THREE-DIMENSIONAL CRACK-FACE WEIGHT FUNCTIONS FOR THE SEMI-INFINITE INTERFACE CRACK

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ABSTRACT

The aim of this paper is to provide the expressions of Bueckner's fundamental three-dimensional crack-face weight functions for a semi-infinite interface *crack in an infinite body. The method of solution avoids the calculation of the full mechanical fields in the elasticity problems implied but concentrates instead on the sole feature of interest, namely the distribution of the stress intensity factors along the crack front. It is inspired from previous works of Gao and Rice on the one hand and Leblond, Mouchrif and Perrin on the other hand. The results are given in a totally explicit, analytic form to the first order in the "bimaterial constant"* ϵ. *Their only non-elementary feature is the appearance of an indefinite integral of the type* $\int \frac{\ell n\ x}{x+a}\,\mathrm{d}x$.

KEYWORDS

3D weight functions, semi-infinite interface crack, Bueckner-Rice theory, perturbation of the front, integrodifferential equations, Fourier transform.

INTRODUCTION

Many authors have given the expressions of Bueckner's fundamental three-dimensional weight functions for a semi-infinite crack in an infinite *homogeneous* (isotropic) elastic body. The most complete work in that field is that of Bueckner (1987) himself, who did not only provide *crack-face* weight functions (that is, for point forces applied on the lips of the crack) but *fully general* weight functions (that is, for point forces applied anywhere in the body). One remarkable feature of Bueckner's method of solution is that he greatly simplified the mathematical treatment by concentrating on the sole feature of interest, namely the distribution of the stress intensity factors (SIFs) along the crack front ("special" method in his terminology), instead of looking for the complete solution of the elasticity problems implied ("general" method).

For a semi-infinite *interface* crack, *i.e.* one lying between two (isotropic) elastic media with different elastic constants, only elementary two-dimensional weight functions are known. The aim of the present paper is to provide three-dimensional ones. Only crack-face weight functions will be calculated. The method of solution will be of "special" type; it is not directly inspired from that of Bueckner (1987), however, but from works of both Gao and Rice (Rice, 1985, Gao and Rice, 1986) and Leblond *et al.* (1995). Considering its complexity, one can reasonably conjecture that any "general" method would be inextricable.

The treatment looks like some kind of big "self-consistent" loop; it consists of two main steps. In the first one, one looks for first-order expressions of the variations of the stress intensity factors along the crack front arising from some small but otherwise arbitrary coplanar perturbation of the crack front. The method is similar to that used in the works of Rice (1985) and Gao and Rice (1986) in that it relies on Rice's formulation of the Bueckner weight function theory, which relates crack-face weight functions to infinitesimal variations of the displacement discontinuity across the crack arising from infinitesimal motions of the crack front. It is more intricate, however, because whereas in the case of a homogeneous body considered by Gao and Rice, the expressions of the crack-face weight functions were known from the beginning, only very basic properties of these functions are known *a priori* for an interface crack. Because of their homogeneity properties, the crack-face weight functions appear in the formulae obtained *in fine* for the variations of the SIFs only through some unknown constants $\gamma_+, \gamma_-, \gamma_{III}, \gamma_z, \gamma$ connected to their asymptotic behaviour when the point of observation of the SIFs goes to infinity.

In the second step, one applies these formulae to some special loadings, namely those which serve for the definition of the crack-face weight functions, and some special motion of the crack front, namely an infinitesimal rotation about the normal to the crack plane. The idea here, which derives from the work of Leblond *et al.* (1995), is that such a movement preserves the *shape* of the crack, although it modifies its *orientation*, so that the resulting variations of the SIFs are expressible in terms of the spatial derivatives of the weight functions. The result consists of integrodifferential equations on these functions, which (of course) involve the constants $\gamma_+, \gamma_-, \gamma_{III}, \gamma_z, \gamma$. Taking the Fourier transforms of these equations in the direction of the crack front, one obtains ordinary differential equations on the Fourier transforms of the weight functions. Performing a first-order expansion with respect to the "bimaterial constant" ϵ, one finds that the solution can be expressed explicitely; this solution of course depends on the constants $\gamma_+, \gamma_-, \gamma_{III}, \gamma_z, \gamma$. Accounting for the fact that (as mentioned above) these quantities are also connected to the Fourier transforms of the weight functions through the asymptotic behaviour of these functions (final "self-consistent" condition), and also for the fact that the Fourier transforms of the weight functions must necessarily vanish at infinity, one gets the values of the constants $\gamma_+, \gamma_-, \gamma_{III}, \gamma_z, \gamma$, and the full (first-order) solution follows from there.

DEFINITIONS AND NOTATIONS

The situation considered is depicted schematically in Fig. 1.a. Materials 1 and 2 occupy the half-spaces $y > 0$ and $y < 0$ respectively. The crack lies on the half-plane $y = 0$, $x < 0$ and the two materials are perfectly bonded on the half-plane $y = 0$, $x > 0$. For any loading, the (real) SIFs $K_I(z)$, $K_{II}(z)$, $K_{III}(z)$ at the point z of the crack front are defined in the same way as in the work of Hutchinson *et al.*; that is, the components of the discontinuity of displacement $[\![\mathbf{u}]\!](x,z) \equiv \mathbf{u}(x,0^+,z) - \mathbf{u}(x,0^-,z)$ at the point (x,z) of the crack are given by

$$\begin{cases} [\![u_y + iu_x]\!](x,z) \sim \dfrac{(1-\upsilon_1)/\mu_1 + (1-\upsilon_2)/\mu_2}{(1/2 + i\epsilon)\cosh(\pi\epsilon)} K(z)\sqrt{\dfrac{|x|}{2\pi}}\,|x|^{i\epsilon}, \; K(z) \equiv (K_I + iK_{II})(z) \\ [\![u_z]\!](x,z) \sim 2\,(1/\mu_1 + 1/\mu_2)\,K_{III}(z)\sqrt{|x|/2\pi} \end{cases} \tag{1}$$

for $|x| \to 0$, where

$$\epsilon \equiv \frac{1}{2\pi} \ell n \frac{\mu_1 + (3-4\upsilon_1)\mu_2}{\mu_2 + (3-4\upsilon_2)\mu_1} \tag{2}$$

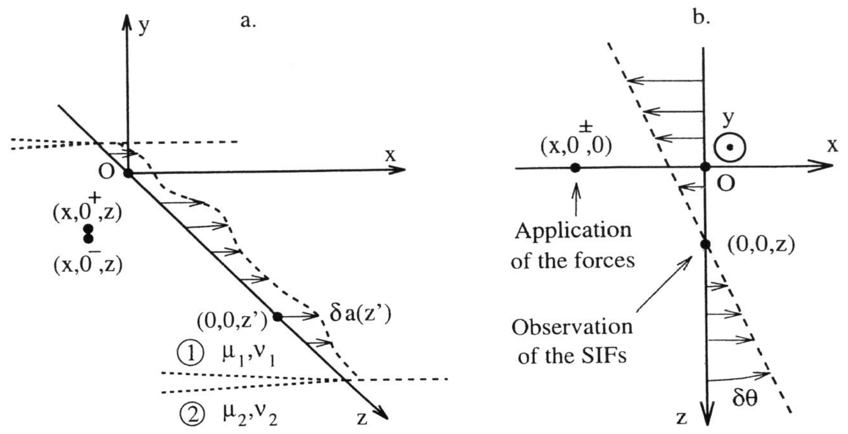

Fig. 1. a: The general problem envisaged. b: Coplanar rotation of the front.

