ON FRACTURE OF AN INFINITE ELASTIC BODY IN COMPRESSION ALONG A CYLINDRICAL DEFECT

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ABSTRACT

A three-dimensional fracture problem of an infinite cracked body is considered. The body is subjected to uniform compression along a circular cylindrical defect. The Griffith-Irwin theory is not applicable to this scheme of loadings since all the stress intensity factors are zero. Instead, a stability criterion within the framework of the three-dimensional linearized stability theory is used. It is based on a mechanism of a local stability loss around a crack-like defect. The general approach makes it possible to use a uniform analytical method in analyzing the problem for compressible and incompressible elastic bodies. It involves the application of an arbitrary form of the elastic potential for large subcritical deformations and for two variants of the small subcritical deformations theory. The numerical analysis of the derived problem for eigenvalues has been carried out for different models of materials. An influence of geometric parameter and the material properties on a value of the critical loadings is studied.

KEYWORDS

Non-classical fracture mechanics, linearized stability theory, local stability loss, compression along a crack, cylindrical crack, critical loadings, problem for eigenvalues.

INTRODUCTION

It is well-known that the problems of fracture mechanics of materials under compression along a crack represent a special class of problems because they cannot be adequately described by the relations of the linear fracture mechanics. As the stress intensity factors, crack opening displacements and energy release rate are independent of these loads (Cherepanov, 1974), the classical fracture criteria of the Griffith–Irwin type, the crack propagation criterion (Wells, 1963; Panasyuk, 1986) and also their generalizations used in the linear fracture mechanics are not applicable to this scheme of loadings.

In the situation when compressive loads are applied to the bodies along a crack, failure may occur by instability. For these non-classical problems, we will use a failure criterion (Guz', 1981) within the framework of linearized stability theory according to which the initiation of the fracture process coincides with a local stability loss near the crack-like defects.

Planar and spatial fracture problems of materials examined on the basis of linearized formulations in the case of compression of solids along a single crack or crack arrays distributed in the same plane were analysed by Wu(1979, 1980), Guz'(1982, 1983); compression of materials along subsurface cracks were examined by a number of authors (Keer *et al.*, 1982; Guz' and Nazarenko, 1985; Nazarenko, 1986a, b); compression along a periodic system of parallel cracks and two internal parallel cracks were studied by Guz' *et al.*, (1984, 1987).

However, until now all these investigations based on the linearized theory were carried out for solids with the plane cracks. In the present paper compression of elastic mediums along a crack located on a cylindrical surface within the linearized stability theory of deformable solids is studying.

PROBLEM FORMULATION

Consider an infinite elastic medium with a circular cylindrical crack of lenght $2a$ and radius b. Let (r, θ, x_3) be the cylindrical coordinates with the x_3- axis, coinciding with the axis of the cylindrical crack. Then, the crack occupies the region: $\{r = b,\ 0 \le \theta < 2\pi,\ -a \le x_3 \le a\}$. The infinite body is subjected to uniform uni-axial pressure in the x_3- direction at infinity.

The disturbance of asymmetric Kirchhoff stress tensor denoted by t is referred to the unit area of the undeformed body. The disturbance of the displacement vector is $\vec{u}$. The symmetric stress tensor in the undisturbed state is denoted by S.

As a result of the compression parallel to the crack axis, a homogeneous subcritical stress and strain state occurs near the crack-like defects

$$S^0_{11} = S^0_{22} = 0\,, \qquad S^0_{33} \neq 0\,, \qquad (S^0_{33} = const)\,;$$
$$u^0_j = \delta_{jm}(\lambda_j - 1)x_m,\quad \lambda_1 = \lambda_2 \neq \lambda_3\,,\quad (\lambda_j = const)\,;\quad (\lambda_3 < 1)\,, \tag{1}$$

where the superscript "0" refers to the initial state so that S^0_{ij} are the components of symmetric stress tensor in the undisturbed state; u^0_j are the components of the displacement vector; x_m are Lagrangian coordinates which coincides with the Cartesian coordinates in the undeformed state; the parameters λ_j are the contractional ratio while δ_{ij} is Kronecker delta.

