ON THE FULL-FIELD ELASTOSTATIC DEFORMATION OF CRACKS USING NON-LINEAR FINITE STRAIN ANALYSIS

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ABSTRACT

Complex potential functions for finite elastostatic deformation gradients are shown to provide a solution to the problem of fully non-linear equilibrium in compressible elastic solids. The strain-energy density function is considered in Knowles and Sternberg form. The foregoing complex functions enable the traction-free crack problem to be solved as a non-linear eigenvalue problem. Finally, the determination of field stresses in finite-sized, homogeneous, and interfacial crack geometries is considered briefly.

KEYWORDS

Finite plane strain analysis, strain-energy density function, complex potential functions, boundary-value problems, interfacial cracks.

INTRODUCTION

Linear Elastic Fracture Mechanics (LEFM) and Elasto-Plastic Fracture Mechanics (EPFM) have played a prominent role in the analysis of cracks in recent years. The former relies on the linearised theory of elasticity that gives rise to the well-known 'singular' field at the crack tip. In EPFM a deformation theory of plasticity can be applied to the problem of a crack undergoing 'small-scale yielding' as shown by Hutchinson (1968) and Rice and Rosengren (1968). Both approaches assume a mechanical response with infinitesimal deformations. Since locally unbounded strains and stresses exist at the crack tip in LEFM problems this is a direct contradiction of the underlying principle behind their derivation.

Despite the doubt that surrounds the derivation of the crack-tip field behaviour LEFM has proved to be a successful 'engineering' tool for the solution of many fracture problems. However, a similar treatment applied to the problem of the interface crack between dissimilar slabs of material, first performed by Williams (1959), leads to unsatisfactory field behaviour consisting of an oscillatory singularity. Herrmann (1989) has shown that if the linear stress-strain law is relinquished in favour of the fully non-linear theory of elasticity, permitting finite deformations, an asymptotic solution to the interfacial crack exists that is free of the oscillatory singularity. The analysis assumes each slab of material is hyperelastic, isotropic and homogeneous satisfying certain asymptotic conditions on its strain-energy density developed by Knowles and Sternberg (1973 & 1983) who had previously considered the plane strain homogeneous crack in an infinite plate. The problem of Neo-Hookean materials has also been considered by many authors such as Ravichandran and Knauss (1989).

The determination of the full-field solution to the non-linear hyperelastic crack problem is of formidable mathematical complexity. The analyses of Knowles and Sternberg (1973), and Herrmann (1989) consider the asymptotic behaviour at the crack tip only. In this paper a solution to the full-field problem is presented to enable the stress state for finite-sized crack geometries to be obtained. Also, this analysis should permit experimental techniques such as photoelasticity (see Nurse and Patterson, 1993) to be used with more confidence in the determination of the asymptotic field behaviour at crack tips.

FINITE ELASTOSTATIC DEFORMATION THEORY

Consider a prismatic body with the middle cross-section occupied by the domain Ω of co-ordinates (x_1, x_2). A plane deformation of the body, which maps Ω onto a domain Ω^* of the same plane, is given by:

$$y_\alpha = y_\alpha(x_1, x_2) = x_\alpha + u_\alpha(x_1, x_2), \quad \text{on } \Omega. \qquad (1)$$

where u_α are the components of the displacement vector $\boldsymbol{u}$[1].

The deformation-gradient tensor associated with (1) is given by $\boldsymbol{F}$, i.e.:

$$F_{\alpha\beta} = \frac{\partial y_\alpha}{\partial x_\beta} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} = \begin{bmatrix} 1 + \partial u_1 / \partial x_1 & \partial u_1 / \partial x_2 \\ \partial u_2 / \partial x_1 & 1 + \partial u_2 / \partial x_2 \end{bmatrix} \qquad (2)$$

where $F_{\alpha\beta}$ are deformation gradients of an elementary slab of material (Fig. 1). Let I and J denote the fundamental scalar invariants of the deformation gradients given in terms of the symmetric positive definite tensor $\boldsymbol{G} = \boldsymbol{F}^T\boldsymbol{F}$ by:

$$I = \mathrm{tr}\boldsymbol{G} = \lambda_1^2 + \lambda_2^2, \quad J = \sqrt{\det \boldsymbol{G}} = \lambda_1\lambda_2, \quad \text{on } \Omega. \qquad (3)$$

where $\lambda_\alpha > 0$ is the value of the local principal extension ratio. The deformation of a homogeneous and isotropic hyperelastic solid is treatable in terms of the stored strain-energy per unit of undeformed volume, or W, as a function of the material co-ordinates in Ω. The function W depends on the invariants I and J via:

$$W(x_1, x_2) = \Theta(I(x_1, x_2), J(x_1, x_2)), \quad \text{for all } (x_1, x_2) \text{ in } \Omega. \qquad (4)$$