is the *bimaterial constant*. Also, the expression of the local energy release rate $G(z)$ is

$$G(z) = \Lambda |K(z)|^2 + \Lambda' K_{III}^2(z), \; \Lambda \equiv \frac{(1-\upsilon_1)/\mu_1 + (1-\upsilon_2)/\mu_2}{4\cosh^2(\pi\epsilon)}, \; \Lambda' \equiv \frac{1}{4}\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right). \tag{3}$$

Finally, the crack-face weight function $h_{pi}(x,z;z')$ $(p = I, II, III;\, i = x,y,z)$ is defined as the pth SIF generated at the point z' of the crack front by unit point forces exerted on the points $(x, 0^\pm, z)$ of the crack lips in the direction $\pm\mathbf{e}_i$. Because of the invariance of the problem in the direction of the crack front, these weight functions depend on z and z' only through their difference: $h_{pi}(x,z;z') \equiv h_{pi}(x,z'-z)$.

ELEMENTARY PROPERTIES OF WEIGHT FUNCTIONS

Parity with respect to $z' - z$

Since both materials are isotropic, a new solution to the equations of elasticity can be generated from an old one by applying a symmetry with respect to the Oxy plane to the geometry, the loading and the displacements. It is easy to see that this implies that the functions $h_{I\alpha}, h_{II\alpha}$ *($\alpha = x,y$) and h_{IIIz} are even with respect to $z' - z$, whereas the functions h_{Iz}, h_{IIz} and $h_{III\alpha}$ ($\alpha = x,y$) are odd.*

Homogeneity

A new solution can also be obtained by multiplying all distances and displacements by any positive factor λ. In such a transformation, the stresses remain unchanged, and point forces, which are homogeneous to a stress times an area, are multiplied by λ^2. It follows from there and linearity that if all distances are multiplied by λ while point forces are kept unchanged, the displacements are multiplied by λ^{-1}. Equations (1) then imply that the SIFs $K(z)$ and $K_{III}(z)$ arising from these point forces are multiplied by $\lambda^{-3/2-i\epsilon}$ and $\lambda^{-3/2}$ respectively. This means that *the functions $(h_{Ii} + ih_{IIi})(x, z'-z)$ and $h_{IIIi}(x,z'-z)$ $(i = x,y,z)$ are positively homogeneous of degree $-\frac{3}{2} - i\epsilon$ and $-\frac{3}{2}$ respectively.*

Bueckner-Rice's theory

Rice's formulation of the theory of Bueckner's weight functions (see e.g. Rice, 1985) for homogeneous cracked materials can be extended in a completely straightforward manner to the case of interface cracks. One thus considers an initial situation where an arbitrary loading generates some distribution of SIFs $K_I(z), K_{II}(z), K_{III}(z)$ along the crack front, and an infinitesimal coplanar perturbation of the crack front characterized by the orthogonal distance $\delta a(z)$ from the initial crack front to the new one (see Fig.1). Adding some point force $\pm \mathbf{P} = \pm P_i \mathbf{e}_i$ at the points $(x, 0^{\pm}, z)$ of the crack lips and letting $\mathbf{P} \to \mathbf{0}$, one then finds that the variation of the displacement discontinuity across the crack arising from the motion of the crack front (in the absence of the point force) is given, to first order in the perturbation, by

$$\begin{aligned} \delta [\![u_i]\!](x,z) &= \int_{-\infty}^{+\infty} \frac{\partial G}{\partial P_i}(z', \mathbf{P}=\mathbf{0}) \delta a(z') \mathrm{d}z' \\ &= 2 \int_{-\infty}^{+\infty} \{\Lambda\, [K_I(z') h_{Ii}(x, z'-z) + K_{II}(z') h_{IIi}(x, z'-z)] + \\ &\quad + \Lambda' K_{III}(z') h_{IIIi}(x, z'-z)\} \, \delta a(z') \mathrm{d}z', \quad i = x, y, z. \end{aligned} \tag{4}$$

Asymptotic behaviour for $|x| \to 0$

Let us consider the case where the function δa is zero except in a very small interval $]z'-\eta, z'+\eta[$ with centre at $z' \neq z$ and not containing z. The integral $2\int_{-\infty}^{+\infty}\{...\}\delta a(z')\mathrm{d}z'$ in eq. (4) can then be replaced by $\{...\}\delta A$, where $\delta A \equiv 2\int_{z'-\eta}^{z'+\eta} \delta a(z'')\mathrm{d}z''$. Now since δa is zero in the vicinity of z, $\delta[\![\mathbf{u}]\!](x,z)$ behaves in the same way as $[\![\mathbf{u}]\!](x,z)$ for $|x| \to 0$ (see e.g. Rice, 1985); that is, $\delta [\![u_y + i u_x]\!](x,z) \propto |x|^{1/2+i\epsilon}$ and $\delta [\![u_z]\!](x,z) \propto |x|^{1/2}$ where the symbol "$\propto$" means "proportional to". Since the SIFs $K_I(z'), K_{II}(z'), K_{III}(z')$ in the expression $\{...\}$ can be varied independently, this implies that

$$(h_{py} + i h_{px})(x, z'-z) \propto |x|^{1/2+i\epsilon}; \; h_{pz}(x, z'-z) \propto |x|^{1/2} \quad (p = I, II, III) \tag{5}$$

for $|x| \to 0$.

Combination of the previous properties

It follows from what precedes that the function $h_{Iy} + i h_{IIy} + i(h_{Ix} + i h_{IIx}) = h_{Iy} + i h_{Ix} + i(h_{IIy} + i h_{IIx})$ is even with respect to $z' - z$, positively homogeneous of degree $-\frac{3}{2} - i\epsilon$, and behaves like $|x|^{1/2+i\epsilon}$ for $|x| \to 0$. The function $h_{Iy} - i h_{IIy} + i(h_{Ix} - i h_{IIx}) = h_{Iy} + i h_{Ix} - i(h_{IIy} + i h_{IIx})$ verifies similar properties except that its degree of homogeneity is $-\frac{3}{2} + i\epsilon$. Let us therefore put

$$\left\{ \begin{aligned} [h_{Iy} + i h_{IIy} + i(h_{Ix} + i h_{IIx})](x, z'-z) &= [h_{Iy} + i h_{Ix} + i(h_{IIy} + i h_{IIx})](x, z'-z) \\ &\equiv (|x|/2\pi)^{1/2} |x|^{i\epsilon} H_+(x, z'-z); \\ [h_{Iy} - i h_{IIy} + i(h_{Ix} - i h_{IIx})](x, z'-z) &= [h_{Iy} + i h_{Ix} - i(h_{IIy} + i h_{IIx})](x, z'-z) \\ &\equiv (|x|/2\pi)^{1/2} |x|^{i\epsilon} H_-(x, z'-z). \end{aligned} \right. \tag{6}$$

The functions H_+ and H_- are even with respect to $z' - z$ and positively homogeneous of degree $-2 - 2i\epsilon$ and -2 respectively. Furthermore their limits $H_+(0, z'-z), H_-(0, z'-z)$ for $|x| \to 0$ are neither zero nor infinite; since they are also even and positively homogeneous of degree $-2 - 2i\epsilon$ and -2, they are of the form

$$H_+(0, z'-z) \equiv \gamma_+ |z'-z|^{-2-2i\epsilon}; \quad H_-(0, z'-z) \equiv \gamma_-(z'-z)^{-2} \tag{7}$$

where γ_+ and γ_- are unknown complex constants.

