STRONG INTERACTION AMONG RANDOMLY DISTRIBUTED MICROCRACKS

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ABSTRACT

The linking process from microcracks to a fatal macroscopic crack is dominated by the strong interaction among neighboring microcracks. The distribution of microcracks modifies the strength and toughness of a material. The present paper focuses on the simple case of collinear microcracks, and quantifies the influence by the statistical distributions of crack lengths and ligament sizes. We predict the scale dependency of the brittle materials. A specimen of large size would have lower strength than a small specimen with the same microcrack density. Furthermore, the strength of a brittle solid decreases as the standard deviation of those distributions increase.

KEYWORDS

Microcracks, strong interaction, statistical distribution, strength, toughness, brittle materials.

INTRODUCTION

Brittle materials are featured by the existence of randomly distributed microcracks. For a solid weakened by microcracks, considerable progresses have been achieved for estimating its overall stiffness. The stiffness can be accurately estimated through the self-consistent method (Budiansky and O'Connell, 1976), the generalized self-consistent method, and Mori-Tanaka's method. The traditional homogenization method, however, is susceptible for the strength estimate of brittle materials.

Experiments (Evans and Wiederhorn, 1984; Wiederhorn and Fuller, 1985) revealed that the strength and toughness of brittle materials are sensitive to the microstructures and exhibit a large Weibull modulus. For a brittle material of fixed crack density and average crack length, the statistical distribution of the strength declines as the specimen size increases, or as the deviation of crack lengths (or ligament lengths) increases.

The scattering in the strength of brittle materials suggests a statistical theory to predict their failure characteristics. Weibull (1939) introduced a fundamental principle known as the "Weakest Link Theory" (WLT), which states that a material breaks when the weakest microcrack in the material leads to a fatal crack. Based on WLT, several statistical models were proposed to predict the failure of brittle materials, such as the works by Batdorf and Crose (1974), and more recently by She and Landes (1993). Those models can explain the strength data of brittle materials to some extent, but nevertheless share a common weakness: the strong interaction among nearby microcracks, as well as its role in the linking process to form a fatal crack, has not been addressed.

In the present work, we evaluate the Stress Intensity Factors (SIF) of collinear microcracks by a method proposed by Kachanov (1987). The accuracy of this evaluation is verified by checking its prediction against the numerical solution of the original system of integral equations. Based on this estimate of strongly interacted microcracks, we propose a statistical method to predict the failure probabilities of an infinite plate containing collinear microcracks. Two simple cases are examined in detail. The first case concerns N collinear microcracks of equal lengths but with randomly distributed ligament sizes; the second case concerns N collinear microcracks of randomly distributed lengths separated by ligaments of equal sizes. Theoretical analysis and examples are presented for each case.

STRESS INTENSITY FACTORS OF COLLINEAR MICROCRACKS

Kachanov (1987) discussed the problem of two collinear cracks of equal length in an otherwise infinite plate loaded by uniform remote tension σ^∞, based on the superposition technique and a self-consistence estimate. His approach can be extended to solve the problem of N collinear microcracks with arbitrary lengths and ligaments. To solve the stress intensity factors, one replaces the original problem by an equivalent configuration: the plate is stress free at infinity but with uniform traction σ^∞ applied along the faces of every microcracks. The latter problem is further reduced to the superposition of N problems, each involving an infinite plate with a single crack at the designated location. The crack faces are loaded by normal tractions yet to be resolved. For the plate with only the i^{th} microcrack, the hypothetical traction $\sigma_i(x)$ is the sum of σ^∞ and the normal stresses induced by the (unknown) tractions applied on the faces of other microcracks. Namely,

$$\sigma_i(x) = \sigma^\infty + \frac{1}{\pi}\sum_{j=1,j\neq i}^{n} \frac{1}{\sqrt{(x-b_j)^2 - a_j^2}} \int_{b_j-a_j}^{b_j+a_j} \frac{\sqrt{a_j^2-\xi^2}}{|x-b_j-\xi|}\sigma_j(\xi)d\xi\,, \quad i = 1,2\ldots N \tag{1}$$

where a_i and b_i denote the half-length and the center location of the i^{th} microcrack. An accurate solution for the above system of integral equations can be obtained through the Chebyshev polynomial technique. To simplify the solution, Kachanov (1991) replaced the non-uniform tractions in the integrand of (1) by their average values. Thus, the system (1) is approximated by

