QUASI-BRITTLE FRACTURE ANALYSIS BY A SYMMETRIC GALERKIN BOUNDARY ELEMENT METHOD

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ABSTRACT

Recent results are outlined concerning simulations of quasi-brittle fracture processes by the classical cohesive-crack model and by a nontraditional symmetric boundary integral equation approach and its Galerkin boundary element discretization involving double hypersingular integrations.

KEYWORDS

Cohesive Fracture Mechanics, Boundary Integral Equations, Symmetric Galerkin Boundary Element Method

INTRODUCTORY REMARKS

The engineering analysis of solids and structures which exhibit constitutive instabilities such as softening behaviour up to fracture, has recourse since many years to idealization based on kinematic discontinuities endowed with interface softening law, linear elasticity being assumed everywhere else. In the Sixties, such concept gave rise to the cohesive crack model for quasi-brittle solids (Barenblatt, 1962; Dugdale, 1967) and to the softening plastic hinge model for frame structures with unstable flexural behaviour (Maier, 1965). The former model has acquired wide popularity in fracture mechanics of concrete and concrete-like materials (see e.g. Bazant and Cedolin, 1991) and represents the constitutive basis of the present contribution.

The computational framework of what follows is provided by a recently developed approach resting on boundary integral equations (BIE) and their space-discretized version endowed with peculiar features, namely by the symmetric Galerkin boundary element method (SGBEM).

Quasi-brittle fracture processes interpreted by means of the cohesive crack model have been successfully analyzed by finite element methods (FEM), see e.g. Carpinteri (1989) and Bocca et al. (1991).

However, since that model confines all non-linearities to a discontinuity locus Γ_d of a dimensionality lesser than that of the problem domain, a BIE approach appears to be

especially suitable for both theoretical investigations and numerical solutions of such kind of mechanical problems. In fact, the dominance of the "linear background" in the cohesive-crack idealization of fracture naturally suggests an approach resting on two conceptual and operative stages: (a) establish a (linear) relationship between static and kinematic variables on Γ_d alone, by capturing in it the geometric and constitutive properties of that background and exploiting superposition of effects: (b) link these variables by an experimentally corroborated constitutive law for the locus Γ_d of possible displacement discontinuities.

Formally, in compact symbols, the two sides (a) and (b) of the above approach can be expressed, respectively, as follows:

$$\Delta \boldsymbol{p}(\boldsymbol{x}) = \int_{\Gamma_d} \boldsymbol{Z}(\boldsymbol{x},\boldsymbol{\xi}) \Delta \boldsymbol{w}(\boldsymbol{\xi}) \, \mathrm{d}\Gamma + \Delta \boldsymbol{p}^E(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma_d \tag{1}$$

$$\Delta \boldsymbol{p} = \Delta \boldsymbol{p}(\Delta \boldsymbol{w}), \quad \boldsymbol{x} \in \Gamma_d \tag{2}$$

Here: $\boldsymbol{w}$ and $\boldsymbol{p}$ represent the displacement jump and traction vectors, respectively, across the discontinuity locus Γ_d: $\boldsymbol{p}^E$ the tractions which would act there in a fictitious purely linear elastic response to the external actions; $\boldsymbol{x}$ and $\boldsymbol{\xi}$ are Cartesian coordinate vectors. Matrix $\boldsymbol{Z}(\boldsymbol{x},\boldsymbol{\xi})$ gathers (Green's) influence functions of the solid or structure conceived as purely elastic and materializes the superposition of effects $\Delta \boldsymbol{p}$ across Γ_d due to $\Delta \boldsymbol{w}$ there in the linear background of the actual problem.

Symbol Δ denotes increments and is used here for three substantially diverse mehanical interpretations, with consequent different definitions of Γ_d: (A) infinitesimal increments (alternatively, rates: e.g. $\Delta \boldsymbol{w} \to \delta \boldsymbol{w} = \dot{\boldsymbol{w}} \delta t$, t being an event-ordering "time" variable); (B) finite increments, over a generally small (but finite) time-step Δt: (C) "total" variables, in a single step from an original reference state to the present one under given external actions.

The rate formulation (A) fully allows for the nonholonomic (history-dependent, irreversible) nature of fracturing processes. It is appropriate for bifurcation and (overall) stability analyses. To these purposes, the integration path Γ_d may reduce to the current "process zone" (i.e. to the surface where displacement jumps are present but the faces still interact, as distinct from the actual crack, along which they do not).

The step formulation (B) required by most procedures of time-marching solutions, presumes holonomy (history-independence) within each step Δt. Hence, it represents an approximation of (A), which is legitimate if localized yielding over Δt is reasonably expected to be "regularly progressive" (no local unloading) because the external actions grow proportionally within the step (or reversibility turns out to be physically warranted). Clearly, when the crack propagation path is a priori unknown, the step-problem is preceded by the determination of the step advancement direction by some suitable criterion. Thus also in this case, the locus Γ_d can be chosen for the step problem by suitably enlarging the current process zone.

The fully-holonomic single-step approach (C) can be adopted, with remarkable computational savings, if two conditions are fulfilled: (i) the whole tip itinerary can be reasonably conjectured a priori (as it often happens, e.g., in the presence of symmetries or for debonding processes in laminates); (ii) manifestations of irreversibility due to local unstressing in the plasticity sense can be a priori ruled out along the fracture process. Then, Γ_d denotes the whole locus of potential displacement discontinuities apt to include all cracks and process zones at the process end.

Focusing now on the integral operator in eq.(1), which transforms displacement jumps acting along Γ_d on the (unloaded, uncracked) elastic body, into the consequent tractions there, the following essential features of it can be proven in general:

$$-\frac{1}{2} \int_{\Gamma_d} \int_{\Gamma_d} \boldsymbol{w}^T(\boldsymbol{\xi}) \boldsymbol{Z}(\boldsymbol{x},\boldsymbol{\xi}) \boldsymbol{w}(\boldsymbol{x}) \, \mathrm{d}\Gamma \, \mathrm{d}\Gamma \geq 0, \quad \forall \boldsymbol{w}; \qquad \boldsymbol{Z}(\boldsymbol{x},\boldsymbol{\xi}) = \boldsymbol{Z}^T(\boldsymbol{\xi},\boldsymbol{x}), \quad \boldsymbol{x} \neq \boldsymbol{\xi} \tag{3}$$

The negative semidefiniteness, eq.(3), of the quadratic form (and its definiteness when the domain Ω is simply connected with respect to Γ_d) stems from the self-evident energy meaning of the r.h.s. of eq.(3), i.e. elastic strain energy associated to the dislocation field $\boldsymbol{w}(\boldsymbol{x})$. The symmetry property eq.$(3)_b$ can be regarded as a consequence of Betti's reciprocity theorem of linear elasticity, $\boldsymbol{x} = \boldsymbol{\xi}$ being excluded in view of the singularity.