Let us also define some functions H_{III}, H_z, H in the following way:

$$\left\{ \begin{aligned} (h_{IIIy} + i h_{IIIx})(x, z'-z) &\equiv (|x|/2\pi)^{1/2} |x|^{i\epsilon} H_{III}(x, z'-z); \\ (h_{Iz} + i h_{IIz})(x, z'-z) &\equiv (|x|/2\pi)^{1/2} H_z(x, z'-z); \\ h_{IIIz}(x, z'-z) &\equiv (|x|/2\pi)^{1/2} H(x, z'-z); \end{aligned} \right. \tag{8}$$

note that the function H is real. Reasonings similar to that just presented show that these functions verify the following properties: *H_{III} and H_z are odd, and H is even, with respect to $z' - z$; they are positively homogeneous of degree $-2 - i\epsilon, -2 - i\epsilon$ and -2 respectively; their limits $H_{III}(0, z'-z), H_z(0, z'-z), H(0, z'-z)$ are of the form*

$$\left\{ \begin{aligned} H_{III}(0, z'-z) &\equiv \gamma_{III} \operatorname{sgn}(z'-z) |z'-z|^{-2-i\epsilon}; \\ H_z(0, z'-z) &\equiv \gamma_z \operatorname{sgn}(z'-z) |z'-z|^{-2-i\epsilon}; \\ H(0, z'-z) &\equiv \gamma (z'-z)^{-2} \end{aligned} \right. \tag{9}$$

where $\operatorname{sgn}(x)$ denotes the sign of x and the unknown constants γ_{III}, γ_z are complex whereas γ is real.

Parity with respect to ϵ

One can also obtain a new solution to the equations of elasticity by applying a symmetry with respect to the Oxz plane to the geometry, the displacements and the loading; it must then be noted, however, that the two materials are interchanged in this transformation so that the sign of the bimaterial constant ϵ changes (see eq. (2)). Detailed inspection of the changes of the various loading conditions and the resulting displacement discontinuities then shows that *$h_{Iy}, h_{IIx}, h_{IIz}, h_{IIIx}, h_{IIIz}$ are even, and $h_{IIy}, h_{IIIy}, h_{Ix}, h_{Iz}$ odd, functions of ϵ.* The definitions (6), (8) of the functions $H_+, H_-, H_{III}, H_z, H$ then imply that they obey the following properties:

$$\left\{ \begin{aligned} & H_{\pm}(-\epsilon; x, z'-z) = \overline{H_{\pm}(\epsilon; x, z'-z)}; \; H(-\epsilon; x, z'-z) = H(\epsilon; x, z'-z) \\ & H_{III}(-\epsilon; x, z'-z) = -\overline{H_{III}(\epsilon; x, z'-z)}; \; H_z(-\epsilon; x, z'-z) = -\overline{H_z(\epsilon; x, z'-z)} \end{aligned} \right. \tag{10}$$

where indications of dependence upon ϵ have temporarily been introduced. By eqs. (7) and (9), the constants $\gamma_+, \gamma_-, \gamma_{III}, \gamma_z, \gamma$ verify similar properties:

$$\gamma_{\pm}(-\epsilon) = \overline{\gamma_{\pm}(\epsilon)}; \; \gamma(-\epsilon) = \gamma(\epsilon); \; \gamma_{III}(-\epsilon) = -\overline{\gamma_{III}(\epsilon)}; \; \gamma_z(-\epsilon) = -\overline{\gamma_z(\epsilon)}. \tag{11}$$

These properties mean that $\mathrm{Re}H_+$, $\mathrm{Re}H_-$, H, $\mathrm{Im}H_{III}$, $\mathrm{Im}H_z$ *are even functions, whereas* $\mathrm{Im}H_+$, $\mathrm{Im}H_-$, $\mathrm{Re}H_{III}$, $\mathrm{Re}H_z$ *are odd functions, of ϵ.* Similarly, $\mathrm{Re}\gamma_+$, $\mathrm{Re}\gamma_-$, γ, $\mathrm{Im}\gamma_{III}$, $\mathrm{Im}\gamma_z$, *considered as functions of ϵ, are even, whereas* $\mathrm{Im}\gamma_+$, $\mathrm{Im}\gamma_-$, $\mathrm{Re}\gamma_{III}$, $\mathrm{Re}\gamma_z$ *are odd.*

VARIATIONS OF THE SIFS RESULTING FROM AN INFINITESIMAL COPLANAR PERTURBATION OF THE CRACK FRONT

Just as in the works of Rice (1985) and Gao and Rice (1986), we begin by considering an infinitesimal coplanar perturbation δa of the crack front which is zero at the point z. The variations $\delta K(z)$, $\delta K_{III}(z)$ of the SIFs at that point can then be directly related (up to some multiplicative factor $|x|^{1/2+i\epsilon}$ or $|x|^{1/2}$) to the variations of the displacement discontinuities $\delta [\![u_y + i u_x]\!](x,z)$,

$\delta \llbracket u_z \rrbracket(x, z)$ for $|x| \to 0$; attention must be paid here to the fact that because of the possible rotation of the crack front, the orthonormal basis "adapted" to the new front at the point z is slightly different from the basis $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ (see Gao and Rice, 1986). Using the fundamental formula (4) of the Bueckner-Rice theory and the definitions (6), (8_1) of the functions H_+, H_- and H_{III}, one thus finds that $\delta K(z)$, for instance, is related to the integrals

$$\int_{-\infty}^{+\infty} H_+(x, z'-z)\,\overline{K(z')}\,\delta a(z')\mathrm{d}z'; \quad \int_{-\infty}^{+\infty} H_-(x, z'-z)\,K(z')\,\delta a(z')\mathrm{d}z';$$
$$\int_{-\infty}^{+\infty} H_{III}(x, z'-z) K_{III}(z')\delta a(z')\mathrm{d}z'$$

in the limit $|x| \to 0$. The problem is thus reduced to the study of such limits.