$$\sigma_i(x) \approx \sigma^\infty + \sum_{j=1,j\neq i}^{n}\left[\frac{|x-b_j|}{\sqrt{(x-b_j)^2 - a_j^2}} - 1\right]\langle\sigma_j(x)\rangle\,, \quad i = 1,2\ldots N \tag{2}$$

where the cusped brackets represent averaging along the faces of respective microcracks. Averaging the system (2) along the faces of the i^{th} microcrack, one obtains a self-consistent estimate:

$$\langle\sigma_i(x)\rangle = \sigma^\infty + \sum_{j=1,j\neq i}^{N}\left\{\left|\frac{\sqrt{(b_i-b_j+a_i)^2 - a_j^2} - \sqrt{(b_i-b_j-a_i)^2 - a_j^2}}{2a_j}\right| - 1\right\}\langle\sigma_j(x)\rangle$$

$$i = 1,2\ldots N \tag{3}$$

The system (3) is a system of N linear algebraic equations to solve for the average tractions $\langle\sigma_i(x)\rangle$, $i = 1,2\ldots N$. After solving $\langle\sigma_i(x)\rangle$ from (3), one can evaluate the non-uniform traction from (2). The stress intensity factors of the i^{th} microcrack are:

$$K_i = \frac{1}{\sqrt{\pi a_i}}\int_{-a_i}^{a_i}\sigma_i(x)\sqrt{\frac{a_i \pm x}{a_i \mp x}}dx \tag{4}$$

where the top (or bottom) sign is for the right (or left) crack tip. Substituting (2) into (4), and evaluating the integral by Chebyshev polynomials, one has the following expression for the right crack tip:

$$K_i = \sigma^\infty\sqrt{\pi a_i} + \frac{\pi a_i}{m}\sum_{j=1,j\neq i}^{N}\langle\sigma_j(x)\rangle\sum_{l=1}^{m}(x_l+1)\left[\frac{\left|x_l + \frac{b_i-b_j}{a_i}\right|}{\sqrt{\left(x_l+\frac{b_i-b_j}{a_i}\right)^2 - \left(\frac{a_j}{a_i}\right)^2}} - 1\right] \tag{5}$$

where m denotes the number of integral points, and $x_l = \cos[(2l-1)\pi/m]$ $(l = 1,2\cdots m)$ are the zeroes of Chebyshev polynomial. The accuracy of this evaluation is verified by checking its prediction against the numerical solution of the integral equation system (1).

STATISTICAL ANALYSIS

This section presents statistical analyses for the problem of N collinear microcracks in an infinite plate under uniform remote tension σ^∞. The half length of a microcrack is denoted by a, and the ligament size between two neighboring microcracks is denoted by c. The statistical distributions for the half-lengths and the ligament sizes of microcracks are described by $f(a)$ and $p(c)$, respectively. $f(a)$ and $p(c)$ are properly normalized, with c_-, a_-, c_+, a_+ being the lower and upper limits of c and a. In the two subsections to follow, we discuss two special cases: (1) collinear microcracks of equal length but with ligaments of distributed sizes; and (2) collinear microcracks of the same ligaments but distributed crack lengths.

1. Equal Length Microcracks with Ligaments of Distributed Sizes

At first, we consider the case of microcracks of equal half-length a_0. The distribution $f(a)$ for this case is a Dirac delta function. Attention is focused on the strong interaction between the two neighboring microcracks, separated by a ligament of size c. To simplify the problem, we approximate the fields associated with the other microcracks by the fields of periodically distributed crack array, separated by average ligaments of size $\bar{c} = \int_{c_-}^{c_+} cp(c)dc$. The SIF at the tips of two interacting microcracks is:

$$K = \sigma^{\infty}\sqrt{\pi a_0}\,F\left(\frac{c}{\bar{c}}, \frac{a_0}{\bar{c}}\right) \tag{6}$$

