THE UNIFIED EQUIVALENT STRESS INTENSITY FACTOR K_θ

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ABSTRACT

In conventional linear elastic fracture mechanics (LEFM), Irwin's (1957) modes I and II stress intensity factors, that is K_I and K_{II}, are adopted. Thus, according to the traditional textbook approach, the stress intensity factors are referred to the θ_0 plane when determining the near field stresses as well as when conducting crack closure analysis. While this approach is adequate from the point of view of determining the stresses in general, it is restrictive in that only self-similar crack propagation may strictly be considered thereby (Lo et al., 1996a). Hence, as a consequence of the apparent oversight of the constraint on Irwin's derivation of the stress intensity factor, an anomaly has arisen in the subsequent traditional approach to shear "fracture". This may be broadly attributed to a rigid adherence to Irwin's notion of a unique stress intensity factor for a given boundary value problem, when taken in conjunction with the association of the mode of fracture (which is actually an opening fracture in the θ_C=−70·5° plane) with the applied shear loading (as referred to the θ_0 plane), as an unqualified extension of Irwin's shear failure criterion for the θ_{0C} plane. Indeed, the latter tendency to associate mode of fracture with mode of loading unconditionally has been unjustifiably carried over to the case of mixed mode fracture. Moreover, in considering both modes of loading, the situation has been aggravated by the effective adoption of K_{IC} as the sole basis for determining fracture. A more recent development has been the application of crack kink analysis to address non self-similar crack propagation. The apparent reason for doing so has - in contrast with traditional non-"kink" analysis - been to retain the notion of an Irwin type of stress intensity factor attaining its critical value as the basis for determining fracture, albeit referred to the generalised θ plane. However, the determination of stress intensity factor by this method of analysis is unduly elaborate, and furthermore the tendency so far has been to focus on the pure mode I fracture toughness only (Lo et al., 1996b), similarly as in the case of the traditional approach. On the other hand, the following discussion will show that by adopting the *unified, equivalent stress intensity factor K_θ*, which for a given boundary value problem would vary from one θ plane to another but remain constant for a particular θ plane, a *unified model* would evolve, which would be able to predict pure as well as mixed mode fracture propagation consistently.

KEYWORDS

Unified model, unified stress intensity factors, mixed mode fracture toughness, brittle clay, aluminium alloy.

INTRODUCTION

It has been shown by Lo et al. (1996a) that Irwin's concept of the modes I and II stress intensity factors may be generalised in terms of the expressions

$$K_I = \lim_{r \to 0} \sigma_{\theta\theta}\sqrt{2\pi r} \quad (1)$$

and

$$K_{II} = \lim_{r \to 0} \tau_{r\theta}\sqrt{2\pi r} \quad (2)$$

respectively, where $\sigma_{\theta\theta}$ is the circumferential stress and $\tau_{r\theta}$ the shear stress, in polar coordinates, and r is the radial distance from the crack tip. However, it appears that Irwin's interpretation of the stress intensity factor is strictly pertinent to the θ_0 plane only, and the extension of its relevance to some other θ plane - which is untenable - has led to various inconsistencies in the subsequent development of fracture mechanics. The oversight of the sole emphasis placed by Irwin's derivation on the θ_0 plane seems to have been perpetrated, in turn, by his implicit assumption that the stress intensity factors K_I and K_{II} are absolute constants of a given boundary value problem, when taken in conjunction with his postulate - based on corresponding closure analyses of the θ_{0C} plane - that a crack would extend whenever K_I or K_{II} reached its critical value of K_{IC} or K_{IIC}, respectively. However, as borne out by the following discussion, not only are the factors strictly pertinent to the θ_0 plane only, and would therefore be more explicitly denoted as K_{I0} and K_{II0} - where the subscript "0" would refer to the same plane, but also the stress intensity factors $K_{I\theta}$ and $K_{II\theta}$, which arise from the formulation of the proposed unified model (vide equations (5) and (6)), would vary with angle θ - where $-\pi \le \theta \le \pi$, and in so doing, constitute appropriate parameters for predicting fracture along the generalised θ_C plane.