Denote by the superscript "1" the quantities referred to the region $r < b$ and by the superscript "2" the quantities referred to the region $r > b$. It is assumed that the crack surfaces are free of stresses. With reference to the foregoing two regions we have the following conditions

$$t^{(1)}_{rr} = 0\,,\ t^{(2)}_{rr} = 0\,,\ t^{(1)}_{r3} = 0\,,\ t^{(2)}_{r3} = 0\,,\ (r = b, -a \le x_3 \le a)\,. \tag{2}$$

At the boundary between the regions "1" and "2" outside the crack ($r = b, |x_3| > a$), the conditions of the stresses and displacements continuity should be satisfied

$$u^{(1)}_r = u^{(2)}_r\,,\quad u^{(1)}_3 = u^{(2)}_3 \quad (r = b\,, |x_3| > a)\,;$$
$$t^{(1)}_{rr} = t^{(2)}_{rr}\,,\quad t^{(1)}_{r3} = t^{(2)}_{r3} \quad (r = b\,, |x_3| > a)\,. \tag{3}$$

Due to symmetry with respect to the plane $x_3 = 0$, the boundary conditions (2) and (3) can be rewritten as follows

$$t^{(1)}_{rr} = t^{(2)}_{rr}\,,\quad t^{(1)}_{r3} = t^{(2)}_{r3}\,,\quad (\rho = 1, 0 \le |\zeta| < \infty)\,; \tag{4}$$
$$u^{(1)}_r - u^{(2)}_r = 0\,,\quad u^{(1)}_3 - u^{(2)}_3 = 0\,,\quad (\rho = 1\,, |\zeta| > \beta)\,; \tag{5}$$
$$t^{(1)}_{rr} = t^{(2)}_{rr} = 0\,,\quad t^{(1)}_{r3} = t^{(2)}_{r3} = 0\,,\quad (\rho = 1\,, |\zeta| < \beta)\,. \tag{6}$$

where

$$\rho = r/b;\quad \zeta = x_3/b;\quad \beta = a/b\,, \tag{7}$$

are the dimensionless values.

General Solution of the Axisymmetric Problem

The general solutions in the case of axisymmetric problem under the homogeneous initial state (1), for unequal roots ($n_1 \neq n_2$) are represented by relations

$$u_r = \frac{\partial \varphi_1}{\partial r} + \frac{\partial \varphi_2}{\partial r}\,;\quad u_3 = m_1 \frac{\partial \varphi_1}{\partial x_3} + m_2 \frac{\partial \varphi_2}{\partial x_3}\,. \tag{8}$$

where the potential functions $\varphi_i(r, x_3), i = 1, 2$ satisfy the following equations

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + n_i \frac{\partial^2}{\partial x_3^2}\right)\varphi_i(r, x_3) = 0\,, \qquad (i = 1, 2)\,. \tag{9}$$

The representations for the components of Kirchhoff stress tensor t are given by

$$t_{r3} = C_{44}\left[d_1 \frac{\partial^2 \varphi_1}{\partial r \partial x_3} + d_2 \frac{\partial^2 \varphi_2}{\partial r \partial x_3}\right]\,;$$
$$t_{rr} = C_{44}\left[-p_1\left(l_1 \frac{\partial}{r\partial r} + \frac{\partial^2}{\partial x_3^2}\right)\varphi_1 - p_2\left(l_2 \frac{\partial}{r\partial r} + \frac{\partial^2}{\partial x_3^2}\right)\varphi_2\right]\,. \tag{10}$$

where the following notations are used

for compressible bodies

$$C_{44} = \omega_{1313}\,,\quad m_i = (\omega_{1111} n_i - \omega_{3113})(\omega_{1133} + \omega_{1313})^{-1}\,;$$
$$d_i = 1 + m_i \omega_{1331} \omega_{1313}^{-1}\,; \qquad (i = 1, 2)\,; \tag{11}$$
$$p_i = (n_i \omega_{1111} + m_i \omega_{1133})\omega_{1313}^{-1}\,;$$
$$l_i = (\omega_{1111} - \omega_{1122})(n_i \omega_{1111} + m_i \omega_{1133})^{-1}\,;$$

for incompressible bodies

$$C_{44} = \text{æ}_{1313}, \qquad m_j = q_{11}q_{33}^{-1}n_j;$$
$$d_j = 1 + \text{æ}_{1331}\text{æ}_{1313}^{-1}m_j; \qquad (j = 1,2);$$
$$p_j = \left[q_{11}q_{33}^{-1}(\text{æ}_{1133} + \text{æ}_{1313})n_j - \text{æ}_{1133}m_j + \text{æ}_{3113}\right]\text{æ}_{1313}^{-1}; \tag{12}$$
$$l_j = (\text{æ}_{1111} - \text{æ}_{1122})\left[q_{11}q_{33}^{-1}(\text{æ}_{1133} + \text{æ}_{1313})n_j - \text{æ}_{1133}m_j + \text{æ}_{3113}\right]^{-1},$$

