is applicable to the case of materials wi cracks. In this connection, it does not exhibit the unacceptable behavior of the self-consiste scheme. A critical review on the method can be found from(Weng, 1990).

This paper constitutes a continuation of analysis completed by the authors of this article(Yu an Qin, 1996) on cracked thermopiezoelectric materials. In contrast to our previous study, th present work focuses on developing a Mori-Tanaka theory for piezoelectric medium containin microcracks with given length and same orientation.

FUNDAMENTAL EQUATIONS

Let us consider a two-dimensional thermopiezoelectric solid, where the material is transversel isotropic and coupling between in-plane stresses and in-plane electric fields takes place. Choosin the x_3-axis as the poling direction, the plane-strain constitutive equations are expressed by

$$q_i = k_{ij} H_j \tag{1}$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{33} \\ \sigma_{13} \\ D_1 \\ D_3 \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{13} & 0 & 0 & e_{31} \\ c_{13} & c_{33} & 0 & 0 & e_{33} \\ 0 & 0 & c_{44} & e_{15} & 0 \\ 0 & 0 & e_{15} & -\kappa_{11} & 0 \\ e_{31} & e_{33} & 0 & 0 & -\kappa_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \\ -E_1 \\ -E_3 \end{Bmatrix} - \begin{Bmatrix} \gamma_{11} \\ \gamma_{33} \\ 0 \\ 0 \\ g_3 \end{Bmatrix} \theta \tag{2}$$

or inversely

$$H_i = \rho_{ij} q_j \tag{3}$$

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{33} \\ 2\varepsilon_{13} \\ -E_1 \\ -E_3 \end{Bmatrix} = \begin{bmatrix} f_{11} & f_{13} & 0 & 0 & p_{31} \\ f_{13} & f_{33} & 0 & 0 & p_{33} \\ 0 & 0 & f_{44} & p_{15} & 0 \\ 0 & 0 & p_{15} & \beta_{11} & 0 \\ p_{31} & p_{33} & 0 & 0 & \beta_{33} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{33} \\ \sigma_{13} \\ D_1 \\ D_3 \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{33} \\ 0 \\ 0 \\ \lambda_3 \end{Bmatrix} \theta \tag{4}$$

and simply, in matrix form

$$\boldsymbol{\Pi} = \mathbf{EZ} - \gamma\theta, \tag{5}$$

$$\mathbf{Z} = \mathbf{F}\boldsymbol{\Pi} + \alpha\theta \tag{6}$$

where σ_{ij}, ε_{ij}, D_j and E_j are stress, strain, electric displacement and electric field respectively, c_{ij} is elastic stiffness, f_{ij} elastic compliance, e_{ij} and p_{ij} are piezoelectric constants, κ_{ij} and β_{ij} dielectric permittivity, g_i and λ_i pyroelectric constants, γ_{ij} and α_{ij} stress-temperature constants and thermal expansion constants respectively, H_j is heat intensity, q_j heat flux, k_{ij} heat conductivity, ρ_{ij} heat resistivity, and θ temperature change.

Since the problem is concerned with the piezoelectric analogue of the uncoupled theory of thermoelasticity where the electric and elastic fields are fully coupled, but the temperature enters the problem only through the constitutive equations. As a result of this, the effective conductivity (or resistivity) and the effective elastoelectric moduli can be determined independently, while the calculation of effective thermal expansion and pyroelectric coefficients requires the information

about both the effective conductivity and effective elastoelectric moduli. The details are presented as follows.

OVERALL MODULI OF A CRACKED THERMOPIEZOELECTRIC MATERIALS

Consider a representative area element(RAE) containing a set of microcracks with the same length $2a$ and same orientation parallel to x_1-axis. This assumption is only for simplifying the ensuing calculation and easy to extend to the case of randomly oriented microcracks.