In addition, it appears that the more recent development of crack kink analysis (Cotterell and Rice, 1980; Hayashi and Nemat-Nasser, 1981) may have been a reaction to the above lack of consensus in traditional fracture mechanics. Accordingly, whereas the notion of an Irwin type of stress intensity factor, which is referred to the current crack extension plane, is maintained as the basis for determining crack propagation, it is, in contrast with the traditional approach, applied to both self-similar and non self-similar crack extensions. In doing so, however, the closure analysis of a non self-similar crack extension becomes unnecessarily elaborate, in that it is necessary to provide an à priori kinked crack, thereby distinguishing it from that of a self-similar crack extension - which is effectively allowed to take place directly from the existing crack tip. On the other hand, non self-similar crack propagation may be addressed at source, that is on the basis of the state of the near field of an existing crack tip, which would thereby be extant, instead of having to analyse the rather more complex boundary value problem of a crack in its kinked state - for which there is no particular conceptual justification, only to have it drawn back to the existing crack tip subsequently in order to simulate the actual propagation (Lo et al., 1996b). Moreover, there has been an undue tendency to focus on the mode I fracture toughness alone - as also in the case of the traditional approach to fracture mechanics.

In view of the above anomalies of conventional fracture mechanics, a unified model will be presented herein, which will essentially be a generalisation of Irwin's work on pure mode fracture along the θ_{0C} plane, so as to cater for either pure or mixed mode fracture along the generalised θ_C plane, where $-\pi \le \theta_C \le \pi$. By means of the model, the foregoing anomalies may be rationalised and hence redressed - a detailed account of which is provided in a recent publication (Lo et al., 1996a). The model has also been verified against known experimental results, a summary of which is depicted by Figs. 2 and 3 of the following discussion.

THE UNIFIED PURE MODE STRESS INTENSITY FACTORS $K_{I\theta}$ AND $K_{II\theta}$

As indicated above, there is no justification in distinguishing between the analysis of a self-similar and non self-similar crack extension. Furthermore, in determining the relationship between critical rate of energy release and fracture toughness, via Irwin's crack closure analysis of the θ_0 plane, the particular definitions of stress intensity factor of equations (1) and (2) are implied. On the above premise, a crack closure analysis of the generalised θ plane will be carried out in the following discussion, and the unified, equivalent stress intensity factor, K_θ, which is based on the proposed unified, pure mode stress intensity factors, $K_{I\theta}$ and $K_{II\theta}$, deduced.

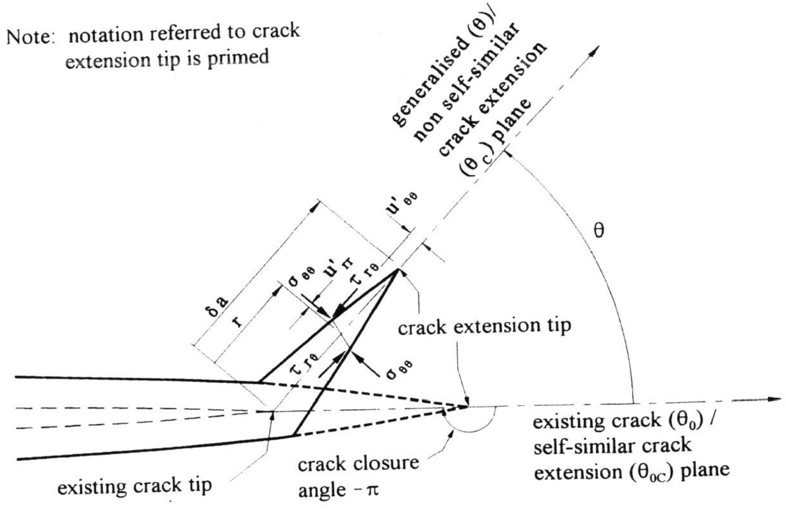

Fig. 1 Closure parameters of crack tip.