The detail expression of F can be obtained from (5), it is a monotonically decreasing function of $c/\bar{c}$. A threshold value of σ^{∞}, denoted by $\sigma^{\infty}_{\text{th}}$, can be defined as

$$\sigma^{\infty}_{\text{th}} = \frac{K_{IC}}{\sqrt{\pi a_0}} F^{-1}\left(\frac{c_-}{\bar{c}}, \frac{a_0}{\bar{c}}\right) \tag{7}$$

where K_{IC} is the fracture toughness of the matrix. If $\sigma^{\infty} < \sigma^{\infty}_{\text{th}}$, fracture cannot occur; if $\sigma^{\infty} \geq \sigma^{\infty}_{\text{th}}$, a number of ligaments will break. Corresponding to σ^{∞} we have a critical size of ligament, denoted by c^1_{cr}, which satisfies the following equation:

$$F\left(\frac{c^1_{cr}}{\bar{c}}, \frac{a_0}{\bar{c}}\right) = \frac{K_{IC}}{\sigma^{\infty}\sqrt{\pi a_0}} \tag{8}$$

The ligaments of sizes less than c^1_{cr} would break. The coalescence of the microcracks modifies the density functions $p(c)$ and $f(a)$ to

$$p_1(c) = \frac{p(c)}{1-\beta} H(c - c^1_{cr}) \tag{9}$$

$$f_1(a) = \frac{1-2\beta}{1-\beta}\delta(a - a_0) + \frac{p(2a - 4a_0)}{1-\beta}[H(2a - 4a_0 - c_-) - H(2a - 4a_0 - c^1_{cr})] \tag{10}$$

In the above expressions, $\beta = \int_{c_-}^{c^1_{cr}} p(c)dc$ denotes the fraction of coalesced ligaments and H denotes the Heaviside step function.

The subsequent failure of the brittle solid is dominated by the further extension of coalesced microcracks. For an initial ligament size c_1 in the range of (c_-, c^1_{cr}), the probability of failure by successive linkages (which do not significantly perturb the ligament distribution $p_1(c)$) can be calculated by a multiplicative formula

$$P_f(a_0, c_1) = \prod_{m=2}^{M} \int_{c^1_{cr}}^{c^m_{cr}} p_1(c)dc \tag{11}$$

The symbol c^m_{cr} denotes the critical size of ligaments for the m^{th} linkage, its calculation is facilitated by (5). The crack length after successive linkages of m times, $2a_m$, is given by

$$2a_m = 4a_0 + 2(m-1)\bar{a}_1 + c_1 + \sum_{k=2}^{m} \bar{c}_k(c_1) \tag{12}$$

where $\bar{c}_k = \int_{c^1_{cr}}^{c^k_{cr}} cp_1(c)dc \Big/ \int_{c^1_{cr}}^{c^k_{cr}} p_1(c)dc$ is the expectation of ligament sizes during the k^{th} linkage, and $2\bar{a}_1 = \int_{a_-}^{a_+} af_1(a)da$ the expectation of crack lengths after the first linkage. The total number of stable microcrack linkages, M, before the emergence of a fatal crack, is determined by

$$a_{M-1} < \frac{1}{\pi}\left(K_{IC}/\sigma^{\infty}\right)^2 \leq a_M \tag{13}$$

The survival probability of one linked microcrack of length $4a_0 + c_1$ is, She and Landes (1993),

$$P_s(a_0, c_1) = 1 - P_f(a_0, c_1) \approx \exp\{-P_f(a_0, c_1)\} \tag{14}$$

The approximation in the last step comes from the fact that the failure probability $P_f(a_0, c_1)$ is usually much smaller than unity.

The total number of microcracks whose length equals to $4a_0 + c_1$ is $Np(c_1)dc_1$. According to WLT, the cumulative survival probability of these microcracks is:

$$P_s = \exp\{-NP_f(a_0, c_1)p(c_1)dc\} \tag{15}$$

Again from WLT, the total survival probability is the product of the survival probabilities of all coalesced microcracks after the first linkage. Thus, one has

$$P_{\text{surv}} = \exp\left\{-N\int_{c_-}^{c^1_{cr}} P_f(a_0, c)p(c)dc\right\} \tag{16}$$

Finally, the failure probability for the brittle solid containing N microcracks is

$$P_{\text{fail}} = 1 - \exp\left\{-N\int_{c_-}^{c^1_{cr}} P_f(a_0, c)p(c)dc\right\} \tag{17}$$

provided that $\sigma^{\infty} \geq \sigma^{\infty}_{\text{th}}$. For the special case of a periodic crack array, the failure probability is reduced to $P_{\text{fail}} = H(\sigma^{\infty} - \sigma^{\infty}_{\text{th}})$.