Effective Conductivity

The effective conductivity k_{ij}^* and the resistivity ρ_{ij}^* are respectively defined by

$$\overline{q}_j = k_{ij}^* \overline{H}_j \tag{7}$$

$$\overline{H}_i = \rho_{ij}^* \overline{q}_j \tag{8}$$

where the superscript "*" denotes the effective value, and overbar denotes its area average. For a cracked medium $\overline{q}_i$ and $\overline{H}_i$ are defined by

$$\overline{H}_i = \overline{H}_i^M + \frac{1}{A}\sum_{k=1}^{N}\int_{l_k} \Delta\theta n_i dc \tag{9}$$

$$\overline{q}_i = \overline{q}_i^M \tag{10}$$

With these two expressions, it can be readily shown that the effective conductivity and resistivity for a cracked medium are determined by(see for example, Yu and Qin, 1996)

$$k_{ij}^* H_j^0 = k_{ij}^M H_j^0 - \frac{k_{ij}^M}{A}\sum_{k=1}^{N}\int_{l_k} \Delta\theta n_j dc = k_{ij}^M H_j^0 - k_{ij}^M \overline{H}_j^c \tag{11}$$

$$\rho_{ij}^* q_j^0 = \rho_{ij}^M q_j^0 + \frac{1}{A}\sum_{k=1}^{N}\int_{l_k} \Delta\theta n_i dc = \rho_{ij}^M q_j^0 + H_i^c \tag{12}$$

where A and N are respectively the area and crack number of the RAE, l_k is the length of the k-th crack, superscript "M" and "0" stand for matrix and constant respectively, n_i is the outward normal to crack surface, and $\Delta\theta$ is the temperature jump across the crack surfaces. From (12) it is immediately seen that the determination of the effective resistivity ρ_{ij}^* requires a knowledge of the jump $\Delta\theta$ in the cracked medium. For a microcrack-weakened medium an exact solution for $\Delta\theta$ is not feasible and approximate methods are usually devised to determine it and thus the effective resistivity ρ_{ij}^*. Here and next subsection, a Mori-Tanaka effective theory is used for evaluating k_{ij}^*(or ρ_{ij}^*) and other effective values.

We start by developing the dilute approximation model and then extend it to the Mori-Tanaka theory. To this end, consider the boundary condition

$$q_i^0 = q^0 \delta_{2i} \tag{13}$$

The jump $\Delta\theta$ in (12) can be given by(see Yu and Qin, 1996, for example)

$$\Delta\theta = \frac{4q^0}{\sqrt{k_{11}^M k_{22}^M}}(a^2 - x^2)^{1/2} \qquad |x| < a \tag{14}$$

where a is the half-length of the crack, and k_{12}=0 is used.

Substituting (14) into (12), the dilute approximation yields

$$\rho^{*}_{22}q^{0}=\rho^{M}_{22}q^{0}+\frac{2\pi\varepsilon}{\sqrt{k^{M}_{11}k^{M}_{22}}}q^{0} \tag{15}$$

where ε is the crack density defined by $\varepsilon = Na^{2}/A$

Alternatively, subjecting the solid to intensity boundary condition with

$$H^{0}_{i}=H^{0}\delta_{2i} \tag{16}$$

From (11) and (14) we have

$$k^{*}_{22}H^{0}=k^{M}_{22}H^{0}-(k^{M}_{22})^{2}\frac{2\pi\varepsilon}{\sqrt{k^{M}_{11}k^{M}_{22}}}H^{0} \tag{17}$$

When ε<<1, we see from (15) and (17) that

$$\frac{k^{*}_{22}}{k^{M}_{22}}=\frac{1}{1+2\pi\varepsilon\sqrt{k^{M}_{22}/k^{M}_{11}}}\approx 1-2\pi\varepsilon\sqrt{k^{M}_{22}/k^{M}_{11}} \tag{18}$$

which indicate the validity of dilute method is only for small values of ε.

Let us now extend the above analysis to Mori-Tanaka method. It will be more convenient to start by imposing intensity boundary condition as defined in (16). Assume the average intensity in the matrix as

$$\overline{H}^{M}_{2}=H^{0}+\tilde{H}^{M} \tag{19}$$

where $\tilde{H}^{M}$ is the perturbed intensity due to the presence of the crack. The Mori-Tanaka method assumes that introduction of one single crack in the matrix will result in a value $\overline{H}^{c}_{2}$ given by

$$\overline{H}^{c}_{2}=2\pi\varepsilon(H^{0}+\tilde{H}^{M})\sqrt{k^{M}_{22}/k^{M}_{11}} \tag{20}$$