The treatment of the first two integrals is simple. One writes $\int_{-\infty}^{+\infty}(\ldots)$ as $\int_{\mathbb{R}-[z-\eta, z+\eta]}(\ldots) + \int_{z-\eta}^{z+\eta}(\ldots)$ where η denotes an arbitrary positive number, which is to be shrunk to zero *in fine*. Performing a first-order Taylor expansion of $\overline{K(z')}\,\delta a(z')$ or $K(z')\delta a(z')$ around the point z in the second term and accounting for the fact that the functions H_+ and H_- are even with respect to $z'-z$, one finds that this term vanishes in the limit $\eta \to 0$. Furthermore, by eqs. (7), the first term tends in the limits $|x| \to 0$, then $\eta \to 0$, toward the Cauchy principal value (PV) of the integrals $\int_{-\infty}^{+\infty} \gamma_+ \overline{K(z')}\,|z'-z|^{-2-2i\epsilon}\delta a(z')\mathrm{d}z'$ and $\int_{-\infty}^{+\infty} \gamma_- K(z')(z'-z)^{-2}\delta a(z')\mathrm{d}z'$.

The treatment of the third integral is more complex because the function H_{III} is not even but odd with respect to $z'-z$. One must reason here in two steps. In the first one, one considers some $\delta a(z')$ which is a linear function of $z'-z$ in some interval $[z-\eta, z+\eta]$ with centre at z, and zero outside it. This allows to determine the behaviour of the integral $\int_{z-\eta}^{z+\eta} H_{III}(x, z'-z)(z'-z)\mathrm{d}z'$ for $|x| \to 0$. The second step consists in considering again an arbitrary δa (still supposed to be zero at the point z, however), splitting the original integral in two as above and taking the limits $|x| \to 0$, then $\eta \to 0$, in both terms. One thus finds, using eq. (9_1), that in the limit $|x| \to 0$, that integral is replaced by the finite part, in the sense of Hadamard (FP), of the integral $\int_{-\infty}^{+\infty} \gamma_{III} K_{III}(z')\mathrm{sgn}(z'-z)|z'-z|^{-2-i\epsilon}\delta a(z')\mathrm{d}z'$, *i.e.* the limit, for $\eta \to 0$, of the same integral but taken over $\mathbb{R} - [z-\eta, z+\eta]$ plus the quantity $\frac{2i}{\epsilon}\gamma_{III} K_{III}(z)\frac{\mathrm{d}\delta a}{\mathrm{d}z}(z)\eta^{-i\epsilon}$.

The hypothesis that δa is zero at the point z is finally removed, just as in the works of Rice (1985) and Gao and Rice (1986), by simply taking, as a reference crack front, that obtained by shifting the original one by a uniform amount $\delta a(z)$. The final result for $\delta K(z)$ reads

$$\delta K(z) = \frac{\mathrm{d}K}{\mathrm{d}a}(z)\delta a(z) + \frac{1+2i\epsilon}{8\cosh(\pi\epsilon)} PV \int_{-\infty}^{+\infty} \left[\gamma_+ \frac{\overline{K(z')}}{|z'-z|^{2i\epsilon}} + \gamma_- K(z')\right] \frac{\delta a(z') - \delta a(z)}{(z'-z)^2}\mathrm{d}z'$$
$$+ \frac{1+2i\epsilon}{4\cosh(\pi\epsilon)}\frac{\gamma_{III}}{1-\upsilon}\, FP \int_{-\infty}^{+\infty} K_{III}(z')\,\mathrm{sgn}(z'-z)\frac{\delta a(z') - \delta a(z)}{|z'-z|^{2+i\epsilon}}\mathrm{d}z. \quad (12)$$

In this equation, $(\mathrm{d}K/\mathrm{d}a)(z)$ denotes the derivative of $K(z)$ with respect to the crack length for a *uniform* shift of the crack front equal to $\delta a(z)$ (note that $\delta K(z)$ reduces to $(\mathrm{d}K/\mathrm{d}a)(z)\delta a(z)$ for a uniform δa). Also, υ denotes the quantity defined by

$$1 - \upsilon \equiv \frac{\Lambda}{\Lambda'} = \frac{(1-\upsilon_1)/\mu_1 + (1-\upsilon_2)/\mu_2}{(1/\mu_1 + 1/\mu_2)\cosh^2(\pi\epsilon)} \quad (13)$$

(the notation is coherent in that for a homogeneous material, υ is identical to Poisson's ratio).

A similar reasoning leads to the following formula for $\delta K_{III}(z)$:

$$\delta K_{III}(z) = \frac{\mathrm{d}K_{III}}{\mathrm{d}a}(z)\delta a(z) + \frac{\gamma}{4}\, PV \int_{-\infty}^{+\infty} K_{III}(z')\frac{\delta a(z') - \delta a(z)}{(z'-z)^2}\mathrm{d}z'$$
$$+ \frac{1-\upsilon}{4}\,\mathrm{Re}\left\{\gamma_z\, FP \int_{-\infty}^{+\infty} \overline{K(z')}\,\mathrm{sgn}(z'-z)\frac{\delta a(z') - \delta a(z)}{|z'-z|^{2+i\epsilon}}\mathrm{d}z'\right\}. \quad (14)$$

INTEGRODIFFERENTIAL EQUATIONS ON THE WEIGHT FUNCTIONS

Let us consider the special loading which consists of two unit point forces applied on the points $(x, 0^\pm, 0)$ of the crack lips in the direction $\pm\mathbf{e}_i$. The pth SIF at the point z of the crack front is just, by definition, $h_{pi}(x, z)$. Now let us consider the motion of the crack front defined by $\delta a(z') \equiv \delta\theta\,(z'-z)$ where $\delta\theta$ is an infinitesimal quantity; this motion just represents an infinitesimal rotation of that front about the axis containing the point z and parallel to $\mathbf{e}_y$ (Fig. 1.b). Since the shape of the crack does not change in the transformation, the new SIFs at the point z are still tied to the weight functions. However, in the frame "adapted" to the new orientation of the crack front, the distance (initially $|x|$) from the point of application of the forces to the front has changed by an infinitesimal quantity, and that (initially z) from this point to that of observation of the SIFs, as measured parallel to the crack front, has also been modified. Furthermore there is also a change in the apparent direction of the point forces in the new "adapted" frame. When accounting for all these changes, one finds that the variations of the SIFs at the point z can be expressed in terms of the spatial derivatives of the weight functions, plus these functions themselves (plus, of course, a multiplicative $\delta\theta$ factor). Using the definitions (6), (8) of the functions H_+, H_-, H_{III}, H_z, H and eqs. (12), (14), one thus gets integrodifferential equations on these functions. Since they are all positively homogeneous of various degrees, one can use Euler's relation to eliminate their derivatives with respect to x. Taking $x = -1$ and putting $W_+(u) \equiv H_+(x = -1, z = u)$, $W_-(u) \equiv H_-(x = -1, z = u)$, etc., one finally obtains the following equations:

$$(1+u^2)W_+'(u) + \left(\frac{3}{2} + i\epsilon\right)uW_+(u) + iW_z(u) = \frac{1+2i\epsilon}{8\cosh(\pi\epsilon)} \times$$
$$\left\{PV \int_{-\infty}^{+\infty} \left[\gamma_+ \frac{W_-(u')}{|u'-u|^{2i\epsilon}} + \gamma_- W_+(u')\right]\frac{\mathrm{d}u'}{u'-u} + \frac{2\gamma_{III}}{1-\upsilon} FP \int_{-\infty}^{+\infty} W_{III}(u')\frac{\mathrm{d}u'}{|u'-u|^{1+i\epsilon}}\right\}; \quad (15)$$

$$(1+u^2)W_-'(u) + \left(\frac{3}{2} - i\epsilon\right)uW_-(u) + i\,\overline{W_z(u)} =$$
$$\frac{1-2i\epsilon}{8\cosh(\pi\epsilon)}\left\{PV \int_{-\infty}^{+\infty} \left[\overline{\gamma_+}\, W_+(u')|u'-u|^{2i\epsilon} + \overline{\gamma_-}\, W_-(u')\right]\frac{\mathrm{d}u'}{u'-u}\right. \quad (16)$$
$$\left. + \frac{2\,\overline{\gamma_{III}}}{1-\upsilon}\, FP \int_{-\infty}^{+\infty} W_{III}(u')\frac{\mathrm{d}u'}{|u'-u|^{1-i\epsilon}}\right\};$$

$$(1+u^2)W_{III}'(u) + \frac{3}{2}uW_{III}(u) + i\,W(u) = \frac{\gamma}{4}\, PV \int_{-\infty}^{+\infty} W_{III}(u')\frac{\mathrm{d}u'}{u'-u}$$
$$+ \frac{1-\upsilon}{8}\left\{\gamma_z\, FP \int_{-\infty}^{+\infty} W_-(u')\frac{\mathrm{d}u'}{|u'-u|^{1+i\epsilon}} + \overline{\gamma_z}\, FP \int_{-\infty}^{+\infty} W_+(u')\frac{\mathrm{d}u'}{|u'-u|^{1-i\epsilon}}\right\}; \quad (17)$$

$$(1+u^2)W_z'(u) + \left(\frac{3}{2}+i\epsilon\right) uW_z(u) + \frac{i}{2}W_+(u) - \frac{i}{2}\overline{W_-(u)} = \frac{1+2i\epsilon}{8\cosh(\pi\epsilon)}\times$$
$$\left\{PV\int_{-\infty}^{+\infty}\left[\gamma_+\frac{\overline{W_z(u')}}{|u'-u|^{2i\epsilon}}+\gamma_- W_z(u')\right]\frac{du'}{u'-u} + \frac{2\gamma_{III}}{1-\upsilon}FP\int_{-\infty}^{+\infty} W(u')\frac{du'}{|u'-u|^{1+i\epsilon}}\right\}; \quad (18)$$

$$(1+u^2)W'(u)+\frac{3}{2}uW(u) - \mathrm{Im}\, W_{III}(u) =$$
$$\frac{\gamma}{4}PV\int_{-\infty}^{+\infty} W(u')\frac{du'}{u'-u} + \frac{1-\upsilon}{4}\mathrm{Re}\left\{\overline{\gamma_z}\, FP\int_{-\infty}^{+\infty} W_z(u')\frac{du'}{|u'-u|^{1-i\epsilon}}\right\}. \quad (19)$$

EQUATIONS ON THE FOURIER TRANSFORMS OF THE WEIGHT FUNCTIONS

Differential equations

Let $\widehat{f}(p) \equiv \int_{-\infty}^{+\infty} f(u)e^{ipu}du$ denote the Fourier transform of any function $f(u)$. Taking the Fourier transforms of eqs. (15) to (19), one obtains the following system of ordinary differential equations on the functions $\widehat{W_+}(p)$, $\widehat{W_-}(p)$, $\widehat{W_{III}}(p)$, $\widehat{W_z}(p)$, $\widehat{W}(p)$:

$$p\widehat{W_+}'' + \left(\frac{1}{2}-i\epsilon\right)\widehat{W_+}' - p\widehat{W_+} + \widehat{W_z} = \frac{1+2i\epsilon}{8\cosh(\pi\epsilon)}\left[-\gamma_+\frac{\sinh(\pi\epsilon)}{\epsilon}\Gamma(1-2i\epsilon)\times\right.$$
$$\left.\times\mathrm{sgn}(p)|p|^{2i\epsilon}\widehat{W_-} - \pi\gamma_-\,\mathrm{sgn}(p)\widehat{W_+} + \frac{4\gamma_{III}}{\epsilon(1-\upsilon)}\cosh\left(\frac{\pi\epsilon}{2}\right)\Gamma(1-i\epsilon)|p|^{i\epsilon}\widehat{W_{III}}\right]; \quad (20)$$

$$p\widehat{W_-}'' + \left(\frac{1}{2}+i\epsilon\right)\widehat{W_-}' - p\widehat{W_-} - \overline{\widehat{W_z}} = \frac{1-2i\epsilon}{8\cosh(\pi\epsilon)}\left[-\overline{\gamma_+}\frac{\sinh(\pi\epsilon)}{\epsilon}\Gamma(1+2i\epsilon)\times\right.$$
$$\left.\times\mathrm{sgn}(p)|p|^{-2i\epsilon}\widehat{W_+} - \pi\,\overline{\gamma_-}\,\mathrm{sgn}(p)\widehat{W_-} - \frac{4\,\overline{\gamma_{III}}}{\epsilon(1-\upsilon)}\cosh\left(\frac{\pi\epsilon}{2}\right)\Gamma(1+i\epsilon)|p|^{-i\epsilon}\widehat{W_{III}}\right]; \quad (21)$$

$$p\widehat{W_{III}}'' + \frac{1}{2}\widehat{W_{III}}' - p\widehat{W_{III}} + \widehat{W} = -\frac{\pi\gamma}{4}\mathrm{sgn}(p)\widehat{W_{III}}$$
$$+\frac{1-\upsilon}{4\epsilon}\cosh\left(\frac{\pi\epsilon}{2}\right)\left[\gamma_z\Gamma(1-i\epsilon)|p|^{i\epsilon}\widehat{W_-} - \overline{\gamma_z}\,\Gamma(1+i\epsilon)|p|^{-i\epsilon}\widehat{W_+}\right]; \quad (22)$$

$$p\widehat{W_z}'' + \left(\frac{1}{2}-i\epsilon\right)\widehat{W_z}' - p\widehat{W_z} + \frac{1}{2}\widehat{W_+} - \frac{1}{2}\overline{\widehat{W_-}} = \frac{1+2i\epsilon}{8\cosh(\pi\epsilon)}\left[\gamma_+\frac{\sinh(\pi\epsilon)}{\epsilon}\Gamma(1-2i\epsilon)\times\right.$$
$$\left.\times\mathrm{sgn}(p)|p|^{2i\epsilon}\,\overline{\widehat{W_z}} - \pi\gamma_-\,\mathrm{sgn}(p)\widehat{W_z} + \frac{4\gamma_{III}}{\epsilon(1-\upsilon)}\cosh\left(\frac{\pi\epsilon}{2}\right)\Gamma(1-i\epsilon)|p|^{i\epsilon}\widehat{W}\right]; \quad (23)$$