Kernel $\boldsymbol{Z}(\boldsymbol{x},\boldsymbol{\xi})$ is deeply rooted in the mathematical theory of elasticity. In fact, when Ω specializes to the unbounded homogeneous elastic space, $\boldsymbol{Z}(\boldsymbol{x},\boldsymbol{\xi})$ coincides with the two-point influence function analytically computed and investigated by Gebbia in 1891. It is also conceptually related to Maysel formula of classical thermoelasticity (see e.g. Nowacki, 1962).

In almost all situations of engineering interest, the Green functions contained in kernel $\boldsymbol{Z}(\boldsymbol{x},\boldsymbol{\xi})$ cannot be obtained analytically, but must be approximated by some space discretization. To this purpose, FEMs exhibit the disadvantage that all nodal displacement except those concerning nodes on Γ_d have to be condensed, i.e. usually a stiffnes submatrix of large size is to invert (cf. e.g. Bocca et al., 1991). In BEMs such condensation concerns only variables on the boundary Γ and, hence, provides approaches naturally suited to numerical simulations of quasi-brittle fracture. However, traditional BEMs (based on Somigliana identity and its approximate enforcement by collocation at nodes) lead to an approximation of kernel $\boldsymbol{Z}(\boldsymbol{x},\boldsymbol{\xi})$ which does not reflect its essential properties eqs.(1) and (2), i.e. which is neither symmetric nor sign-semidefinite (cf. e.g. Cruse, 1988; Cen and Maier, 1992). On the contrary, these properties are preserved in the transition from the continuum formulation eq.(1), to its BIE and BE counterparts, if the SGBEM of concern herein is adopted. The consequences of this fact (besides the aforementioned computational advantages common to all BEMs with respect to FEM) appear to be especially beneficial in the specific area of quasi-brittle fracture mechanics and in the neighbouring and partially analogous area of unilateral contact and delamination analysis.

The main benefits of SGBEM with respect to traditional BEMs can be summarized as follows: (α) bifurcation and overall stability criteria can be related to mechanical and algebraic features of a symmetric (generally non-definite) matrix; (β) the once-for-all factorization and inversion of the coefficient matrix for the boundary variables to condense are considerably alleviated by its symmetry; (γ) when interface constitutive laws, either in rates or in holonomic piecewise linear terms, are associated to the discretized BIEs, the problem is reduced to a complementarity problem or, alternatively, to generally nonconvex mathematical programming; (δ) solutions are characterized by variational (extremum or saddle point) theorems.

The above positive features of the SGBEM as applied to problems governed by eqs.(1) and (2) have been pointed out and elucidated in earlier papers (Maier et al., 1991 and 1993; Cen et al. 1991; Bolzon et al., 1994 and 1995), where pertinent results can be found in detail. Also important and to some extent still unanswered questions concerning multiplicity of solutions (Bolzon et al., 1996) and the analytical description of softening interface laws dictated by experiments can only be mentioned herein.

In what follows, focus will be on the generation of BIEs which exhibit a symmetric (self-adjoint) integral operator, on the regularization and space-integrations of the hypersingular

kernels involved, on the evaluation of stresses at crack tips and, finally, on incremental problems resting on a broad class of interface models for Γ_d.

The regularization of singularities is crucial in all applications of SGBEMs (see e.g. Sirtori et al., 1992; Kane et al., 1992; Frangi and Novati, 1996) and, hence, only some peculiar aspects of the present context will be considered herein.

The SGBEM initiated in elasticity by Sirtori (1979) and Hartmann et al. (1985) and developed in plasticity by Maier and Polizzotto (1987), Polizzotto (1988) and others, nowadays constitutes the subject of a fairly abundant, fast-growing literature surveyed in a paper in preparation by Bonnet et al. (1997)

SYMMETRIC INTEGRAL EQUATIONS AND CONSISTENT INFLUENCE MATRIX FOR THE DISCONTINUITY LOCUS Γ_d

The considered homogeneous solid or structure occupies (say in $\Re^3$) the volume $\bar{\Omega} = \Omega \cup \Gamma$, Ω being an open domain and Γ its boundary and "closure", assumed as smooth (with unique outward normal $\boldsymbol{n}$ everywhere). It contains the locus Γ_d of possible increments of displacement discontinuities $\Delta \boldsymbol{w}$ at the end of a fracture process. To be specific, this process is thought of below as holonomic according to the approach labelled (C) in the preceding Section, so that total variables can be used, Δ can be dropped and Γ_d acquires its most comprehensive meaning.

Let the homogeneous body in point be inbedded in the homogeneous elastic space Ω_∞ and assume that surface forces $\boldsymbol{F}^*$ (static or "single layer" sources) and displacement jumps $\boldsymbol{D}^*$ (kinematic or "double layer" sources) act on Ω_∞ over Γ, the latter sources also over Γ_d. Consider the following effects in Ω_∞ due to the above sources: (α) displacements, to identify with data $\bar{\boldsymbol{u}}$ in the actual body, in points $\boldsymbol{x}^-$ of Ω at infinitesimal distances from the constrained boundary Γ_u (and forming a surface Γ_u^-): (β) tractions, to identify with data $\bar{\boldsymbol{p}}$ in points $\boldsymbol{x}^-$ of Ω at infinitesimal distances from the free (unconstrained) boundary $\Gamma_p = \Gamma - \Gamma_u$ (and constituting surface Γ_p^-): (γ) tractions across Γ_d.

Due to the linearity of Ω_∞ the above cumulative effects can be expressed by superposition through influence (Green's) functions G of Ω_∞.