Using the average intensity theorem(Hashin, 1983), (9) becomes

$$H^{0}=\left(1+2\pi\varepsilon\sqrt{k^{M}_{22}/k^{M}_{11}}\right)(H^{0}+\tilde{H}^{M}) \tag{21}$$

which yields

$$\tilde{H}^{M}=\frac{-2\pi\varepsilon\sqrt{k^{M}_{22}/k^{M}_{11}}}{1+2\pi\varepsilon\sqrt{k^{M}_{22}/k^{M}_{11}}}H^{0} \tag{22}$$

which, in conjunction with (11) provides the following expression for k^{*}_{22} as

$$\frac{k^{*}_{22}}{k^{M}_{22}}=\frac{1}{1+2\pi\varepsilon\sqrt{k^{M}_{22}/k^{M}_{11}}} \tag{23}$$

It is found that the equation (23) is identical to the first expression in (18). However, the validity of (23) extends to large values of ε to non-dilute conditions.

Effective Elastoelectric, Thermal Expansion and Pyroelectric Constants

Consider again the representative area element. The effective material properties are defined in the following fashion

$$\overline{\mathbf{\Pi}}=\mathbf{E}^{*}\overline{\mathbf{Z}}-\gamma^{*}\overline{\theta}\quad\text{or}\quad\overline{\mathbf{Z}}=\mathbf{F}^{*}\overline{\mathbf{\Pi}}+\alpha^{*}\overline{\theta} \tag{24}$$

where $\mathbf{F}^{*}$ is the "inverse" of $\mathbf{E}^{*}$, and α=$\mathbf{F}\gamma$.

For a cracked medium, $\overline{\mathbf{\Pi}}$ and $\overline{\mathbf{Z}}$ in (24) can be given by

$$\overline{\mathbf{\Pi}}=\overline{\mathbf{\Pi}}^{M} \tag{25}$$

$$\overline{\mathbf{Z}}=\overline{\mathbf{Z}}^{M}+\overline{\mathbf{Z}}^{c} \tag{26}$$

where

$$\overline{Z}^{c}_{ij}=\frac{1}{2A}\sum_{k=1}^{N}\int_{l_k}\{[1+H(i-3)]\Delta\breve{U}_{,i}\breve{n}_{j}+H(2-j)\Delta\breve{U}_{,j}\breve{n}_{i}\}dc \tag{27}$$

where $\mathbf{\breve{n}}=\{\breve{n}_{1},\breve{n}_{2},0\}^{T}$, $\mathbf{U}=\{u_1\ u_2\ \phi\}^{T}$, u_i is elastic displacement, ϕ electric potential, $H(x)$ is the Heaviside step function, and superscript "˘" means the variable measured in a coordinate system local to crack line. Through use of (25~27), one gets

$$E^{*}_{iJKl}Z^{0}_{Kl}=E^{M}_{iJKl}Z^{0}_{Kl}-E^{M}_{iJKl}\overline{Z}^{c}_{Kl} \tag{28}$$

$$F^{*}_{IjkL}\Pi^{0}_{kL}=F^{M}_{IjkL}\Pi^{0}_{kL}+\overline{Z}^{c}_{Ij} \tag{29}$$

As was done earlier, we begin with developing the dilute formulation and then extend to Mori-Tanaka method. Finally, a comparison is made to see if these two methods yield the identical results in the case of piezoelectric medium.

<u>The dilute approximation</u>. We start by imposing stress boundary conditions given as $\sigma^{0}_{21}=\sigma^{0}_{1}$ and others equal 0. In this case (29) reduces to (see Yu and Qin, 1996, for example):

$$f^{*}_{44}=f^{M}_{44}+\pi\varepsilon X^{M}_{11}/2 \tag{30}$$

where X_{11} is the component of following matrix(Yu and Qin, 1996)

$$\mathbf{X}=-2\,\mathrm{Im}(\mathbf{AB}^{-1}) \tag{31}$$

and the superscript "-1" maens the inverse of matrix, "Im" stands for the imaginary part, **A** and **B** are well-defined matrices(see Chung and Ting, 1995, for example), p_i are the roots of following equation(see Yu and Qin, 1996, for details)

$$a_0p^{6}+a_1p^{4}+a_2p^{2}+a_3=0 \tag{32}$$

Successively, let $\sigma^{0}_{22}=\sigma^{0}_{2}$ or $D^{0}_{2}=D^{0}$ and others equal 0, one may obtain