To begin with, consider the closure of the crack extension in Fig. 1, which is orientated in the generalised θ direction and subjected to mixed modes I and II loading, for which

$$G_\theta = \frac{1}{\delta a}\int_0^{\delta a} [\sigma_{\theta\theta}(r,\theta)u'_{\theta\theta}(\delta a - r, -\pi) + \tau_{r\theta}(r,\theta)u'_{rr}(\delta a - r, -\pi)]\,dr\,, \quad (3)$$

where G_θ is the mixed mode rate of energy release in the θ plane, "a" the existing crack length, θ the direction of the plane of interest with respect to existing crack plane θ_0=0, and $u'_{\theta\theta}$ and u'_{rr} the circumferential and radial displacements in polar coordinates, respectively. Next, substituting the expressions given by Irwin (1958) for $\sigma_{\theta\theta}$, $u'_{\theta\theta}$, $\tau_{r\theta}$ and u'_{rr} in equation (3), we would have

$$G_\theta = \frac{1}{\delta a}\int_0^{\delta a}\left(\left[\frac{K_{I\theta}}{\sqrt{2\pi r}}\right]\left[\frac{(1+\nu)(1+\kappa)}{4}\sqrt{\frac{\delta a-r}{2\pi}}K'_I\right] + \left[\frac{K_{II\theta}}{\sqrt{2\pi r}}\right]\left[\frac{(1+\nu)(1+\kappa)}{4}\sqrt{\frac{\delta a-r}{2\pi}}K'_{II}\right]\right)dr\ , \tag{4}$$

where ν is Poisson's ratio and κ a known function of ν. Note that equation (4) is in the same format as the traditional analysis of the θ_{0C} plane, except that in view of equation (1), and by the same token equation (2), $K_{I\theta}$ and $K_{II\theta}$ would represent the modes I and II stress intensity factors corresponding to the generalised θ plane, given as

$$K_{I\theta} = \lim_{r\to0}\ \sigma_{\theta\theta}\sqrt{2\pi r} = K_I\cos^3\frac{\theta}{2} - 3K_{II}\sin\frac{\theta}{2}\cos^2\frac{\theta}{2} \tag{5}$$

and

$$K_{II\theta} = \lim_{r\to0}\ \tau_{r\theta}\sqrt{2\pi r} = K_I\sin\frac{\theta}{2}\cos^2\frac{\theta}{2} + K_{II}\cos\frac{\theta}{2}\left(1-3\sin^2\frac{\theta}{2}\right) \tag{6}$$

respectively, rather than Irwin's K_I and K_{II} which pertain to the θ_{0C} plane only, while

$$K'_I = \lim_{(r-\delta a)\to0}\ \sigma_{\theta\theta}\sqrt{2\pi(r-\delta a)} \tag{7}$$

and

$$K'_{II} = \lim_{(r-\delta a)\to0}\ \tau_{r\theta}\sqrt{2\pi(r-\delta a)}\ , \tag{8}$$

also similarly as in the traditional analysis of the θ_{0C} plane, although now referred to the generalised θ plane.

In other words, $K_{I\theta}$ and $K_{II\theta}$ are the counterparts, in the generalised θ plane, to the traditional stress intensity factors K_I and K_{II}, of the θ_0 plane, and therefore a necessary condition would be that when $\theta = 0$, $K_{I\theta} = K_I$ and $K_{II\theta} = K_{II}$, which is indeed confirmed by corresponding equations (5) and (6), respectively. Hence, as indicated earlier, K_I and K_{II} would be more appropriately referred to as K_{I0} and K_{II0} respectively, where the additional subscript "0" would refer specifically to the θ_0 plane.

On the above basis, since $K'_I = K_{I\theta}$ and $K'_{II} = K_{II\theta}$ when $\delta a \to 0$, we would have

$$G_\theta = \frac{(1+\nu)(1+\kappa)}{4E}\left(K_{I\theta}^{\ 2} + K_{II\theta}^{\ 2}\right)\ , \tag{9}$$

where "E" is Young's modulus. Furthermore, since, for pure mode I crack propagation in the generalised θ_C plane, $K_{II\theta} = 0$, and $K_{I\theta} \to K_{IC}$ as $G_\theta \to G_{IC}$, we would obtain

$$G_{IC} = \frac{(1+\nu)(1+\kappa)}{4E}K_{IC}^{\ 2}\ , \tag{10}$$

as in the case of Irwin's (1957) analysis of mode I crack propagation along the θ_{0C} plane. Likewise, for pure mode II crack propagation along the generalised θ_C plane, $K_{I\theta} = 0$, and $K_{II\theta} \to K_{IIC}$ as $G_\theta \to G_{IIC}$, so that

$$G_{IIC} = \frac{(1+\nu)(1+\kappa)}{4E}K_{IIC}^{\ 2}\ , \tag{11}$$

as in the case of Irwin's analysis of mode II crack propagation along the θ_{0C} plane. Hence, the fundamental, physical requirement, that the relationships specified in equations (10) and (11) should be applicable to any θ_C plane in an isotropic, homogeneous medium, may be satisfied by the adoption of the unified stress intensity factors $K_{I\theta}$ and $K_{II\theta}$, as defined by equations (5) and (6) respectively.