Note that representations (8), (10) are formally similar to representations of the general solutions of the linear elasticity theory for the transversely isotropic body. They turn into the latter only when $\omega_{1313} = \omega_{3113} = \omega_{1331}$. But this symmetry condition is not realized in the case under consideration.

RESEARCH DESIGN AND METHODS

In order to obtain a system of dual integral equations, the theory of the Fourier-Hankel integral transforms is used. Then the derived integral system by use of the series expansion method is reduced to a system of the linear algebraic equations.

Derivation of dual integral equations

The potential functions $\varphi_1(r, x_3)$ and $\varphi_2(r, x_3)$ referring to the regions "1" and "2" can be represented in the form of Fourier-Hankel integral transforms that satisfy the regularity conditions for the displacements and stresses as $r \to \infty$. Referred to the region "1" we have

$$C_{44}\varphi_i^{(1)} = \int_0^\infty a_i(\lambda) I_0(\sqrt{n_i}\lambda\rho)\cos\zeta\lambda d\lambda\,, \quad (i = 1,2). \tag{13}$$

Similarly, to the region "2"

$$C_{44}\varphi_i^{(2)} = \int_0^\infty b_i(\lambda) K_0(\sqrt{n_i}\lambda\rho)\cos\zeta\lambda d\lambda\,, \quad (i = 1,2), \tag{14}$$

where $a_i(\lambda), b_i(\lambda), i = 1,2$ are unknown functions; I_0 and K_0 are the modified Bessel functions.

Substituting (13), (14) into (8), (10) and making use of the boundary conditions (4), we determine the unknown functions $b_i(\lambda)$ in the terms of $a_i(\lambda), i = 1,2$. From the remaining boundary conditions (5), (6), the following system of dual integral equations is obtained with respect to the new unknown functions $K_r(\lambda)$ and $K_z(\lambda)$

$$\begin{aligned} &\int_0^\infty \lambda\Big[\alpha_{11}K_r(\lambda) + \overline{\psi_{11}}(\lambda)\; K_r(\lambda) + \\ &\quad + \overline{\psi_{12}}(\lambda)\; K_z(\lambda)\Big]\cos\zeta\lambda d\lambda = 0\,, \\ &\qquad\qquad (0 \le \zeta < \beta); \\ &\int_0^\infty \lambda\Big[\alpha_{22}K_z(\lambda) + \overline{\psi_{21}}(\lambda)\; K_r(\lambda) + \\ &\quad + \overline{\psi_{22}}(\lambda)\; K_z(\lambda)\Big]\sin\zeta\lambda d\lambda = 0\,, \end{aligned} \tag{15}$$

$$\begin{aligned} &\int_0^\infty K_r(\lambda)\cos\zeta\lambda d\lambda = 0\,, \\ &\qquad\qquad (\beta < \zeta < \infty)\,. \\ &\int_0^\infty K_z(\lambda)\sin\zeta\lambda d\lambda = 0\,, \end{aligned} \tag{16}$$

where α_{11} and α_{22} are the following constans

$$\alpha_{11} = \frac{\sqrt{n_1}d_1p_2 - \sqrt{n_2}d_2p_1}{2(d_1 - d_2)\sqrt{n_1}\sqrt{n_2}}\,; \quad \alpha_{22} = \frac{\sqrt{n_1}d_1p_2 - \sqrt{n_2}d_2p_1}{2(m_1p_2 - m_2p_1)}\,. \tag{17}$$

and

$$\overline{\psi_{11}}(\lambda) = \overline{\psi_{22}}(\lambda) = O(1/\lambda^2)\,; \quad \overline{\psi_{12}}(\lambda) = \overline{\psi_{21}}(\lambda) = O(1/\lambda)\,. \tag{18}$$

The obtained dual integral equations will be reduced to a system of linear algebraic equations.