$$\begin{Bmatrix}f^{*}_{33}\\ p^{*}_{33}\end{Bmatrix}=\begin{Bmatrix}f^{M}_{33}\\ p^{M}_{33}\end{Bmatrix}+\frac{\pi\varepsilon}{2}\begin{Bmatrix}X^{M}_{22}\\ X^{M}_{32}\end{Bmatrix} \tag{33}$$

or

$$\beta^{*}_{33}=\beta^{M}_{33}+\pi\varepsilon X^{M}_{33}/2 \tag{34}$$

Alternatively, subjecting the medium to DEP boundary conditions given as $2\varepsilon_{12}=\gamma^{0}$, or $\varepsilon^{0}_{22}=\varepsilon^{0}$ or $Z^{0}_{42}=-E^{0}$ and others equal 0, (28) yields

$$\begin{Bmatrix}c^{*}_{44}\\ e^{*}_{15}\end{Bmatrix}=(1-\pi\varepsilon X^{M}_{11}c^{M}_{44}/2)\begin{Bmatrix}c^{M}_{44}\\ e^{M}_{15}\end{Bmatrix} \tag{35}$$

or

$$\begin{Bmatrix}c^{*}_{13}\\ c^{*}_{33}\\ e^{*}_{33}\end{Bmatrix}=(1-\pi\varepsilon(X^{M}_{22}c^{M}_{33}+X^{M}_{23}e^{M}_{33})/2)\begin{Bmatrix}c^{M}_{13}\\ c^{M}_{33}\\ e^{M}_{33}\end{Bmatrix} \tag{36}$$

or

$$\begin{Bmatrix} e_{31}^* \\ \varepsilon_{33}^* \end{Bmatrix} = (1 - \pi\varepsilon(X_{32}^M e_{33}^M - X_{33}^M \varepsilon_{33}^M)/2) \begin{Bmatrix} e_{31}^M \\ \varepsilon_{33}^M \end{Bmatrix} \tag{37}$$

Eqns. (30) and (33~37) constitute the governing equations of dilute method.

The Mori-Tanaka theory. It is convenient to begin with considering the displacement boundary conditions given as $2\varepsilon_{12}=\gamma^0$ and others equal 0. In this case the jump $\Delta\mathbf{U}$ in (27) can be found in (Yu and Qin, 1996)

$$\{\Delta\mathbf{U}\} = \begin{Bmatrix} \Delta u_1 \\ \Delta u_2 \\ \Delta\phi \end{Bmatrix} = \begin{Bmatrix} X_{11}^M \\ X_{21}^M \\ X_{31}^M \end{Bmatrix} (a^2 - x^2)^{1/2} c_{44}^M \gamma^0 \tag{38}$$

Similar to the heat conduction analysis in previous section, we assume the average strain in the matrix as

$$\bar{\gamma}^M = \gamma^0 + \tilde{\gamma}^M \tag{39}$$

Substituting (39) into (26) leads to

$$\tilde{\gamma}^M = \frac{-\pi\varepsilon X_{11}^M c_{44}^M / 2}{1 + \pi\varepsilon X_{11}^M c_{44}^M / 2} \gamma^0 \tag{40}$$

which, in conjunction with (28) yields

$$\begin{Bmatrix} c_{44}^* \\ e_{15}^* \end{Bmatrix} = \frac{1}{1 + \pi\varepsilon X_{11}^M c_{44}^M / 2} \begin{Bmatrix} c_{44}^M \\ e_{15}^M \end{Bmatrix} \tag{41}$$

Similarly we can obtain other effective elastoelectric constants by letting $\varepsilon_{22}^0 = \varepsilon^0$ or $Z_{42}^0 = -E^0$ and others equal 0, which gives

$$\begin{Bmatrix} c_{13}^* \\ c_{33}^* \\ e_{33}^* \end{Bmatrix} = \frac{1}{1 + \pi\varepsilon(X_{22}^M c_{33}^M + X_{23}^M e_{33}^M)/2} \begin{Bmatrix} c_{13}^M \\ c_{33}^M \\ e_{33}^M \end{Bmatrix} \tag{42}$$

or

$$\begin{Bmatrix} e_{31}^* \\ \varepsilon_{33}^* \end{Bmatrix} = \frac{1}{1 + \pi\varepsilon(X_{32}^M e_{33}^M - X_{33}^M \varepsilon_{33}^M)/2} \begin{Bmatrix} e_{31}^M \\ \varepsilon_{33}^M \end{Bmatrix} \tag{43}$$