$$p\widehat{W}''+\frac{1}{2}\widehat{W}'-p\widehat{W}+\mathrm{Re}\widehat{W_{III}} = -\frac{\pi\gamma}{4}\mathrm{sgn}(p)\widehat{W} - \frac{1-\upsilon}{2\epsilon}\cosh\left(\frac{\pi\epsilon}{2}\right)\mathrm{Re}\left[\overline{\gamma_z}\,\Gamma(1+i\epsilon)|p|^{-i\epsilon}\widehat{W_z}\right] \quad (24)$$

where Γ denotes the gamma function and indications of dependence of the functions $\widehat{W_+}$, etc. upon p have been discarded for shortness. The derivation of these equations is elementary except that it requires to evaluate integrals of the form $FP\int_0^{+\infty} e^{ix}dx/x^{1+i\alpha}$, which give rise to the Γ function.

Asymptotic behaviour for $p \to 0$

It is easy to calculate the values of the functions $\widehat{W_+}$, $\widehat{W_-}$, $\widehat{W_{III}}$, $\widehat{W_z}$, $\widehat{W}$ for $p=0$, because they are identical to the integrals, from $-\infty$ to $+\infty$, of the functions W_+, W_-, W_{III}, W_z, W, which

are tied to the well-known values of the *two-dimensional* weight functions for the semi-infinite interface crack.

This information can be refined in the following way. Consider the asymptotic behaviour of $W_+(u)$ for $u \to +\infty$ for instance; because of the homogeneity property of the function $H_+(x,z)$, $W_+(u) \equiv H_+(-1,u) = H_+(-u.u^{-1}, u.1) = u^{-2-2i\epsilon}H_+(-u^{-1},1) \sim u^{-2-2i\epsilon}H_+(0,1) = \gamma_+ u^{-2-2i\epsilon}$ (by eq. (7_1)); similar results hold for the other functions. Also, note that because of the parity properties of these functions, *the Fourier transforms* $\widehat{W_+}$, $\widehat{W_-}$, $\widehat{W}$ *are even whereas* $\widehat{W_{III}}$ *and* $\widehat{W_z}$ *are odd*. Transforming then the integrals from $-\infty$ to $+\infty$ defining the derivatives of these Fourier transforms into integrals from 0 to $+\infty$ and accounting for the behaviour of $W_+(u)$, etc. for $u \to +\infty$, one obtains the asymptotic behaviour of $\widehat{W_+}'(p)$, etc., for $p \to 0$. Accounting for the values of $\widehat{W_+}$, etc. for $p=0$, one finally gets:

$$\widehat{W_+}(p) = -\gamma_+\frac{\sinh(\pi\epsilon)}{\epsilon}\frac{\Gamma(1-2i\epsilon)}{1+2i\epsilon}|p|^{1+2i\epsilon} + o(p);\quad \widehat{W_-}(p) = 4\cosh(\pi\epsilon) - \pi\gamma_-|p| + o(p);$$
$$\widehat{W_{III}}(p) = -2\frac{\gamma_{III}}{\epsilon}\cosh\left(\frac{\pi\epsilon}{2}\right)\frac{\Gamma(1-i\epsilon)}{1+i\epsilon}\mathrm{sgn}(p)|p|^{1+i\epsilon} + C_{III}\,p + o(p);$$
$$\widehat{W_z}(p) = -2\frac{\gamma_z}{\epsilon}\cosh\left(\frac{\pi\epsilon}{2}\right)\frac{\Gamma(1-i\epsilon)}{1+i\epsilon}\mathrm{sgn}(p)|p|^{1+i\epsilon} + C_z p + o(p);\quad \widehat{W}(p) = 2 - \pi\gamma|p| + o(p) \quad (25)$$

where C_{III} and C_z are extra unknown constants.

Equations (25) can be considered as the "self-consistent" conditions which close the system: they say that *in addition to intervening in the differential equations for the functions* $\widehat{W_+}$, *etc., the constants* γ_+, *etc. are also tied to their asymptotic behaviour for* $p \to 0$.

Consequences on the constants γ_-, γ_{III}, γ_z, C_{III}, C_z

Inserting eqs. (25) into eqs. (20) to (24) and identifying terms of dominant order, one gets some informations about the constants γ_-, γ_{III} and γ_z, plus the values of the constants C_{III} and C_z :

$$(1+2i\epsilon)\gamma_- \in \mathbb{R};\ (1+2i\epsilon)\gamma_{III}+(1-\upsilon)\cosh(\pi\epsilon)\gamma_z = 0;\ C_{III} = -4;\ C_z = \frac{4\cosh(\pi\epsilon)}{1-2i\epsilon}. \quad (26)$$

ZEROTH ORDER SOLUTION

Since the bimaterial constant ϵ is always small in practice, it is reasonable to look for an expansion of the solution of eqs. (20) to (26) in powers of that parameter: $X \equiv X^0 + \epsilon X^1 + \epsilon^2 X^2 + \ldots$ where X denotes any quantity. At order 0, *i.e.* for $\epsilon = 0$, this solution is expected to be identical to the well-known one for a homogeneous material, that is, with the present notations:

$$W_+^0(u) = -\frac{4\upsilon}{\pi(2-\upsilon)}\frac{1-u^2}{(1+u^2)^2};\quad W_-^0(u) = \frac{4}{\pi(1+u^2)} + \frac{4\upsilon}{\pi(2-\upsilon)}\frac{1-u^2}{(1+u^2)^2};$$
$$W_{III}^0(u) = W_z^0(u) = i\frac{8\upsilon}{\pi(2-\upsilon)}\frac{u}{(1+u^2)^2};\quad W^0(u) = \frac{2}{\pi(1+u^2)} - \frac{4\upsilon}{\pi(2-\upsilon)}\frac{1-u^2}{(1+u^2)^2}, \quad (27)$$

which implies, by the definition of the Fourier transform chosen and eqs. (25) and (26), that

$$\begin{cases}\left(\widehat{W_+^0}, \widehat{W_-^0}, \widehat{W_{III}^0}, \widehat{W_z^0}, \widehat{W^0}\right)(p) = e^{-|p|}\left(-\frac{4\upsilon|p|}{2-\upsilon}, 4+\frac{4\upsilon|p|}{2-\upsilon}, -\frac{4\upsilon p}{2-\upsilon}, -\frac{4\upsilon p}{2-\upsilon}, 2-\frac{4\upsilon|p|}{2-\upsilon}\right)\\ (\gamma_+^0, \gamma_-^0, \gamma_{III}^0, \gamma_z^0, \gamma^0) = \frac{1}{\pi(2-\upsilon)}(4\upsilon, 8-8\upsilon, 0, 0, 4+2\upsilon);\ (\gamma_{III}^1, \gamma_z^1) = \frac{1}{2-\upsilon}(-4+4\upsilon, 4)\end{cases} \quad (28)$$

(note that knowledge of the zeroth order functions $\widehat{W^0_{III}}$ and $\widehat{W^0_z}$ does not only imply knowledge of the constants γ^0_{III} and γ^0_z $(=0)$ but also of the first order ones γ^1_{III} and γ^1_z, since γ_{III} and γ_z are divided by ϵ in eqs. $(25_{3,4})$). It is indeed easy, though tedious, to check that this solution does satisfy eqs. (20) to (24) for $\epsilon = 0$. The real or imaginary character of the functions W^0_+, etc. and the constants γ^0_+, etc. could be anticipated *a priori* from the parity properties with respect to ϵ mentioned above.