In contrast, the traditional notion that $K_I = K_{I0} = K_{IC}$ or $K_{II} = K_{II0} = K_{IIC}$ unconditionally, when fracture takes place due to an applied, pure opening or shearing mode of loading in the far field (as referred to the θ_0 plane), respectively, would be untenable in that, for such fractures to occur in the generalised θ_C plane, *the relevant physical relationship given by either equation (12) or (10) could not be satisified by mere substitution of the corresponding traditional fracture criterion, indicated above, in the right member expression of equation (4).* Thus, it would appear that, for a given boundary value problem, Irwin implicitly extended his proposition of K_I and K_{II} as absolute constants for determining the near field stresses of an existing crack tip, to K_I and K_{II} as absolute constants which may be employed in determining crack propagation when $K_I = K_{IC}$ or $K_{II} = K_{IIC}$, when he found by closure analysis that K_{IC} and K_{IIC} could be considered as material constants on an equal footing with G_{IC} and G_{IIC} respectively, not realising that his derivation of stress intensity factor was, from the outset, restricted to the θ_0 plane.

THE MIXED MODE FRACTURE TOUGHNESS K_C AND UNIFIED EQUIVALENT STRESS INTENSITY FACTOR K_θ

In general, a mixed mode of fracture could occur under either a pure or mixed mode of applied loading, in which case, referring to equation (9), we could write

$$G_C = \frac{1}{4E}(1+\nu)(1+\kappa)K_C^2\ , \tag{12}$$

where G_C would therefore represent the mixed mode critical rate of energy release and K_C the mixed mode fracture toughness, as defined by the unified model. Thus, by the same token as for the relationships of equations (10) and (11), K_C may be construed as a valid alternative parameter to G_C for determining crack propagation. Consequently, the traditional fracture parameters K_{IC}, K_{IIC}, G_{IC} and G_{IIC}, which have hitherto been strictly applicable to pure mode fracture in the θ_{0C} plane only, would be generalised in terms of the unified parameters K_C and G_C, for mixed mode fracture along an arbitrary θ_C plane. Furthermore, in view of equations (9) and (12), we would have

$$K_{I\theta}^2 + K_{II\theta}^2 = K_C^2 \tag{13}$$

at fracture, where the values of $K_{I\theta}^{\ 2}$ and $K_{II\theta}^{\ 2}$ in the left member of equation (13), which is indicative of the loading energy, may be obtained via equations (5) and (6) respectively. On the other hand, the right member parameter reflects the fracture energy, which is a material property (for simplicity of reference, the indicative energy terms on the left and right members of equation (13) will hereinafter be abbreviated as "loading energy" and "fracture energy", respectively). In determining the mixed mode fracture toughness, K_C, it is evident that the limiting condition of pure mode I fracture along the generalised θ_C plane - for which $K_{II\theta}$=0 and therefore

$$K_{I\theta} = K_{IC} \tag{14}$$

- and that of pure mode II fracture along the same plane - for which $K_{I\theta}=0$ and hence

$$K_{II\theta} = K_{IIC} \tag{15}$$

- would have to be satisfied initially. Thereafter, an appropriate variation of K_C would have to be prescribed between the two limiting conditions. One possible approach would be to convert the component pure mode "loading energy" terms of equation (13) into equivalent mixed mode "loading energy", in direct proportion to their respective "fracture energies", that is

$$K_{I\theta}^2 + K_{II\theta}^2 = K_{I\theta}^2 \frac{K_C^2}{K_{IC}^2} + K_{II\theta}^2 \frac{K_C^2}{K_{IIC}^2} \, . \tag{16}$$

Hence, in view of equation (13), the mixed mode fracture criterion would be defined as

$$\left(\frac{K_{I\theta}}{K_{IC}}\right)^2 + \left(\frac{K_{II\theta}}{K_{IIC}}\right)^2 = 1 \tag{17}$$