[1]Letters in **boldface** designate first or second-order tensors, as well as the matrix of their scalar components.

Knowles and Sternberg (1973), showed that W has a suitable asymptotic representation given by:

$$\Theta(I, J) = \left[AI + BJ + CIJ^{-2} + A + C + D\right]^n + O\left(R^{n-1}\right), \quad \text{where } R = \sqrt{I^2 + J^2} \qquad (5)$$

which satisfies the inequalities of Baker and Ericksen (1954), and of Coleman and Noll (1959). The parameters A, B, C and n are material dependent and must satisfy $A>0$, $0<B<2A$, $C>0$, and $1/2<n<\infty$ (Knowles and Sternberg, 1973). The parameter D ensures that the strain-energy density vanishes in the undeformed state.

Let σ denote the 'pseudo' two-dimensional Piola stress tensor defined on Ω. The stress-deformation relationships are given by:

$$\sigma_{\alpha\beta} = 2\frac{\partial\Theta}{\partial I}\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} + \frac{\partial\Theta}{\partial J}\begin{bmatrix} F_{22} & -F_{21} \\ -F_{12} & F_{11} \end{bmatrix}, \quad \text{on } \Omega. \qquad (6)$$

The 'actual' or Cauchy stress defined on Ω is denoted by τ and given by:

$$\tau_{\alpha\beta} = \tau_{\beta\alpha} = \frac{2}{J}\frac{\partial\Theta}{\partial I}\begin{bmatrix} F_{11}F_{11} + F_{12}F_{12} & F_{11}F_{21} + F_{12}F_{22} \\ F_{11}F_{21} + F_{12}F_{22} & F_{22}F_{22} + F_{21}F_{21} \end{bmatrix} + \frac{\partial\Theta}{\partial J}\delta_{\alpha\beta}, \quad \text{on } \Omega. \qquad (7)$$

where $\delta_{\alpha\beta}$ is the Kroneker delta.

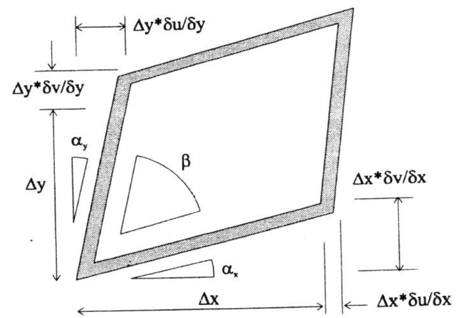

Figure 1: Two-dimensional frame element originally consisting of two mutually perpendicular members of unit length Δx and Δy.

DETERMINATION OF FULL-FIELD SOLUTIONS

Equilibrium

The equations of equilibrium are[2]:

$$\sigma_{\alpha\beta,\beta} = 0, \quad \text{on } \Omega. \tag{8}$$

For the sake of brevity in the proceeding theory consideration will be made to the case of n=1 which represents material with a small degree of hardening. The equilibrium equations in terms of (2), (5) and (6) become:

$$\begin{aligned}
\sigma_{1\beta,\beta} = &-4CJ\left[\frac{\partial J}{\partial x_1}\left(1+\frac{\partial u_1}{\partial x_1}\right)+\frac{\partial J}{\partial x_2}\left(\frac{\partial u_1}{\partial x_2}\right)\right]+2J^2\left[AJ^2+C\right]\left(\frac{\partial^2 u_1}{\partial x_1^2}+\frac{\partial^2 u_1}{\partial x_2^2}\right)+\\
&6CI\left[\frac{\partial J}{\partial x_1}\left(1+\frac{\partial u_2}{\partial x_2}\right)+\frac{\partial J}{\partial x_2}\left(-\frac{\partial u_2}{\partial x_1}\right)\right]-2CJ\left[\frac{\partial I}{\partial x_1}\left(1+\frac{\partial u_2}{\partial x_2}\right)+\frac{\partial I}{\partial x_2}\left(-\frac{\partial u_2}{\partial x_1}\right)\right]\\
&=0\\
\sigma_{2\beta,\beta} = &-4CJ\left[\frac{\partial J}{\partial x_1}\left(\frac{\partial u_2}{\partial x_1}\right)+\frac{\partial J}{\partial x_2}\left(1+\frac{\partial u_2}{\partial x_2}\right)\right]+2J^2\left[AJ^2+C\right]\left(\frac{\partial^2 u_2}{\partial x_1^2}+\frac{\partial^2 u_2}{\partial x_2^2}\right)+\\
&6CI\left[\frac{\partial J}{\partial x_1}\left(-\frac{\partial u_1}{\partial x_2}\right)+\frac{\partial J}{\partial x_2}\left(1+\frac{\partial u_1}{\partial x_1}\right)\right]-2CJ\left[\frac{\partial I}{\partial x_1}\left(-\frac{\partial u_1}{\partial x_2}\right)+\frac{\partial I}{\partial x_2}\left(1+\frac{\partial u_1}{\partial x_1}\right)\right]\\
&=0
\end{aligned} \tag{9}$$

on Ω. With a view to solving for these non-linear differential equations in u_α expression (9) can be made linear in I and J by adopting the harmonic equations:

$$\partial^2 u_1/\partial x_1^{\,2}+\partial^2 u_1/\partial x_2^{\,2}=0,\quad \partial^2 u_2/\partial x_1^{\,2}+\partial^2 u_2/\partial x_2^{\,2}=0 \tag{10}$$

Separating terms in I and J and substituting for expression (10) in (9), (to eliminate for $u_{\alpha,22}$), enables the equilibrium equations to written in the following matrix form:

$$\boldsymbol{DH}=0 \tag{11}$$

where $\boldsymbol{D}$ and $\boldsymbol{H}$ are 4×4 and 4×1 matrices respectively, and are given by:

[2]The summation convention for subscripts is now employed with the comma between subscripts denoting partial differentiation with respect to that variable after the comma.

$$D_{11}=3\left(1+\frac{\partial u_2}{\partial x_2}\right)^2-3\left(\frac{\partial u_2}{\partial x_1}\right)^2$$

$$D_{12}=3\left(1+\frac{\partial u_1}{\partial x_1}\right)\left(\frac{\partial u_2}{\partial x_1}\right)-3\left(1+\frac{\partial u_2}{\partial x_2}\right)\left(\frac{\partial u_1}{\partial x_2}\right)$$

$$D_{13}=-6\left(1+\frac{\partial u_2}{\partial x_2}\right)\left(\frac{\partial u_2}{\partial x_1}\right)$$

$$D_{14}=3\left(1+\frac{\partial u_1}{\partial x_1}\right)\left(1+\frac{\partial u_2}{\partial x_2}\right)+3\left(\frac{\partial u_2}{\partial x_1}\right)\left(\frac{\partial u_1}{\partial x_2}\right)$$

$$D_{21}=-4\left(1+\frac{\partial u_1}{\partial x_1}\right)\left(1+\frac{\partial u_2}{\partial x_2}\right)-4\left(\frac{\partial u_2}{\partial x_1}\right)\left(\frac{\partial u_1}{\partial x_2}\right)$$

$$D_{22}=4\left(1+\frac{\partial u_1}{\partial x_1}\right)\left(\frac{\partial u_1}{\partial x_2}\right)-4\left(1+\frac{\partial u_2}{\partial x_2}\right)\left(\frac{\partial u_2}{\partial x_1}\right)$$

$$D_{23}=4\left(1+\frac{\partial u_1}{\partial x_1}\right)\left(\frac{\partial u_2}{\partial x_1}\right)-4\left(1+\frac{\partial u_2}{\partial x_2}\right)\left(\frac{\partial u_1}{\partial x_2}\right)$$