FIRST ORDER SOLUTION

Since $W(u) \equiv H(-1, u)$ is even with respect to ϵ, $W^1(u)$ is zero, so that its Fourier transform $\widehat{W^1}(p)$ is also zero; similarly, γ^1 is zero. Accounting for these properties and expanding eqs. (20) to (23) to first order in ϵ, one gets

$$
\begin{aligned}
&p\widehat{W^1_+}'' + \frac{1}{2}\widehat{W^1_+}' + \left(\frac{\pi}{8}\gamma^0_- - p\right)\widehat{W^1_+} + \widehat{W^1_z} + \frac{\pi}{8}\gamma^0_+\widehat{W^1_-} - \frac{\gamma^1_{III}}{2(1-\upsilon)}\widehat{W^1_{III}} = i\widehat{W^0_+}' - i\frac{\pi}{4}\gamma^+_0 \times \\
&\widehat{W^0_-}\ell\text{np} - \frac{\pi}{8}(2i(C+1)\gamma^0_+ + \gamma^1_+)\widehat{W^0_-} + i\frac{\gamma^1_{III}}{2(1-\upsilon)}\widehat{W^0_{III}}\ell\text{np} + \frac{i(C+2)\gamma^1_{III} + \gamma^2_{III}}{2(1-\upsilon)}\widehat{W^0_{III}};
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
&p\widehat{W^1_-}'' + \frac{1}{2}\widehat{W^1_-}' + \left(\frac{\pi}{8}\gamma^0_- - p\right)\widehat{W^1_-} + \widehat{W^1_z} + \frac{\pi}{8}\gamma^0_+\widehat{W^1_+} + \frac{\gamma^1_{III}}{2(1-\upsilon)}\widehat{W^1_{III}} = -i\widehat{W^0_-}' + i\frac{\pi}{4}\gamma^+_0 \times \\
&\widehat{W^0_+}\ell\text{np} + \frac{\pi}{8}(2i(C+1)\gamma^0_+ + \gamma^1_+)\widehat{W^0_+} + i\frac{\gamma^1_{III}}{2(1-\upsilon)}\widehat{W^0_{III}}\ell\text{np} + \frac{i(C+2)\gamma^1_{III} + \gamma^2_{III}}{2(1-\upsilon)}\widehat{W^0_{III}};
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
&p\widehat{W^1_{III}}'' + \frac{1}{2}\widehat{W^1_{III}}' + \left(\frac{\pi}{4}\gamma^0 - p\right)\widehat{W^1_{III}} + \frac{(1-\upsilon)\gamma^1_z}{4}\left(\widehat{W^1_+} - \widehat{W^1_-}\right) = \\
&\frac{1-\upsilon}{4}\left(i\gamma^1_z\ell\text{np} + iC\gamma^1_z + \gamma^2_z\right)\left(\widehat{W^0_+} + \widehat{W^0_-}\right);
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
&p\widehat{W^1_z}'' + \frac{1}{2}\widehat{W^1_z}' + \left(\frac{\pi}{8}(\gamma^0_+ + \gamma^0_-) - p\right)\widehat{W^1_z} + \frac{1}{2}(\widehat{W^1_+} + \widehat{W^1_-}) = i\widehat{W^0_z}' + i\frac{\pi}{4}\gamma^0_+\widehat{W^0_z}\ell\text{np} \\
&+\frac{\pi}{8}(2i(C+1)\gamma^0_+ + \gamma^1_+)\widehat{W^0_z} + i\frac{\gamma^1_{III}}{2(1-\upsilon)}\widehat{W^0}\ell\text{np} + \frac{i(C+2)\gamma^1_{III} + \gamma^2_{III}}{2(1-\upsilon)}\widehat{W^0}
\end{aligned} \tag{32}
$$

where eq. (26_1) has been used and C is Euler's constant (the appearance of which arises from the property $\Gamma'(1) = -C$). Only positive values of p are considered here for simplicity; this is sufficient because of the parity properties of the Fourier transforms $\widehat{W_+}$, etc. with respect to p mentioned above.

Quite remarkably, the system of coupled equations (29) to (32) on the functions $\widehat{W^1_+}$, $\widehat{W^1_-}$, $\widehat{W^1_{III}}$, $\widehat{W^1_z}$ can be uncoupled by considering the functions $\widehat{P}$, $\widehat{Q}$, $\widehat{R}$, $\widehat{S}$ defined, for $p > 0$, by

$$
\begin{aligned}
&\widehat{P}(p) \equiv (\widehat{W^1_+} + \widehat{W^1_-} - 2\widehat{W^1_z})(p);\ \widehat{Q}(p) \equiv (\widehat{W^1_+} + \widehat{W^1_-} + 2\widehat{W^1_z})(p); \\
&\widehat{R}(p) \equiv (\widehat{W^1_+} - \widehat{W^1_-} - 2\widehat{W^1_{III}})(p);\ \widehat{S}(p) \equiv \left((1-\upsilon)(\widehat{W^1_+} - \widehat{W^1_-}) + 2\widehat{W^1_{III}}\right)(p);
\end{aligned} \tag{33}
$$

indeed, combination of eqs. (29) to (32) yields the following equations (for $p > 0$) on these functions:

$$
\left\{
\begin{aligned}
&\left[p\frac{d^2}{dp^2} + \frac{1}{2}\frac{d}{dp} - \left(\frac{1}{2} + p\right)\right]\begin{pmatrix}\widehat{P}(p)\\ \widehat{R}(p)\end{pmatrix} = \begin{pmatrix}\mathcal{P}(p)\\ \mathcal{R}(p)\end{pmatrix} \\
&\left[p\frac{d^2}{dp^2} + \frac{1}{2}\frac{d}{dp} + \frac{3}{2} - p\right]\begin{pmatrix}\widehat{Q}(p)\\ \widehat{S}(p)\end{pmatrix} = \begin{pmatrix}\mathcal{Q}(p)\\ \mathcal{S}(p)\end{pmatrix}
\end{aligned}
\right. \tag{34}
$$

where $\mathcal{P}(p)$, $\mathcal{Q}(p)$, $\mathcal{R}(p)$, $\mathcal{S}(p)$ are "second members" the expression of which is easily deduced from the right-hand sides of eqs. (29) to (32). Remarkably again, each of these equations can be integrated analytically by elementary means. The solution for the function $\widehat{P}$, for instance, reads