In order to recover in the body $\Omega + \Gamma$ its actual state under loads, on Γ_d the sources $\boldsymbol{D}^*$ are identified with the (unknown) relative displacements there, and on Γ two circumstances are made explicit in the traction and displacement jumps, i.e. in $\boldsymbol{F}^* = \boldsymbol{p}(\boldsymbol{\xi}^-) - \boldsymbol{p}(\boldsymbol{\xi}^+)$ and $\boldsymbol{D}^* = \boldsymbol{u}(\boldsymbol{\xi}^+) - \boldsymbol{u}(\boldsymbol{\xi}^-)$, respectively:
(i) the exterior domain $\Omega' = \Omega_\infty - \bar{\Omega}$ is undeformed, i.e. $\boldsymbol{u}(\boldsymbol{\xi}^+) = \boldsymbol{0}$, $\boldsymbol{p}(\boldsymbol{\xi}^+) = \boldsymbol{0}$ in any $\boldsymbol{\xi}^+$;
(ii) $\boldsymbol{p}(\boldsymbol{\xi}^-) = \bar{\boldsymbol{p}}$ on Γ_p, $\boldsymbol{u}(\boldsymbol{\xi}^-) = \bar{\boldsymbol{u}}$ on Γ_u, $\boldsymbol{p}(\boldsymbol{\xi}^-) = \boldsymbol{p}$ on Γ_u, $\boldsymbol{u}(\boldsymbol{\xi}^-) = \boldsymbol{u}$ on Γ_p, denoting by barred and unbarred symbols data and unknowns, respectively.

Through the outlined provisions, the expressions of the effects (α), (β) and (γ) yield the following integral equations, respectively:

$$\int_{\Gamma_u} \boldsymbol{G}_{uu}(\boldsymbol{x},\boldsymbol{\xi})\boldsymbol{p}(\boldsymbol{\xi})\,\mathrm{d}\Gamma - \int_{\Gamma_p} \boldsymbol{G}_{up}(\boldsymbol{x},\boldsymbol{\xi})\boldsymbol{u}(\boldsymbol{\xi})\,\mathrm{d}\Gamma + \int_{\Gamma_d} \boldsymbol{G}_{up}(\boldsymbol{x},\boldsymbol{\xi})\boldsymbol{w}(\boldsymbol{\xi})\,\mathrm{d}\Gamma = \bar{\boldsymbol{f}}_u(\boldsymbol{x}), \quad \forall \boldsymbol{x} \in \Gamma_u^- \tag{4}$$

$$-\int_{\Gamma_u} \boldsymbol{G}_{pu}(\boldsymbol{x},\boldsymbol{\xi})\boldsymbol{p}(\boldsymbol{\xi})\,\mathrm{d}\Gamma + \int_{\Gamma_p} \boldsymbol{G}_{pp}(\boldsymbol{x},\boldsymbol{\xi})\boldsymbol{u}(\boldsymbol{\xi})\,\mathrm{d}\Gamma - \int_{\Gamma_d} \boldsymbol{G}_{pp}(\boldsymbol{x},\boldsymbol{\xi})\boldsymbol{w}(\boldsymbol{\xi})\,\mathrm{d}\Gamma = \bar{\boldsymbol{f}}_p(\boldsymbol{x}), \quad \forall \boldsymbol{x} \in \Gamma_p^- \tag{5}$$

$$\int_{\Gamma_u} \boldsymbol{G}_{pu}(\boldsymbol{x},\boldsymbol{\xi})\boldsymbol{p}(\boldsymbol{\xi})\,\mathrm{d}\Gamma - \int_{\Gamma_p} \boldsymbol{G}_{pp}(\boldsymbol{x},\boldsymbol{\xi})u(\boldsymbol{\xi})\,\mathrm{d}\Gamma + \int_{\Gamma_d} \boldsymbol{G}_{pp}(\boldsymbol{x},\boldsymbol{\xi})\boldsymbol{w}(\boldsymbol{\xi})\,\mathrm{d}\Gamma = \bar{\boldsymbol{f}}_d(\boldsymbol{x}) + \boldsymbol{p}(\boldsymbol{\xi}), \quad \forall \boldsymbol{x} \in \Gamma_d \tag{6}$$

On the r.h.s. of eqs.(4)-(6), $\bar{\boldsymbol{f}}$ denotes given fields which gather effects due to external actions (boundary data $\bar{\boldsymbol{p}}$ and $\bar{\boldsymbol{u}}$ and possible domain data, i.e. body forces and imposed strains).

As well expected, energy meanings and Betti's reciprocity can be shown to induce in the kernels $\boldsymbol{G}$ of Ω_∞ the properties expressed by eqs.$(3)_{a,b}$ for kernel $\boldsymbol{Z}(\boldsymbol{x},\boldsymbol{\xi})$ of $\bar{\Omega}$, namely:

$$\boldsymbol{G}_{uu}(\boldsymbol{x},\boldsymbol{\xi}) \text{ positive definite;} \quad \boldsymbol{G}_{pp}(\boldsymbol{x},\boldsymbol{\xi}) \text{ negative (semi)definite;} \tag{7}$$

$$\boldsymbol{G}_{hk}(\boldsymbol{x},\boldsymbol{\xi}) = \boldsymbol{G}_{kh}^T(\boldsymbol{\xi},\boldsymbol{x}), \quad \boldsymbol{x} \neq \boldsymbol{\xi}, \quad h,k = u,p \tag{8}$$

For the two and three dimensional homogeneous isotropic elastic space, the Green functions in $\boldsymbol{G}_{uu}$ and $\boldsymbol{G}_{pu}$ coincide with Kelvin's (1848) fundamental solutions of Navier equation for unit forces in $\boldsymbol{\xi}$, $\boldsymbol{G}_{up}$ and $\boldsymbol{G}_{pp}$ with Gebbia's (1891) fundamental solutions for displacement jumps "concentrated" in $\boldsymbol{\xi}$. Therefore they are all known in simple analytical forms and can be integrated, provided their singularities for $\boldsymbol{x} = \boldsymbol{\xi}$ are adequately dealt with as discussed in the next Section.

Now let a Galerkin weighted residual enforcement of the integral equations (4)-(6) be adopted for their discrete approximate algebrization. This implies, first, to model the unknown fields, which can be expressed as follows:

$$\boldsymbol{p}(\boldsymbol{x}) = \boldsymbol{N}_p(\boldsymbol{x})\boldsymbol{P}' \text{ on } \Gamma_u; \quad \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{N}_u(\boldsymbol{x})\boldsymbol{U} \text{ on } \Gamma_p; \quad \boldsymbol{w}(\boldsymbol{x}) = \boldsymbol{N}_w(\boldsymbol{x})\boldsymbol{W} \text{ on } \Gamma_d \tag{9}$$

where matrices $\boldsymbol{N}$ contain interpolation functions (which are conceived as defined over the whole boundary portions Γ_p, Γ_u, Γ_d and vanishing outside the "support" of the relevant nodes) and vectors $\boldsymbol{P}'$, $\boldsymbol{U}$ and $\boldsymbol{W}$ gather all the variables on the respective Γ portions. Second, the Galerkin discretization requires to employ the same functions chosen for the interpolation of an unknown field (say $\boldsymbol{u}$ on Γ_p) also as weight functions for the weighted average enforcement of the equation written over the relevant locus (Γ_p).