Solution of the Dual Integral Equations

The possible way of solving of the system (15), (16) is as follows. The unknown functions are chosen in such manner that the equations (16) will be satisfied automatically. Therefore, one can solve these integral equations by use of the series expansion method. That is, we make use of the following expansions

$$K_r(\lambda) = \sum_{j=0}^{\infty} a_j\lambda^{-1}J_{2j+1}(\beta\lambda)\,; \quad K_z(\lambda) = \sum_{j=0}^{\infty} b_j\lambda^{-1}J_{2j+2}(\beta\lambda)\,, \tag{19}$$

where a_j, b_j are unknown constants; J_i are Bessel functions of the first kind. After substituting equations (19) into (16), we find that the both integral equations are automatically satisfied.

From the remaining equations (15), making use of the following expansions for functions $\cos\zeta\lambda$ and $\sin\zeta\lambda$

$$\begin{aligned} &\cos\zeta\lambda = \sum_{i=0}^{\infty} \varepsilon_i J_{2i}(\beta\lambda)\cos 2i\varphi\,; \quad \varepsilon_0 = 1\,, \ \varepsilon_i = 2 \ (i \ge 1)\,; \\ &\sin\zeta\lambda = \sum_{i=0}^{\infty} 2J_{2i+1}(\beta\lambda)\sin(2i+1)\varphi\,; \quad \varphi = \arcsin(\zeta/\beta)\,, \end{aligned} \tag{20}$$

we obtain the system of the linear algebraic equations with respect to the unknown a_j and b_j

$$\begin{aligned} &\sum_{j=0}^{\infty} a_j(\alpha_{11}R_{ij}^* + R_{ij}) + \sum_{j=0}^{\infty} b_jT_{ij} = 0\,, \quad i = 0,1,2,3,4,\ldots\,; \\ &\sum_{j=0}^{\infty} a_jP_{ij} + \sum_{j=0}^{\infty} b_j(\alpha_{22}Q_{ij}^* + Q_{ij}) = 0\,, \quad i = 0,1,2,3,4,\ldots\,. \end{aligned} \tag{21}$$

where $R_{ij}, \ldots, Q_{ij}$ are infinite integrals.

For the existence of a non-trivial solution of the homogeneous system of the algebraic equations (21), the determinant of this system must be set equal to zero.

Hence, for compressible and incompressible bodies we have, respectively, the following relations

$$\det \|d_{kl}(\beta, \lambda_3, \omega)\| = 0\,; \quad \det \|d_{kl}(\beta, \lambda_3, æ)\| = 0\,, \quad (k, l = 1, 2, \ldots)\,. \tag{22}$$

Therefore, the problem for eigenvalues with respect to the parameter $\lambda_3 < 1$ is obtained.

NUMERICAL RESULTS AND CONCLUSIONS

The problem reduced to the eigen-value problems for equations (22) are solved numerically. In solving equations (22), we must evaluate the infinite integrals denoted by $R_{ij}, \ldots, Q_{ij}$, $(i, j = 0, 1, 2, \ldots)$. For this purpose we separate a finite and an asymptotic part of these integrals. The former one is evaluated by use of Gaussian-quadrature formulae. The latter part of the integrals after rearranging can be written in the following form

$$F_k(m, n, y_0) = \int_{y_0}^{\infty} \frac{J_m(y)J_n(y)}{y^k} dy\,, \qquad (k = 1, 2)\,, \tag{23}$$

and it has been evaluated by use of the following formulae (Kasano H. *et al.*, 1981)

$$F_1(m, n, x_0) = \frac{2\sin[\frac{(m-n)\pi}{2}]}{\pi(m^2 - n^2)} + \frac{x_0}{m^2 - n^2}\Big[J_{m+1}(x_0)J_n(x_0) - J_m(x_0)J_{n+1}(x_0)\Big] - \frac{J_m(x_0)J_n(x_0)}{m+n}\,; \qquad (m \neq n)\,;$$

$$F_1(m, m, x_0) = \frac{1}{2m}\Big[J_0^2(x_0) + 2\sum_{s=1}^{m-1} J_s^2(x_0) + J_m^2(x_0)\Big]\,; \quad (m = n > 0)\,;$$

$$\ldots \text{ and so on} \tag{24}$$

Numerical results are obtained for elastic solids with the Lame potential as well as for the series of the uni-directionally fiber reinforced composite materials, for example, such as e.glass–epoxy and graphite–epoxy.