$$
\widehat{P}(p) = e^p \int_0^p \frac{e^{-2q}}{\sqrt{q}}dq \int_0^q \frac{\mathcal{P}(r)e^r}{\sqrt{r}}dr + Ae^p \int_0^p \frac{e^{-2q}}{\sqrt{q}}dq + A'e^p \tag{35}
$$

where A and A' are arbitrary constants, and a similar formula of course holds for $\widehat{R}$. Examination of the behaviour of $\widehat{P}(p)$ for $p \to 0^+$ reveals that A and A' must be zero. Since the term in factor of e^p must then vanish for $p \to +\infty$ (otherwise $\widehat{P}(p)$ would not tend to zero at infinity), one gets the following necessary condition on the "second member" $\mathcal{P}$:

$$
\int_0^{+\infty} \frac{e^{-2q}}{\sqrt{q}}dq \int_0^q \frac{\mathcal{P}(r)e^r}{\sqrt{r}}dr = 0; \tag{36}
$$

a similar condition holds for $\mathcal{R}$. Also, the solution for $\widehat{Q}$ is

$$
\widehat{Q}(p) = e^{-p}\sqrt{p}\int_0^p \frac{e^{2q}}{q^{3/2}}dq \int_0^q \mathcal{Q}(r)e^{-r}dr + Be^{-p}\sqrt{p}\int_0^p \frac{e^{2q}}{\sqrt{q}}dq - \frac{B}{2}e^p + B'e^{-p}\sqrt{p} \tag{37}
$$

and similarly for $\widehat{S}$. Again, the constants B and B' must be zero for $\widehat{Q}(p)$ to behave as desired for $p \to 0^+$. It is then easy to see that the condition

$$
\int_0^{+\infty} \mathcal{Q}(r)e^{-r}dr = 0 \tag{38}
$$

is necessary for $\widehat{Q}$ to vanish at infinity; a similar result holds for $\mathcal{S}$.

Writing down explicitly conditions (36), (38) for the "second members" $\mathcal{P}, \mathcal{R}, \mathcal{Q}, \mathcal{S}$ and using eqs. $(26_{1,2})$, one gets, after some algebra, the values of the unknown constants γ^1_+, γ^1_-, γ^2_{III}, γ^2_z :

$$
\gamma^1_+ = i\frac{8\ \ell\text{n}\ 2}{\pi}\frac{\upsilon}{2-\upsilon};\ \gamma^1_- \doteq -i\frac{16}{\pi}\frac{1-\upsilon}{2-\upsilon};\ \gamma^2_{III} = 4i(1-\ell\text{n}\ 2)\frac{1-\upsilon}{2-\upsilon};\ \gamma^2_z = 4i\frac{1+\ell\text{n}\ 2}{2-\upsilon}. \tag{39}
$$

The "second members" $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}$ are then fully known; from there follow the functions $\widehat{P}$, $\widehat{Q}$, $\widehat{R}$, $\widehat{S}$, then the functions $\widehat{W^1_+} + \widehat{W^1_-}$, $\widehat{W^1_+} - \widehat{W^1_-}$, $\widehat{W^1_{III}}$, $\widehat{W^1_z}$, and finally the functions $W^1_+ + W^1_-$, $W^1_+ - W^1_-$, W^1_{III}, W^1_z. The calculations required are long and the Fourier inversion is somewhat tricky. The final results read as follows:

$$
\begin{aligned}
(W^1_+ + W^1_-)(u) = \frac{2i}{\pi}\Bigg\{&\text{Re}\ \chi(u) + \frac{1}{1+u^2}\ \ell\text{n}\frac{1+u^2}{4} \\
&+\frac{2\upsilon}{2-\upsilon}\frac{1}{(1+u^2)^2}\left[2(u^2-1) + (1-u^2)\ \ell\text{n}\frac{1+u^2}{4} + 4u\tan^{-1}u\right]\Bigg\};
\end{aligned} \tag{40}
$$

$$
(W^1_+ - W^1_-)(u) = -\frac{4i}{\pi(2-\upsilon)}\left[(1-\upsilon)\text{Re}\ \chi(u) + \frac{1}{1+u^2}\ \ell\text{n}\frac{1+u^2}{4}\right]; \tag{41}
$$

$$W^1_{III}(u) = \frac{2(1-\upsilon)}{\pi(2-\upsilon)}\left[\mathrm{Im}\,\chi(u) - \frac{2}{1+u^2}\tan^{-1}u\right]; \tag{42}$$

$$W^1_z(u) = \frac{1}{\pi}\left\{-\mathrm{Im}\,\chi(u) + \frac{2}{1+u^2}\tan^{-1}u + \frac{4\upsilon}{2-\upsilon}\frac{1}{(1+u^2)^2}\left[-2u + u\,\ell n\frac{1+u^2}{4} + (u^2-1)\tan^{-1}u\right]\right\}, \tag{43}$$

where

$$\chi(u) \equiv \frac{i}{(1+iu)^{3/2}}\int_{-\infty}^{u} \ell n\left(\frac{1+it}{2}\right)\frac{dt}{(1+it)^{1/2}(1-it)}. \tag{44}$$

(This expression cannot be put in a more explicit form since it is easily shown to be reducible to indefinite integrals of the form $\int \frac{\ell n x}{x+a}dx$, which are known not to be expressible in terms of elementary functions). Again, the real or imaginary character of the various constants and functions could be predicted *a priori*.

Since the functions H_+, etc. are homogeneous of various degrees, they can be expressed in terms of their restrictions W_+, etc. to the line $x = -1$; from there and eqs. (6), (8) follow the final expressions of the crack-face weight functions (to first order in ϵ) :

$$\left\{\begin{array}{l} (h_{Iy} + ih_{IIy})(x,z) = \frac{1}{2\sqrt{2\pi}|x|^{3/2}}\left[(1 - i\epsilon\,\ell n|x|)(W^0_+ + W^0_-) + \epsilon(W^1_+ - W^1_-)\right] \\ (h_{Ix} + ih_{IIx})(x,z) = -\frac{i}{2\sqrt{2\pi}|x|^{3/2}}\left[(1 - i\epsilon\,\ell n|x|)(W^0_+ - W^0_-) + \epsilon(W^1_+ + W^1_-)\right] \\ (h_{IIIy} + ih_{IIIx})(x,z) = \frac{1}{\sqrt{2\pi}|x|^{3/2}}\left[W^0_{III} + \epsilon W^1_{III}\right] \\ (h_{Iz} + ih_{IIz})(x,z) = \frac{1}{\sqrt{2\pi}|x|^{3/2}}\left[(1 - i\epsilon\,\ell n|x|)W^0_z + \epsilon W^1_z\right] \\ h_{IIIz}(x,z) = \frac{1}{\sqrt{2\pi}|x|^{3/2}}W^0 \end{array}\right. \tag{45}$$

where all functions W^0_+, etc., the expressions of which have been given above (eqs. (27) and (40) to (44)), are to be taken at the point $u \equiv z/|x|$.

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