This discretization procedure entails double integrations,e.g. of the type:

$$\int_{\Gamma_p}\int_{\Gamma_p} \boldsymbol{N}_p^T(\boldsymbol{x})\boldsymbol{G}_{up}(\boldsymbol{x},\boldsymbol{\xi})\boldsymbol{N}_u(\boldsymbol{\xi})\,\mathrm{d}\Gamma\,\mathrm{d}\Gamma; \quad \int_{\Gamma_u}\int_{\Gamma_p} \boldsymbol{N}_u^T(\boldsymbol{x})\boldsymbol{G}_{pp}(\boldsymbol{x},\boldsymbol{\xi})\boldsymbol{N}_u(\boldsymbol{\xi})\,\mathrm{d}\Gamma\,\mathrm{d}\Gamma \tag{10}$$

When all these integrations are suitably performed (see subsequent Section), eqs.(4) and (5) together, and eq.(6) provide two linear algebraic equation systems which read, respectively:

$$\boldsymbol{A}\boldsymbol{X} + \boldsymbol{C}\boldsymbol{W} = \boldsymbol{B}_\Gamma \tag{11}$$

$$\boldsymbol{C}^T\boldsymbol{X} + \hat{\boldsymbol{G}}\boldsymbol{W} = \boldsymbol{B}_d + \boldsymbol{P} \tag{12}$$

where:

$$\boldsymbol{A} = \boldsymbol{A}^T, \quad \hat{\boldsymbol{G}} = \hat{\boldsymbol{G}}^T, \quad \boldsymbol{P} = \int_{\Gamma_d} \boldsymbol{N}_w^T(\boldsymbol{x})\boldsymbol{p}(\boldsymbol{\xi})\,\mathrm{d}\Gamma \tag{13}$$

Vectors $\boldsymbol{B}$ are generated by weighted integrations of data fields $\bar{\boldsymbol{f}}_h$, $h = u,p,d$.

Eq.(11) can be solved with respect to vector $\boldsymbol{X}$ which gathers the unknown vectors $\boldsymbol{P}'$ and $\boldsymbol{U}$ and can be substituted into eq.(12), to yield:

$$\boldsymbol{P} = \boldsymbol{Z}\boldsymbol{W} + \boldsymbol{P}^E \tag{14}$$

Clearly, eq.(14) is the discrete counterpart of eq.(1) in the continuum approach to quasi-brittle fracture analysis.

It might easily be shown that the influence coefficient matrix $\boldsymbol{Z}$ does exhibit the essential properties, eqs.$(3)_{a,b}$, of the kernel $\boldsymbol{Z}(\boldsymbol{x},\boldsymbol{\xi})$, namely it is negative definite (or semidefinite

if Γ_d alters the original simply connected topology of Ω) and exhibits symmetry. These circumstances (referred to herein by the adjective "consistent"), represent the main distinctive features of the SGBEM (not exhibited by traditional BEMs), togheter with the underlying symmetry of matrix $\boldsymbol{A}$, eq.$(13)_a$, and the consequent reduced computing burden in its factorization and inversion.

REGULARIZATION OF SINGULAR INTEGRALS AND COMPUTATION OF TIP STRESSES

In view of the singularities of the Green functions, integrations like those in eqs.(10) embody mathematical and numerical difficulties which turn out to be more severe than in traditional BEMs and have contributed to delay developments and implementations of the SGBEM. Therefore it is worth briefly discussing below at least the approach adopted in the SGBEM by Sirtori et al. (1992) and Frangi and Novati (1996) to regularization purposes (regularization means here singularity reduction). Other approaches are presented e.g. in Kane and Balakrishna (1993).

With reference to two dimensional situations, the former integrand in eq.(10) exhibits the ("strong") singularity $O(r^{-1})$ (with $r = \| \boldsymbol{x} - \boldsymbol{\xi} \|$) occurring in traditional BEMs as well. In fact, the kernel $\boldsymbol{G}_{up}$ in eq.$(10)_b$ for the i^{th} displacement component in $\boldsymbol{x}$ due to the j^{th} kinematic source component in $\boldsymbol{\xi}$, in isotropic elasticity reads:

$$G^{ij}_{up}(\boldsymbol{x},\boldsymbol{\xi}) = -\frac{1}{4\pi(1-\nu)r}\left\{\left[(1-2\nu)\delta_{ij} + 2\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j}\right]\frac{\partial r}{\partial x_k}m_k - (1-2\nu)\left(\frac{\partial r}{\partial x_i}m_j - \frac{\partial r}{\partial x_j}m_i\right)\right\} \tag{15}$$

where ν and E denote the Poisson's coefficient and the elastic modulus, respectively, $\boldsymbol{m}$ defines the unit outward normal to Γ at $\boldsymbol{\xi}$; δ_{ij} is the Kronecker symbol, with $i,j,k = 1,2$. The expression (15) can be derived through the reciprocity relation (8) from kernel $\boldsymbol{G}_{pu}$ (which governs tractions due to unit forces in Ω) and, hence, emanates through differentiation from Kelvin's classical fundamental solution $\boldsymbol{G}_{uu}$ in terms of displacement, which is endowed with weak singularity (logarithmic in 2D).

As for the kernel $\boldsymbol{G}_{pp}$, peculiar of the SGBEM of concern here, its lengthy explicit expression can be formulated in terms of $\boldsymbol{G}_{up}$ as follows (n_k denoting outward normal in $\boldsymbol{x}$):

$$G^{ij}_{pp}(\boldsymbol{x},\boldsymbol{\xi}) = \frac{E\nu}{(1+\nu)(1-2\nu)}\frac{\partial G^{kj}_{up}(\boldsymbol{x},\boldsymbol{\xi})}{\partial x_k}n_i + \frac{E}{2(1+\nu)}\left[\frac{\partial G^{ij}_{up}(\boldsymbol{x},\boldsymbol{\xi})}{\partial x_k} + \frac{\partial G^{kj}_{up}(\boldsymbol{x},\boldsymbol{\xi})}{\partial x_i}\right]n_k \tag{16}$$