Eqns. (41~43) constitute the governing equations for determining the effective elastoelectric moduli of a cracked body. It is found from (35~37) and (41~43) that these two methods give almost identical results for small values of ε, but the Mori-Tanaka method can give almost accurate results for relatively large values of ε. Once k_{ij}^* and $\mathbf{E}^*$ is obtained, the effective α^* can be determined by (Yu and Qin, 1996)

$$\begin{Bmatrix} \alpha_{22}^* \\ \gamma_3^* \end{Bmatrix} = \begin{Bmatrix} \alpha_{22}^M \\ \gamma_3^M \end{Bmatrix} - \frac{\pi\varepsilon}{2} \begin{Bmatrix} b_2^* \\ b_3^* \end{Bmatrix} \tag{44}$$

where $\mathbf{b}=\{b_1\, b_2\, b_3\}^T$ is defined by(Yu and Qin,1996)

$$\mathbf{b} = -2\,\mathrm{Im}(\mathbf{A}_0 - \mathbf{A}\mathbf{B}^{-1}\mathbf{B}_0) \tag{45}$$

where

$$\mathbf{A}_0 = \mathbf{D}^{-1}(p_1^*)\{ \gamma_1 + p_1^* \gamma_2 \}$$

$$\mathbf{D}(p) = \mathbf{Q} + (\mathbf{R} + \mathbf{R}^T) p + \mathbf{T} p^2$$

$$\mathbf{B}_0 = \mathbf{R}^T \mathbf{A}_0 + \mathbf{T} \mathbf{A}_0 p_1^*$$

and where

$$\gamma_1 = \{ \gamma_{11}\; 0\; 0 \}^T, \quad \gamma_2 = \{ 0\; \gamma_{22}\; g_3 \}^T, \quad p_1^* = i\sqrt{k_{11}/k_{22}}$$

$$\mathbf{Q} = \begin{bmatrix} c_{11} & 0 & 0 \\ 0 & c_{44} & e_{15} \\ 0 & e_{15} & -\kappa_{11} \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} 0 & c_{13} & e_{31} \\ c_{44} & 0 & 0 \\ e_{15} & 0 & 0 \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} c_{44} & 0 & 0 \\ 0 & c_{33} & e_{33} \\ 0 & e_{33} & -\kappa_{33} \end{bmatrix}$$

NUMERICAL RESULTS

As illustration we consider a cracked piezoelectric ceramic($BatiO_3$)(Dunn, 1993, or Yu and Qin, 1996). The normalized effective modulus c_{44}^*/c_{44}^M, is presented in Fig. 1 for various values of crack density ε. The self-consistent and GSC solutions are also given in Fig. 1 for comparison. It is found again that the curve for Mori-Tanaka method are slightly higher than that for GSCM.

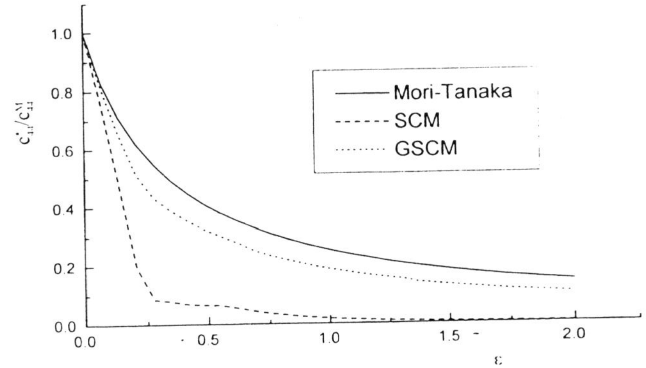

Fig. 1 Effective modulus vs crack density

COMCLUSIONS

This investigation presented a Mori-Tanaka formulation to calculate the effective material constants of microcrack-weakened thermopiezoelectric medium. In contrast the GSCM in our previous paper, the presented version of Mori-Tanaka's method is of striking simplicity and needs very little computing effort. Although the results are confined to the case of plane strain and all microcracks with the same length and same orientation, it is easy to extend the procedure to other plane problems, such as the case of $u_3 = u_3(x_1, x_2) \neq 0$ and the cracks being randomly oriented.

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