(vide Fig. 2). Furthermore, since $K_{I\theta}$ and $K_{II\theta}$ would be known quantities, it would only be necessary to determine the fracture angle, θ_C, in order to establish the point of fracture on the corresponding envelope. This may be achieved by maximising the loading energy, a detailed account of which is provided elsewhere (Lo et al., 1996a). However, notwithstanding the foregoing derivation, Lo et al. also proposed a more generalised form of the fracture envelope which would, in principle, cater for practically any form of brittle material behaviour. Nevertheless, it is noteworthy that the unified fracture envelope and corresponding fracture surface of Figs. 2 and 3, respectively, satisfy a wide range of material behaviour, as exhibited by brittle clay, plexiglass, aluminum alloy and steel.

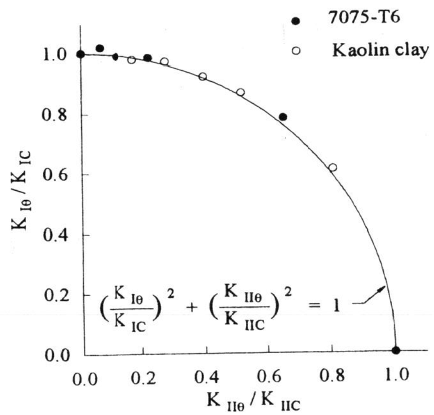

Fig. 2 Unified envelope for 7075-T6 aluminium and brittle kaolin clay.