Let s_x and s_ξ denote the arc-length coordinates defining the position of the field point $\boldsymbol{x}$ and of the source point $\boldsymbol{\xi}$. The first step in the regularization process consists in identifying auxiliary kernels $\boldsymbol{G}_{u\varphi}, \boldsymbol{G}_{\varphi u}$ and $\boldsymbol{G}_{\varphi\varphi}$ such that:

$$\boldsymbol{G}_{pu}(\boldsymbol{x},\boldsymbol{\xi}) = \frac{\partial}{\partial s_x}\boldsymbol{G}_{\varphi u}(\boldsymbol{x},\boldsymbol{\xi}) \qquad \boldsymbol{G}_{up}(\boldsymbol{x},\boldsymbol{\xi}) = \frac{\partial}{\partial s_\xi}\boldsymbol{G}_{u\varphi}(\boldsymbol{x},\boldsymbol{\xi}) \tag{17}$$

$$\boldsymbol{G}_{pp}(\boldsymbol{x},\boldsymbol{\xi}) = \frac{\partial}{\partial s_x}\frac{\partial}{\partial s_\xi}\boldsymbol{G}_{\varphi\varphi}(\boldsymbol{x},\boldsymbol{\xi}) \tag{18}$$

The auxiliary kernel $\boldsymbol{G}_{\varphi u}$ was established by Ghosh et al. (1986) and, neglecting inessential constants, reads:

$$G^{ij}_{\varphi u}(\boldsymbol{x},\boldsymbol{\xi}) = -\frac{1}{4\pi(1-\nu)}\left[2(1-\nu)\delta_{ij}\alpha + e_{ik}\frac{\partial r}{\partial x_k}\frac{\partial r}{\partial x_j} - (1-2\nu)e_{ij}\log r\right] \tag{19}$$

where α is the angle between a reference direction and the distance vector $\boldsymbol{r}$ and e_{ij} $(i,j = 1,2)$ is the permutation symbol. The angle α varies continously with the vector $\boldsymbol{r}$ throughout the double integration process. Since $\boldsymbol{G}_{up}(\boldsymbol{x},\boldsymbol{\xi}) = \boldsymbol{G}^T_{pu}(\boldsymbol{\xi},\boldsymbol{x})$, the second auxiliary kernel turns out to be: $\boldsymbol{G}_{u\varphi}(\boldsymbol{x},\boldsymbol{\xi}) = \boldsymbol{G}^T_{\varphi u}(\boldsymbol{\xi},\boldsymbol{x})$. As for the third kernel $\boldsymbol{G}_{\varphi\varphi}$, its expression reads:

$$G^{ij}_{\varphi\varphi}(\boldsymbol{x},\boldsymbol{\xi}) = \frac{E}{4\pi(1+\nu)(1-\nu)}\left(\delta_{ij}\log r - \frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j}\right) \tag{20}$$

Let us now consider the integrals in eq.(10). Using eqs.(17) and (18) and integrating by parts, the following identities are obtained:

$$\int_{\Gamma_p}\int_{\Gamma_p} \boldsymbol{N}^T_u(\boldsymbol{x})\boldsymbol{G}_{pp}(\boldsymbol{x},\boldsymbol{\xi})\boldsymbol{N}_p(\boldsymbol{\xi})\,\mathrm{d}\Gamma\,\mathrm{d}\Gamma = \int_{\Gamma_p}\int_{\Gamma_p}\frac{\partial}{\partial s_x}[\boldsymbol{N}^T_u(\boldsymbol{x})]\boldsymbol{G}_{\varphi\varphi}(\boldsymbol{x},\boldsymbol{\xi})\frac{\partial}{\partial s_\xi}[\boldsymbol{N}_p(\boldsymbol{\xi})]\,\mathrm{d}\Gamma\,\mathrm{d}\Gamma$$

$$\int_{\Gamma_u}\int_{\Gamma_p} \boldsymbol{N}^T_p(\boldsymbol{x})\boldsymbol{G}_{up}(\boldsymbol{x},\boldsymbol{\xi})\boldsymbol{N}_u(\boldsymbol{\xi})\,\mathrm{d}\Gamma\,\mathrm{d}\Gamma = -\int_{\Gamma_u}\int_{\Gamma_p}\boldsymbol{N}^T_p(\boldsymbol{x})\boldsymbol{G}_{u\varphi}(\boldsymbol{x},\boldsymbol{\xi})\frac{\partial}{\partial s_\xi}[\boldsymbol{N}_u(\boldsymbol{\xi})]\,\mathrm{d}\Gamma\,\mathrm{d}\Gamma$$

As a conclusion of the above hints on regularization, the following circumstances are worth noticing. (i) The above results apply if $\boldsymbol{N}_u(\boldsymbol{x})$ and $\boldsymbol{N}_u(\boldsymbol{\xi})$ are C^0 on Γ_p and vanish at the end-points of Γ_p itself. (ii) Every term inside the integrands is weakly singular, and not only the overall expression. (iii) The tangential derivatives are computed through differentiation of the shape functions modelling the relevant fields.

In evolutive analyses where the crack path is not a-priori known, the conventional growth-direction search techniques require the computation of stresses at the crack tip.

This reduces to the computation of the displacement gradient since, from Hooke's law and compatibility: $\sigma_{ij} = D_{ijkl}\frac{\partial u_k}{\partial x_l}$.

The collocation displacement equation in tensorial notation reads:

$$u_i(\boldsymbol{\xi}) = \int_\Gamma (G^{ji}_{uu}(\boldsymbol{x},\boldsymbol{\xi})t_j(\boldsymbol{x}) - G^{ji}_{pu}(\boldsymbol{x},\boldsymbol{\xi})u_j(\boldsymbol{x}))\,\mathrm{d}\Gamma + \int_{\Gamma_d} G^{ji}_{pu}(\boldsymbol{x},\boldsymbol{\xi})w_j(\boldsymbol{x})\,\mathrm{d}\Gamma \tag{21}$$

The displacement gradient equation is obtained by differentiating eqs.(21) with respect to the ξ_k coordinate after applying eq.(17):

$$\frac{\partial u_i(\boldsymbol{\xi})}{\partial \xi_k} = \int_\Gamma\left(\frac{\partial G^{ji}_{uu}(\boldsymbol{x},\boldsymbol{\xi})}{\partial \xi_k}t_j(\boldsymbol{x}) + \frac{\partial G^{ji}_{\varphi u}(\boldsymbol{x},\boldsymbol{\xi})}{\partial \xi_k}\frac{\partial}{\partial s_x}[u_j(\boldsymbol{x})]\right)\mathrm{d}\Gamma - \int_{\Gamma_d}\frac{\partial G^{ji}_{\varphi u}(\boldsymbol{x},\boldsymbol{\xi})}{\partial \xi_k}\frac{\partial}{\partial s_x}[w_j(\boldsymbol{x})]\,\mathrm{d}\Gamma \tag{22}$$