$$D_{24}=-2\left(1+\frac{\partial u_1}{\partial x_1}\right)^2-2\left(1+\frac{\partial u_2}{\partial x_2}\right)^2+2\left(\frac{\partial u_1}{\partial x_2}\right)^2+2\left(\frac{\partial u_2}{\partial x_1}\right)^2$$

$$D_{31}=3\left(1+\frac{\partial u_1}{\partial x_1}\right)\left(\frac{\partial u_2}{\partial x_1}\right)-3\left(1+\frac{\partial u_2}{\partial x_2}\right)\left(\frac{\partial u_1}{\partial x_2}\right)$$

$$D_{32}=3\left(\frac{\partial u_1}{\partial x_2}\right)^2-3\left(1+\frac{\partial u_1}{\partial x_1}\right)^2$$

$$D_{33}=3\left(1+\frac{\partial u_1}{\partial x_1}\right)\left(1+\frac{\partial u_2}{\partial x_2}\right)+3\left(\frac{\partial u_2}{\partial x_1}\right)\left(\frac{\partial u_1}{\partial x_2}\right)$$

$$D_{34}=-6\left(1+\frac{\partial u_1}{\partial x_1}\right)\left(\frac{\partial u_1}{\partial x_2}\right)$$

$$D_{41}=4\left(1+\frac{\partial u_1}{\partial x_1}\right)\left(\frac{\partial u_1}{\partial x_2}\right)-4\left(1+\frac{\partial u_2}{\partial x_2}\right)\left(\frac{\partial u_2}{\partial x_1}\right)$$

$$D_{42}=-4\left(1+\frac{\partial u_1}{\partial x_1}\right)\left(1+\frac{\partial u_2}{\partial x_2}\right)+4\left(\frac{\partial u_2}{\partial x_1}\right)\left(\frac{\partial u_1}{\partial x_2}\right)$$

$$D_{43}=-2\left(1+\frac{\partial u_1}{\partial x_1}\right)^2-2\left(1+\frac{\partial u_2}{\partial x_2}\right)^2+2\left(\frac{\partial u_1}{\partial x_2}\right)^2+2\left(\frac{\partial u_2}{\partial x_1}\right)^2$$

$$D_{44}=-4\left(1+\frac{\partial u_1}{\partial x_1}\right)\left(\frac{\partial u_2}{\partial x_1}\right)+4\left(1+\frac{\partial u_2}{\partial x_2}\right)\left(\frac{\partial u_1}{\partial x_2}\right)$$

$$\boldsymbol{H}=\begin{bmatrix}\dfrac{\partial^2 u_1}{\partial x_1^2}\\[2mm]\dfrac{\partial^2 u_2}{\partial x_1^2}\\[2mm]\dfrac{\partial^2 u_1}{\partial x_1\partial x_2}\\[2mm]\dfrac{\partial^2 u_2}{\partial x_1\partial x_2}\end{bmatrix}$$

A solution to the equilibrium equations exists if the determinant $|\boldsymbol{D}|$ is equal to zero. A non-linear expression in terms of the deformation gradients is yielded which applies a further restriction on the field distribution of stress.

Complex Potential Functions

Due to expression (10) the displacements $u_\alpha(x_1, x_2)$ can now be expressed as functions of the complex variable $z = x_1 + ix_2$ and will have conjugate pairs given by the Cauchy-Riemann relationships. The in-plane extension ratios $\lambda_{\alpha\beta}$ can also be expressed in terms of complex functions (Nurse, 1995). Let:

$$\Lambda_1(z) = (1 + \partial u_1(z) / \partial x_1) - i(\partial u_1(z) / \partial x_2)$$
$$\Lambda_2(z) = (1 + \partial u_2(z) / \partial x_2) + i(\partial u_2(z) / \partial x_1) \qquad (12)$$

then:

$$\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} = \begin{bmatrix} \Re e(\Lambda_1) & -\Im m(\Lambda_1) \\ \Im m(\Lambda_2) & \Re e(\Lambda_2) \end{bmatrix}, \quad I(z) = |\Lambda_1|^2 + |\Lambda_2|^2, \quad J(z) = \frac{1}{2}\left(\Lambda_1\overline{\Lambda_2} + \overline{\Lambda_1}\Lambda_2\right) \qquad (13)$$