The values which enter into the foregoing relations are presented below.

Lamé potential (compressible body)

$$\Phi = \frac{1}{2}\lambda A_1^2 + \mu A_2\,. \tag{25}$$

where A_1, A_2 are algebraic invariants of the Green's strain tensor; λ, μ are parameters of Lamé. The values (11) for the Lamé potential (25) can be written in the following form

$$\begin{aligned} m_1 &= 1 + 4\nu(1+\nu)(\lambda_3 - 1)\,, & m_2 &= 1\,; \\ d_1 &= 2 + 4\nu(1+\nu)(\lambda_3 - 1), & d_2 &= 2\,; \\ p_1 &= 2 + 2(1+2\nu)(1+\nu)(\lambda_3 - 1)\,, & p_2 &= 2 + 2(1+\nu)(\lambda_3 - 1)\,; \\ l_1 &= 1/(1 + (1+2\nu)(1+\nu)(\lambda_3 - 1))\,, & l_2 &= 1/(1 + (1+\nu)(\lambda_3 - 1))\,. \end{aligned} \tag{26}$$

The dependence of the critical shortenings λ_3^{cr} on the dimensionless parameter β^{-1} is presented in Fig.1. The plot shows that the curves of the critical shortening λ_3^{cr} increase as $\beta^{-1} \to \infty$. The curves 1, 2 and 3 correspond to values of Poisson's ratio $\nu = 0.1$; 0.2 and 0.3, respectively.

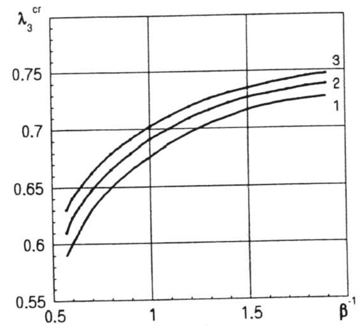

Fig.1

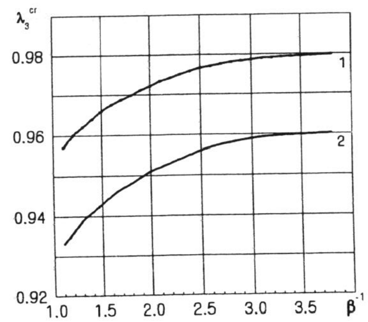

Fig.2

Composites (compressible materials, small subcritical deformation theory)

The results for uni-directionally fiber reinforced composites made of e.glass–epoxy (a fibre concentration is 0.65) and graphite–epoxy (a fibre concentration is 0.5) are obtained. It is assumed that minimum crack dimensions in the composites are essentially larger than the dimensions of its structural elements, i.e. macrocracks are considered. Under such assumptions, the composites are regarded to posses transverse isotropy.

$$\begin{aligned} &\omega_{1111} = \frac{(1 - e\nu'^2)e}{(1+\nu)(1-\nu-2e\nu'^2)}E'\,, \quad \omega_{1133} = \frac{e}{g}E'\,; \\ &\omega_{1313} = \frac{e}{g}E'\,, \qquad \omega_{3113} = \left(\frac{e}{g} + \lambda_3 - 1\right)E'\,; \\ &\omega_{1133} = \frac{e\nu'}{(1-\nu-2e\nu'^2)}E'\,, \qquad e = \frac{E}{E'}\,, \quad g = \frac{E}{G'}\,; \\ &\omega_{3333} = \left(\frac{1-\nu}{(1-\nu-2e\nu'^2)} + \lambda_3 - 1\right)E'\,, \quad \omega_{1221} = \frac{e}{2(1+\nu)}\,. \end{aligned} \tag{27}$$

Variations of the critical values of λ_3^{cr} for composites with $\nu = 0.3, \nu' = 0.2, E/E' = 0.8$ and different ratios of E/G' are shown in Fig.2. The curves 1 and 2 correspond to $E/G' = 40$ and $E/G' = 20$, respectively. Here, E, G and ν are Young's modulus, shear modulus and Poisson's ratio in the isotropic plane, while E', G' and ν' are the corresponding ones in the principal material direction. As indicated by the results, the ratio E/G' and parameter β^{-1} have a strong effect on the critical values of λ_3^{cr}.

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