If eq.(22) is collocated at the crack tip without any additional hypothesis on $\boldsymbol{w}$, the last hypersingular integral does not yield a finite value. This is physically correct in linear fracture mechanics where strains are not bounded at crack tips (e.g. Cruse, 1988). In cohesive fracture mechanics, on the contrary, it is well known that not only the displacement discontinuity does vanish at the crack tip, but also its tangential derivative. Let $\frac{\partial}{\partial s_x}w_j$ be $C^{0,\alpha}$ continuous (and w_j be $C^{1,\alpha}$) at the crack tip, i.e. $|\frac{\partial}{\partial s_x}w_j| = O(r^\alpha)$ with $0 < \alpha \leq 1$. Then $\frac{\partial}{\partial \xi_k}[G^{ji}_{\varphi u}]\frac{\partial}{\partial s_x}w_j$ behaves like $O(r^{\alpha-1})$ (i.e. it is only weakly singular) and can be numerically integrated. This implies that nonconventional modelling of the $\boldsymbol{w}$ field has to be employed at crack tips. Cubic hermitian shape functions or suitable combinations of quadratic lagrangian shape functions can however comply with the condition imposed.

The above considerations are intended to concisely specify some of the main singularity-related questions in the SGBEM and to corroborate the remark that at present they can be satisfactorily answered to numerical implementation purposes.

INTERFACE LAWS IN LOCAL AND GENERALIZED VARIABLES

The constitutive models, eq.$(2)_b$, attributed to the locus of potential displacement discontinuities as long as its two faces interact (i.e. with $p \neq 0$), in many practical situations belong to the vast category of time-independent, path-dependent (nonholonomic), nonassociative plasticity models described by the following set of relations, borrowed from plasticity theory of continua:

$$\Delta \boldsymbol{w} = \frac{\partial \boldsymbol{\psi}^T}{\partial \boldsymbol{\sigma}}(\boldsymbol{p}, \boldsymbol{q})\Delta\boldsymbol{\lambda}, \qquad \Delta \boldsymbol{s} = -\frac{\partial \boldsymbol{\psi}^T}{\partial \boldsymbol{q}}(\boldsymbol{p}, \boldsymbol{q})\Delta\boldsymbol{\lambda} \tag{23}$$

$$\boldsymbol{\phi}(\boldsymbol{p}, \boldsymbol{q}) \leq 0, \qquad \Delta\boldsymbol{\lambda} \geq 0, \qquad \boldsymbol{\phi}^T \Delta\boldsymbol{\lambda} = 0 \qquad \boldsymbol{q} = \frac{\partial \boldsymbol{\pi}}{\partial \boldsymbol{s}}(\boldsymbol{s}) \tag{24}$$

In the jargon of platicity, $\boldsymbol{\phi}$, $\boldsymbol{\psi}$ and $\boldsymbol{\lambda}$ are (equidimensional) m-vectors of yield functions, plastic potentials and plastic multipliers, respectively; $\boldsymbol{q}$ and $\boldsymbol{s}$ denote m'-vectors of static and kinematic internal variables, respectively. Eqs.(23) and (24) cover as special cases: (A) processes of infinitesimal amplitudes (or in rates) for $\Delta \to \delta$, with $\boldsymbol{\phi}$ that can be replaced by $\delta\boldsymbol{\phi}$ (or $\dot{\boldsymbol{\phi}}$) in eq.(24); (B) finite-step problems over Δt formulated according to some finite-difference time-integration scheme, namely with $\boldsymbol{p} = \bar{\boldsymbol{p}} + \alpha\Delta\boldsymbol{p}$, $\boldsymbol{q} = \bar{\boldsymbol{q}} + \alpha\Delta\boldsymbol{q}$ in the arguments of $\boldsymbol{\psi}$ and $\boldsymbol{\phi}$, marking by bars known quantities at the step-starting instant $\bar{t}$ (Euler's forward scheme for $\alpha = 0$; backward difference scheme for $\alpha = 1$).

In both these interpretations Γ_d is basically the process zone at $\bar{t}$ (exactly or with suitable adjustements dictated by the step amplitude and by the space-modelling to see later) and m is the number of the yield modes which may be activated in the step.

It is worth noting that the interface model, eqs.(23) and (24), is usually formulated in a local reference system (for reasons which are obvious in special cases, e.g. for mode I cohesive cracks). However, for brevity, coordinate transformations are not explicitly considered here (see e.g. Maier et al., 1993).

The option (C) of holonomic analysis in total variables and single step is not naturally amenable to interface laws of the kind (23)-(24) when cracks with separate noninteracting faces are expected. Therefore, for space limitations, only the interpretations (A) and (B) will be considered henceforth.

The BIE description (4)-(6) of the linear background equation (1), through space modelling (9) and Galerkin approximate enforcement, have led to the linear algebraic equations (11) and (12) and to their condensation (14) in generalized variables $\boldsymbol{W}$ and $\boldsymbol{P}$ concerning locus Γ_d alone. The former vector $\boldsymbol{W}$ was introduced for modelling, eq.$(9)_c$; the latter $\boldsymbol{P}$, eq.$(13)_c$, originated by the Galerkin weighted-residuals approximation.

Along a process zone Γ_d with two faces marked by $^+$ and $^-$ so that $\boldsymbol{w} = \boldsymbol{u}^+ - \boldsymbol{u}^-$ and $\boldsymbol{p} = \boldsymbol{p}^- = -\boldsymbol{p}^+$, the energy dissipation rate reads:

$$\int_{\Gamma_d} (\boldsymbol{p}^{+T}\dot{\boldsymbol{u}}^+ + \boldsymbol{p}^{-T}\dot{\boldsymbol{u}}^-)\, \mathrm{d}\Gamma = -\int_{\Gamma_d} \boldsymbol{p}^T \dot{\boldsymbol{w}}\, \mathrm{d}\Gamma = -\boldsymbol{P}^T\dot{\boldsymbol{W}} \tag{25}$$

The second equality arising from eqs.$(9)_c$ and $(13)_c$ entails that the dot product of work-conjugate variables and its energy meaning are preserved in passing from local to generalized variables. This circumstance holds for modelled tractions $\boldsymbol{p}$, if the interpolation functions adopted in $\boldsymbol{N}_p$ fulfill an orthogonality condition with those for displacement jumps $\boldsymbol{w}$ in $\boldsymbol{N}_w$:

$$\boldsymbol{p}(\boldsymbol{x}) = \boldsymbol{N}_P(\boldsymbol{x})\boldsymbol{P}, \qquad \int_{\Gamma_d} \boldsymbol{N}_p^T \boldsymbol{N}_w\, \mathrm{d}\Gamma = \boldsymbol{I}, \quad \boldsymbol{N}_p = \boldsymbol{N}_w \Big(\int_{\Gamma_d} \boldsymbol{N}_w^T \boldsymbol{N}_w\, \mathrm{d}\Gamma\Big)^{-1} \tag{26}$$

where $\boldsymbol{I}$ denotes identity matrix and the third equation provides a convenient (but not unique) way of satisying the second one by deriving $\boldsymbol{N}_p$ from $\boldsymbol{N}_w$.