SOLUTION TO THE TRACTION-FREE CRACK PROBLEM

Anticipating a neo-singularity at z=0 the crack faces are prescribed to be traction-free on the real axis to the left of the origin, i.e.:

$$\sigma_{\alpha 2}(x_1, 0+) = \sigma_{\alpha 2}(x_1, 0-) = 0, \quad x_1 \leq 0 \qquad (14)$$

The complex functions $\Lambda_1(z)$ and $\Lambda_2(z)$ are assumed to admit the following expressions:

$$\Lambda_1(z) = \frac{C_1}{z^m}, \quad \Lambda_2(z) = \frac{C_2}{z^m} \qquad (15)$$

where C_1 and C_2 are generally complex. The field solution is obtained by the non-linear eigenvalue problem of the form:

$$2(AJ^3 + CJ)\Re e(\Lambda_2) + (BJ^3 - 2CI)\Re e(\Lambda_1) = 0$$
$$2(AJ^3 + CJ)\Im m(\Lambda_1) + (BJ^3 - 2CI)\Im m(\Lambda_2) = 0, \quad z = re^{\pm i\pi} \qquad (16)$$

where the invariants I and J must be expressed in terms of $\Lambda_1(z)$ and $\Lambda_2(z)$ using expression (13).

DISCUSSION

General

Consideration is now drawn to the problem of determining the field stresses for finite-sized crack geometries with known boundary conditions. Knowles and Sternberg (1973) sought an asymptotic representation of the near-tip field stresses using the solution to the linearised problem of the crack in an infinite plate of Inglis (1913).

The deformation is assumed to be small in the far-field region, i.e. $|x| >> 0$, and, therefore, the stress-deformation relationship behaves linearly:

$$\sigma_{\alpha,\beta} = \tau_{\alpha,\beta} = 2\mu\left[\frac{\upsilon}{1-2\upsilon}\delta_{\alpha,\beta}u_{\rho,\rho} + \frac{1}{2}\left(u_{\alpha,\beta} + u_{\beta,\alpha}\right)\right] \qquad (17)$$

where μ is the elastic shear modulus and υ is Poisson's ratio. Using expression (12) the stresses can be written in the following complex form:

$$\sigma_{22} - \sigma_{11} + 2i\sigma_{12} = 2\mu(\Lambda_2 - \Lambda_1)$$
$$\sigma_{22} + \sigma_{11} = 2\mu\frac{\upsilon}{1-2\upsilon}(\Re e(\Lambda_2 + \Lambda_1) - 2) \qquad (18)$$

In the manner used by Knowles and Sternberg for the infinite plate the finite geometry problem may be linearised in terms of the boundary stresses. Any loading at the boundary of the crack problem is permitted in the infinitesimal strain range. Expression (18) can be used to solve for the boundary-value problem using methods similar to those described by Muskhelishvili (1953).

The Interfacial Crack

It is now assumed that each material half is represented by a pair of complex functions, i.e. $\Lambda_1^U(z)$ and $\Lambda_2^U(z)$ for the upper half, and $\Lambda_1^L(z)$ and $\Lambda_2^L(z)$ for the lower half. Herrmann (1989) adopted the same approach as Knowles and Sternberg (1973) in the determination of the solution for the interfacial crack with loading applied at infinity.

To preserve the continuity of the deformation and stresses across the interface there exists bond conditions (Herrmann, 1989):

$$y_\alpha(x_1, 0+) = y_\alpha(x_1, 0-), \quad \sigma_{\alpha 2}(x_1, 0+) = \sigma_{\alpha 2}(x_1, 0-), \quad x_1 > 0 \qquad (19)$$

Another eigenvalue problem results from expression (19) that must be solved in conjunction with (16) for a traction-free crack. The aforementioned approach to the solution of finite geometries is also applicable. However, in finite geometries for interface problems it must be noted that two ends of the interface may exist where boundary conditions are prescribed. For an accurate solution to the field stresses consideration may have to be paid to the existence of a pair of 'coupled' neo-singularities at the ends of the interface.

CONCLUSIONS

Formulation of the non-linear elastostatic problem for a traction-free crack is presented in terms of complex potential functions for the deformation gradients. The determination of full-field solution to finite-sized geometries is possible by expressing the boundary values in terms of linearised stresses.

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