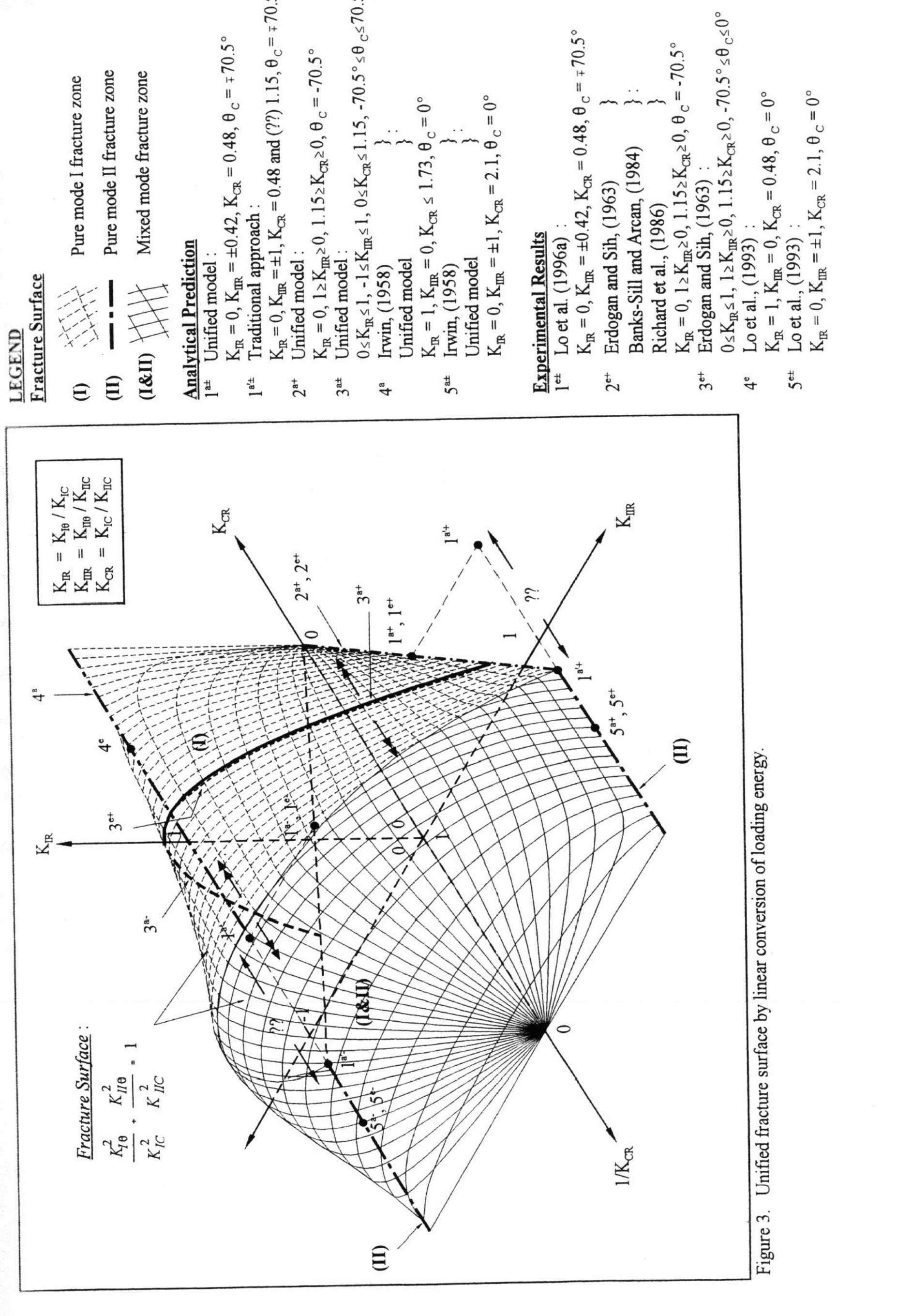

Figure 3. Unified fracture surface by linear conversion of loading energy.

As a final point, equation (13) indicates that for generalised mixed mode fracture to occur along an arbitrary plane, θ_C, from the crack tip,

$$K_\theta = K_C , \tag{18}$$

where

$$K_\theta = \pm\sqrt{K_{I\theta}^2 + K_{II\theta}^2} . \tag{19}$$

Thus, K_θ has the connotation of a mixed mode stress intensity factor from the point of view of its attaining the fracture toughness, K_C, during mixed mode fracture. However, K_θ does not comply with the other characteristic requirement of a stress intensity factor, that is as a means by which the magnitudes of near field stresses may be determined by simple factoring of the terms in r and θ of their respective standard expressions (to do this, it would be necessary to employ $K_{I\theta}$ and $K_{II\theta}$ instead). Nevertheless, since, as indicated above, K_θ does have the attributes of a stress intensity factor from the standpoint of its corresponding fracture criterion, it would be appropriate to refer to it as the *unified, equivalent stress intensity factor*.

REFERENCES

Cotterell, B. and Rice, J. R. (1980). Slightly curved or kinked cracks. *Int. J. Fracture*, 16,155-169.

Banks-Sills, L. and Arcan, M. (1984). Mode II fracture toughness testing with application to PMMA. In: *Application of Fracture Mechanics to Materials and Structures* (G. C. Sih, ed.), pp. 337-344, Kluwer, Boston.

Erdogan, F. & Sih, G. C. (1963). On the crack extension in plates under plane loading and transverse shear. *ASME, J. Basic Engrg.*,85, 519-527.

Hayashi, H. and Nemat-Nasser, S. (1981). Energy-release rate and crack kinking under combined loading. *ASME, J. Appl. Mech.*, 48, 520-524.

Irwin, G. R. (1957). Analysis of stresses and strains near the end of a crack traversing a plate. *ASME, J. Appl. Mech.*, 24, 361-364.

Irwin, G. R. (1958). Fracture. In: *Handbuch der Physik* (S. Flugge, ed.),Vol. 6, pp. 551-590. Springler-Verlag, Berlin.

Lo, K. W., Chua, K. H. and Tamilselvan, T. (1993). A fracture mechanical model for hard soil. In: *Geotechnical Engineering of Hard Soils - Soft Rocks* (A. Agnostopoulos et al., eds.), Vol. 1, pp, 647-651. A. A. Balkema, Rotterdam.

Lo, K. W., Tamilselvan, T., Chua, K. H. and Zhao, M. M. (1996a). A unified model for fracture mechanics. *Engrg Fracture Mech.*, 54, 189-210.

Lo, K. W., Tamilselvan, T., Zhao, M. M. and Chua, K. H. (1996b). A new treatise on fracture mechanics. *Research Report No CE017/96*, Dept. of Civil Engrg., Nat. Univ. of Singapore.

Richard, H. A., Tenhaeff, D. and Hahn, H. G. (1986). Critical survey of mode II fracture specimens. In: *The Mechanism of Fracture* (V. S. Goel, ed.), 89-95, ASM, USA.