The above provisions applied to the BE generation of the discrete elasticity relationship between conjugate quantities on Γ_d, for consistency must be applied also to the inelastic interface constitution between the same quantities, eqs.(23) and (24). However, this constitutive model involves other pairs of conjugate variables, namely: $\boldsymbol{s}, \boldsymbol{q}$; $\boldsymbol{\lambda}, \boldsymbol{\phi}$; $\boldsymbol{\lambda}, \boldsymbol{\psi}$. Let the same provisions, eqs.(25) and (26), be adopted also in the discretization of these variables over Γ_d (i.e. $\boldsymbol{s} = \boldsymbol{N}_s\boldsymbol{S}$, $\boldsymbol{q} = \boldsymbol{N}_q\boldsymbol{Q}$, $\boldsymbol{\lambda} = \boldsymbol{N}_\lambda\boldsymbol{\Lambda}$, $\boldsymbol{\phi} = \boldsymbol{N}_\phi\boldsymbol{\Phi}$, $\boldsymbol{\psi} = \boldsymbol{N}_\psi\boldsymbol{\Psi}$).

Thus, the local interface law (23) and (24) can be shown to induce the following interface law in terms of "generalized variables in Prager sense" (this expression labelling the above multifield, "mixed" space-modelling rests on historical reasons) with the conservation of the original essential features (such as convexity and, if applicable, normality):

$$\Delta\boldsymbol{W} = \frac{\partial \boldsymbol{\Psi}^T}{\partial \boldsymbol{P}}(\boldsymbol{P}, \boldsymbol{Q})\Delta\boldsymbol{\Lambda}, \quad \Delta\boldsymbol{S} = -\frac{\partial \boldsymbol{\Psi}^T}{\partial \boldsymbol{Q}}(\boldsymbol{P}, \boldsymbol{Q})\Delta\boldsymbol{\Lambda}, \quad \Delta\boldsymbol{Q} = -\frac{\partial \boldsymbol{\Pi}}{\partial \boldsymbol{S}}(\boldsymbol{S}) \tag{27}$$

$$\boldsymbol{\Phi} = \boldsymbol{\Phi}(\bar{\boldsymbol{P}}, \bar{\boldsymbol{Q}}) + \frac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{P}^T}(\boldsymbol{P}, \boldsymbol{Q})\Delta\boldsymbol{P} + \frac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{Q}^T}(\boldsymbol{P}, \boldsymbol{Q})\Delta\boldsymbol{Q} \leq \boldsymbol{0} \quad \Delta\boldsymbol{\Lambda} \geq \boldsymbol{0}, \quad \boldsymbol{\Phi}^T\Delta\boldsymbol{\Lambda} = \boldsymbol{0} \tag{28}$$

In eq.$(28)_a$ the yield functions, like often in plasticity, have been assumed positively homogeneous of order one and re-formulated accordingly with $\boldsymbol{P} = \boldsymbol{P} + \alpha\Delta\boldsymbol{P}$, $\boldsymbol{Q} = \boldsymbol{Q} + \alpha\Delta\boldsymbol{Q}$, α depending on the chosen integration scheme.

SOME ASPECTS OF A COMPUTATIONAL THEORY AND CONCLUSIONS

After the space and time discretizations, outlined in the preceding Sections, the fracture processes occurring on Γ_d in the time step Δt as part of the inelastic structural response to external action increments turns out to be governed by the association of the non linear relationship (27)-(28) to the linear equations:

$$\Delta\boldsymbol{P} = \boldsymbol{Z}\Delta\boldsymbol{W} + \Delta\boldsymbol{P}^E \tag{29}$$

The input is vector $\Delta\boldsymbol{P}^E$ resulting from a preliminary linear elastic analysis and capturing the load increments. The solution of problem (27)-(29) is briefly discussed below on the basis of a distinction of crucial importance from both the mathematical and mechanical stanpoint: (I) situations in which the gradients of $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$ are constant (i.e. do not depend on the increments: $\boldsymbol{P} = \bar{\boldsymbol{P}}$, $\boldsymbol{Q} = \bar{\boldsymbol{Q}}$) and the internal variable potential $\boldsymbol{\Pi}$ is quadratic; (II) cases in which the above specialization does not hold.

From the mechanical standpoint, the category (I) is rather rich in the quasi-brittle fracture context (richer than in plasticity). In fact, it includes: (α) infinitesimal processes ($\Delta \to \delta$) in rates; (β) finite step problems formulated by Euler's foreward difference time-integration scheme; (γ) piecewise linear (PWL) models, i.e. constructed or approximated by linear $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$ and hardening rules. Particular interface laws of kind (γ) are the popular cohesive-crack model in the opening mode with a PWL softening branch of traction decay to zero (without or with "break point").

From the mathematical standpoint, all cases in class (I) can be cast into the format of linear complementarity problem (LCP):

$$-\boldsymbol{\Phi} = \boldsymbol{M}\Delta\boldsymbol{\Lambda} - \boldsymbol{B} \geq \boldsymbol{0}, \quad \Delta\boldsymbol{\Lambda} \geq \boldsymbol{0}, \quad \boldsymbol{\Phi}^T\Delta\boldsymbol{\Lambda} = \boldsymbol{0} \tag{30}$$

$$\boldsymbol{M} = \boldsymbol{H} - \frac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{P}^T}\boldsymbol{Z}\frac{\partial \boldsymbol{\Psi}^T}{\partial \boldsymbol{P}} \tag{31}$$

The LCP (31) is arrived at by substituting eqs.(27) and (29) into eq.$(28)_a$. Vector $\boldsymbol{B}$ gathers data and reduces to $\Delta \boldsymbol{P}^E$ when $\boldsymbol{\Phi}(\bar{\boldsymbol{P}}, \bar{\boldsymbol{Q}}) = \boldsymbol{0}$, like in all rate problems. Matrix $\boldsymbol{M}$, eq.(31), is generally nonsymmetric and indefinite. In fact, its former addend $\boldsymbol{H}$ is negative definite or nondefinite, as it reflects the constitutive softening of the process zone; the latter is symmetric and positive semidefinite (because of its generation by the present SGBEM) whenever normality holds (i.e. $\boldsymbol{\Phi} = \boldsymbol{\Psi}$) as it obviously holds in uniaxial constitution for opening-mode cohesive models.