Figure 1 plots the failure probabilities versus the normalized strength $\sigma^{\infty}/\sigma^{\infty}_{\text{th}}$. We prescribe $p(c)$ by a normal distribution. It peaks at $c = \bar{c}$ and truncated below at $c_- = 0.05\bar{c}$. The dimensionless standard deviation s is normalized with respect to $\bar{c}$. The calculations are conducted under a crack density of $a_0/\bar{c} = 4$. The left graph is plotted under a fixed standard deviation of $s = 0.2$, with different curves corresponding to the N values of 100, 200, 300, 500 and 1000. The failure probability slowly takes off as $\sigma^{\infty} > \sigma^{\infty}_{\text{th}}$, then undergoes a transition stage, and finally approach the asymptote of $P_{\text{fail}} = 1$. The graph depicts that the transition strength level for a brittle solid decreases as the number of microcracks increase. Accordingly, the present model is capable of predicting the scaling effect (or dependence on the size of the specimen) of brittle solids. In the right graph of Fig. 1, the number of microcracks is fixed at 300, with different curves corresponding to the s values of 0.1, 0.15, 0.2, 0.25 and 0.3. Under the same crack density, a brittle solid with non-uniform ligament sizes would have a strength considerably lower than the one with relatively uniform ligament sizes.

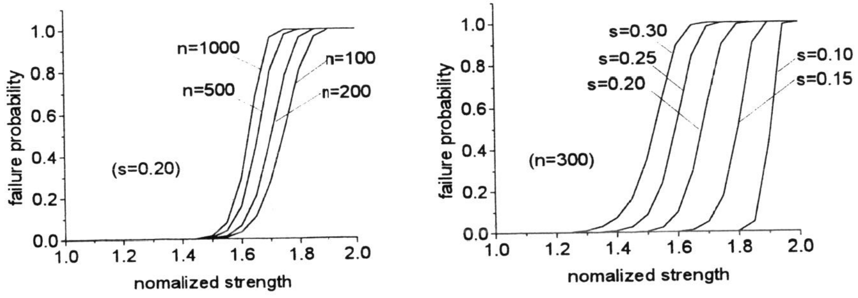


Fig. 1 Failure probability vs normalized strength of brittle solids. Left: $s = 0.2$; right: N=300.

2. Collinear Microcracks of Distributed Lengths

We next discuss microcracks separated by ligaments of fixed size c_0, while $f(a)$ is a certain distribution function with average crack length of $2\bar{a}$. Attention is focused on the local interaction between two neighboring cracks of lengths $2a_L$ (long microcrack) and $2a_S$ (short microcrack), with $a_L \geq a_S$. The interacting SIF at the tip of long microcrack is:

$$K = \sigma^{\infty}\sqrt{\pi a_L}\, G\left(\frac{\bar{a}}{c_0}, \frac{a_L}{c_0}, \frac{a_R}{c_0}\right) \tag{18}$$

G can be computed through (6), and is a monotonically increasing function of both a_L / c_0 and a_S / c_0. The threshold value of σ^{∞}, $\sigma^{\infty}_{\text{th}}$, refers to the special case of $a_L = a_S = a_+$. If $\sigma^{\infty} < \sigma^{\infty}_{\text{th}}$, no ligament breaks. Otherwise, a number of ligaments will break, and there exists an $a^* \leq a_+$, such that

$$G\left(\frac{\bar{a}}{c_0}, \frac{a^*}{c_0}, \frac{a^*}{c_0}\right) = \frac{K_{IC}}{\sigma^{\infty}\sqrt{\pi a^*}} \tag{19}$$

For a long microcrack with half length of $a_L \geq a^*$, one can find a critical value of a_S, denoted by a_S^{cr}, which meets the following equation:

$$G\left(\frac{\bar{a}}{c_0}, \frac{a_L}{c_0}, \frac{a_S^{cr}}{c_0}\right) = \frac{K_{IC}}{\sigma^{\infty}\sqrt{\pi a_L}} \tag{20}$$

The long microcrack can link with any short microcracks whose lengths are larger than $2a_S^{cr}$. Therefore, the probability of microscopic ligament failure is:

$$p_c = \int_{a^*}^{a_+} f(a_L)\int_{a_S^{cr}}^{a_L} f(a)\mathrm{d}a\, \mathrm{d}a_L \tag{21}$$