The expected absence in LCP (30) of the special features (symmetry and positiveness) of the LCPs occurring in classical contexts (such as Drucker-stable plasticity and Signorini-Fichera unilateral contacts in elasticity), makes the numerical solution harder and its underlying theory poorer. However, the following circumstances of computational relevance can be stated for problem (30) and, hence, for all the aforementioned approaches (α)-(γ) to quasi-brittle fracture analysis:

(a) Problem (30) is equivalent to a generally non convex quadratic programming problem in $\Delta \boldsymbol{\Lambda}$ only, all the solutions of which are vertices of the (polyhedrical) feasible domain defined by the linear inequalities $(30)_{a,b}$.

(b) There is a finite number of solutions if, and only if, all principal minors of matrix $\boldsymbol{M}$ are non-zero. (c) All solutions can be computed in a finite number of operations (or it can be shown that no solution exists) by an enumerative tree-search procedure consisting of a sequence of linear programming problems.

(d) Problem (30) can be transformed into a system of nonsmooth equations in $\Delta \boldsymbol{\Lambda}$.

(e) The system (d) can be numerically solved by the unconstrained minimization of a (piecewise quadratic) norm using an iterative generalized damped Newton method.

(f) In LCP (30) in rates with symmetric $\boldsymbol{M}$, overall strict stability (in the sense of positive second-order work) holds if $\boldsymbol{M}$ is positive definite; if, and only if, it is copositive.

(g) When problem (30) in rates is solved (and the advancement direction has been established), for every solution (if any) one can trivially find the load factor increment leading to the activation of a new yield mode (with advancement of process zone and/or crack). Thus fracture simulations with cohesive models can be performed by steps consisting of LCP in rates, and by linear expansion of the rate solution (possibly preceded by a tip-direction search).

In the above list, statement (a) can be easily proved (Maier et al., 1993). Statement (b) rephrases a mathematical theorem by Murty (Cottle et al., 1992), here of interest in view of bifurcations. The finite termination method (c), devised in operations research (Judice and Mitra, 1988) and implemented to the present purpose (Bolzon et al., 1995), requires a computational effort rapidly increasing with the number of variables which is proportional to the node number on Γ_d. The mathematical result (d), due to Robinson (1992), gave rise to the "Path method" mentioned in (e) and to the relevant computer code by Dirkse and Ferris (1995) employed by Bolzon et al. (1995, 1996): with respect to (c) the greater cost-effectiveness of solution turned out to be partly compensated for by difficulties in finding all solutions trying different inizializations . The stability criterion (f) was pointed out in Maier et al. (1991). Finally, the marching solution procedure (g) has been adopted and tested in the illustrative example presented below.

The two-notch tensile specimen of Fig. 1_a with linear softening in the mode I cohesive crack model on Γ_d along the horizontal axis of symmetry has been analyzed by the SGBEM. The results plotted in Figures 1_b and 2_b evidence the bifurcation (into 3 rate solutions) and the relevant processes. The extensive local unloadings occurring in later stage of each nonsymmetric fracture invalidate the holonomic (single step) solution, which is illustrated

in Fig. 2_b for comparison, since holonomic solutions still amount to solve LCPs of kind (30), in view of the PWL nature of the adopted cohesive crack model.

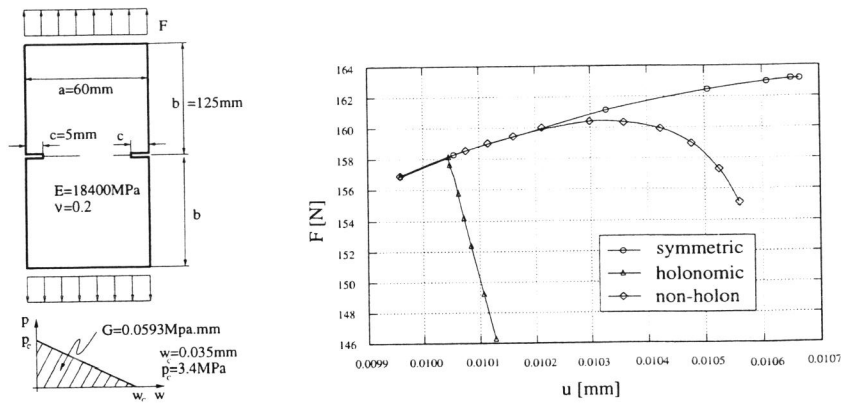

Figure 1: Two-notch tensile specimen: geometry, material parameters and load versus displacement plots (symmetric, nonsymmetric holonomic and nonsymmetric nonholonomic).

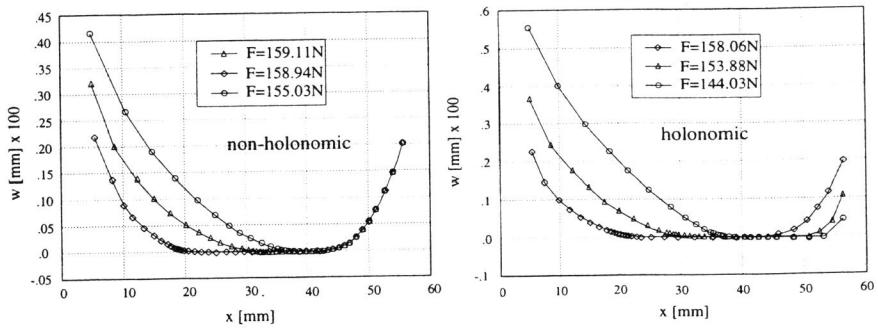

Figure 2: Opening displacement between notches versus point position on Γ_d: holonomic and non holonomic solutions.

As for the approaches of class (II) resting on the interface model of eqs.(23) and (24), the latter in its full nonlinearity, several developments not dealt with herein, can be mutuated from softening nonassociative plastic analysis (by implicit integration schemes) and possibly from recent results concerning nonlinear complementarity problems (see e.g. Ferris and Tin Loi, 1996).

These topics, togheter with various computational aspects of singular integrations outlined earlier, represent some of several desirable prospects of future research in the title subject.

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