After the first step linkage, the long microcracks of length $2a_L$ change to the ones of length $2a_L + c_0 + 2a_S$, with a_S taking distributed values. Moreover, the same number of short microcracks disappear. The probability density function for the microcrack lengths change to

$$f_1(a) = \frac{f(a)}{1-p_c}\left[1 - H(a-a^*)\int_{a_S^{cr}}^{a} f(a)\mathrm{d}a - \int_{\max(a^*,a)}^{a_+} H\left(a - a_{cr}^1(a_L)\right)f(a_L)\mathrm{d}a_L\right]$$
$$+\frac{1}{1-p_c}\int_{A^*}^{a_+} f(a_L)f\left(a-a_L-\frac{c_0}{2}\right)H\left(2a_L - a + \frac{c_0}{2}\right)H\left(a - a_L - \frac{c_0}{2} - a_{cr}^1(a_L)\right)\mathrm{d}a_L \tag{22}$$

In (22), the first line of $f_1(a)$ denotes the distribution density of unlinked microcracks, whereas the second line denotes the distribution density of coalesced microcracks. It is those coalesced microcracks that are most likely to create a fatal crack.

Consider a coalesced microcrack of length $2a_1$, its probability of failure by successive linkages can be calculated by the following multiplicative formula

$$P_f(c_0, a_1) = \prod_{m=1}^{M}\int_{a_{cr}^m}^{\infty} f_1(a)\mathrm{d}a \tag{23}$$

where $2a_{cr}^m$ denotes the critical crack length for the m^{th} linkage, its calculation is facilitated by (5). The crack length after successive linkages of m times, $2a_m$, is given by

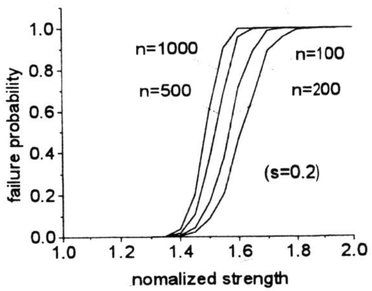
$$2a_m = 2a_1 + (m-1)c_0 + 2\sum_{k=2}^{m}\overline{a_k} \tag{24}$$

where $2\overline{a_k} = 2\int_{a_{cr}^k}^{a_+} af_1(a)\mathrm{d}a \Big/ \int_{a_{cr}^k}^{a_+} f_1(a)\mathrm{d}a$ is the expected length of the microcrack that links to the extending crack during the k^{th} linkage. The total number of stable microcrack linkages, M, before the emergence of a fatal crack, is still determined by (13). Following the same procedure in the previous subsection, the failure probability for the brittle solid containing N microcracks is

$$P_{\mathrm{fail}} = 1 - \exp\left\{ -N(1-p_c) \int_{a_+}^{2a_- + \frac{1}{2}c_0} P_{\mathrm{f}}(c_0, a) f_1(a) \mathrm{d}a \right\} \tag{25}$$

provided that $\sigma^\infty \geq \sigma^\infty_{\mathrm{th}}$.

Figure 2 plots the failure probabilities versus the normalized strength $\sigma^\infty / \sigma^\infty_{\mathrm{th}}$. We prescribe $f(a)$ by a normal distribution. The distribution peaks at $\bar{a} = 4c_0$, and truncated at $a_- = 0$ and $a_+ = 8c_0$. The dimensionless standard deviation s is normalized with respect to $\bar{a}$. The left graph is plotted under a fixed standard deviation of $s = 0.2$, with different curves corresponding to the N values of 100, 200, 500 and 1000, whilst the the right graph fixes the number of microcracks at 300, with different curves corresponding to the s values of 0.1, 0.15, 0.2, 0.25 and 0.3. Trends similar to those in Fig. 1 are predicted for collinear microcracks of distributed lengths.



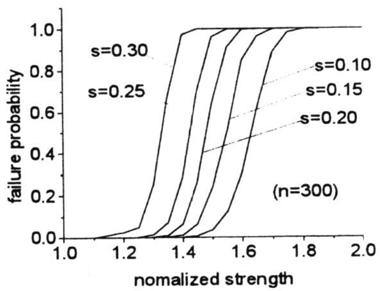


Fig. 2 Failure probability vs normalized strength of brittle solids. Left: $s = 0.2$; right: N=300.

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