NEW DIRECTIONS : ANALYSIS OF CRACKING LOCALIZATION AND ULTIMATE STRENGTH OF STRUCTURES IN CIVIL ENGINEERING

H. HORII and J. INOUE
Department of Civil Engineering, University of Tokyo
Bunkyo-ku, Tokyo 113, JAPAN

ABSTRACT

In the present paper, a theory and an analysis method for cracking localization are presented. The method to judge the initiation of localization and to find the localized solution is presented through consideration of spring-crack element models. It is shown that the problem is reduced to a minimization problem under constraint conditions. The theory is generalized and applied to problems of cracking localization in concrete. The proposed method can reproduce the localization of initially distributed cracks into a single crack in a four-point bending test. It is shown that to capture the ultimate strength of structures treatment of the localization of cracks or shear failure planes is necessary in many cases, and minimization problems under constraint conditions are the problems to be solved.

KEYWORDS

Fracture mechanics of concrete, cracking localization, ultimate strength of structures.

INTRODUCTION

Fracture Mechanics of Concrete: Fracture mechanics is one of important study fields which has brought remarkable contributions. After the great success in brittle materials, fracture mechanics has been growing by extending its subjects. Fracture mechanics of concrete has been studied extensively for more than 10 years (Mihashi *et al.*, ed., 1993; Karihaloo, 1995; Shar *et al.*, 1995; Wittmann, ed., 1995), while the pioneering work by Hillerborg *et al.* (1976) is traced back to 1970's. Mode I crack growth in concrete is characterized by the stress transmission across the crack surfaces and the material property is represented by a tension-softening curve, the relationship between the transmitted stress and the crack opening displacement. The success in fracture mechanics of concrete is not limited in the modeling and material characterization, but it bears fruit in design methods and development of new materials.

Fracture Mechanics Based Design Method: A fracture mechanics based design method of steel-fiber-reinforced concrete (SFRC) tunnel linings proposed by Nanakorn and Horii (1996a) is adopted in a design provision in Japan. The design method is studied based on FEM analysis with an element having a displacement discontinuity and a tension-softening curves obtained by the back analysis method proposed by Nanakorn and Horii (1996b). Existence of a crack and transmission of stress by fibers are considered in the estimation of the maximum resultant forces of the critical cross-section.

Development of new materials: Fracture mechanics also plays an important role in development of new materials. Li (1996) has developed strain-hardening cementitious composites, named Engineered Cementitious Composite, ECC. An analytical tool is necessary for the evaluation of the performance of new materials used in structural members. Kabele and Horii (1996) proposed such a numerical method for ECC. Both multiple and localized cracking, which are characteristic of this class of materials, are represented in the model. A plasticity-based modeling method is used for the distributed multiple cracking, while discrete modeling is employed for the localized cracks. Fracture tests of pseudo strain-hardening cement-based composites are analyzed, and predicted fracture energy is compared with reported experimental results.

Cracking Localization and Ultimate Strength of Structures: Tremendous damage of structures in Kobe earthquake (The Great Hanshin Earthquake) demonstrated the new role and direction of fracture mechanics. The shear failure of RC (Reinforced Concrete) columns led to catastrophic collapse of structures as shown in Fig.1. RC structures must be designed so that shear capacity of their cross sections is larger than its bending capacity. However, the estimation of the shear capacity which depends on the size of the structure and many other factors is based on experimental data and no analytical methods exist which can predict the shear capacity as a result of numerical computations, except that Okamura and Maekawa's group recently succeeded in predicting the shear failure by FEM for RC structures (An, 1996).

To capture the shear failure of RC columns, the direct application of fracture mechanics of concrete is not sufficient, but the treatment of cracking localization is required. For structures without a notch or a stress concentrator, the cracking condition is satisfied at many places and conventional numerical methods without judgment of localization predict cracks distributed in a region. In reality, the cracking is localized and discrete cracks are formed. It is not just a matter of numerical techniques, but a fundamental difficulty lies behind this problem.

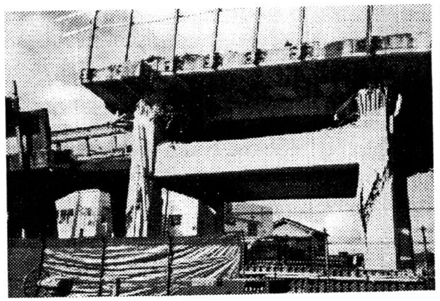

Figure 1: Collapse of Shinkansen RC rigid frame viaduct by shear failure.

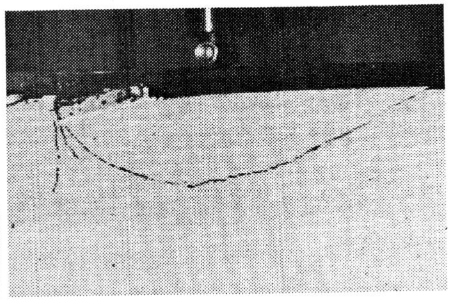

Figure 2: Propagation of shear failure planes in artificial soft rock.

The same problem is encountered in the numerical analysis of formation and propagation of shear failure planes (shear bands, sliding surfaces), which is important for example in the safety assessment of rock foundation of important structures. If displacement discontinuities are introduced when the failure condition is satisfied, then the shear failure planes are distributed over a region. In reality, the shear failure planes are localized as shown in Fig.2.

The allowable stress design method has been replaced with the ultimate state design method in various fields. This implies the necessity of analytical tools which can capture the ultimate strength of structures. Since the ultimate strength of structures are governed by the formation and propagation of displacement discontinuities in many cases, the associated localization problem possesses essential importance.

Localization phenomena are observed in variety of materials with different mechanisms of inelastic behaviors. Their global behaviors have common features. Hence it is expected that the localization phenomena in different materials are captured by a unified theory. Localization phenomena are classified into two categories; localization in material level and that in structural level. Strain localization in metals, which has been a hot topic in solid mechanics (Needleman and Tvergaard, 1992; Tomita, 1994), belongs to the former category, while the cracking localization discussed in this paper is classified as localization in structural level.

Two types of localization phenomena share a common principle. They often appear in the same problem. For example, the formation of the shear failure plane is the result of strain localization, while the progress and elastic unloading of multiple shear failure planes are governed by the localization in structural level. In this paper, attention is paid to the localization in structural level, and the localization in material level is treated with a simple failure condition.

LOCALIZATION IN SIMPLE SPRING-CRACK ELEMENT MODELS

In this section, simple models with spring and crack element are studied in order to establish a theory to judge the bifurcation from a fundamental or "homogeneous" solution to a bifurcated or "localized" solution and to select the "localized" solution that nature chooses.

Model with a Spring and a Single Crack Element

Fig.3 shows a model which consists of a spring with a spring constant k and a crack element. The crack element starts to open when the force f reaches the capacity f_c, and after that the force changes as a linear function of the crack opening displacement α with a modulus A, as shown in Fig.4. The crack opening displacement α is a monotonically increasing function and its increment must be non-negative. When A is positive and the crack element shows a softening behavior, the crack element has two choices with a decreasing force: to follow the solid line called loading path or to follow the dashed line called unloading path in Fig.4. The problem is to find α for a given displacement $\bar{u}$. $\bar{u}$ is assumed to be monotonically increasing, and hence it is not necessary to consider the unloading path in this problem.

The solution of this problem can be easily obtained from equilibrium condition:

$$f_c - A\alpha = k(\bar{u} - \alpha) \rightarrow \alpha = \frac{f_c - k\bar{u}}{A - k} \tag{1}$$

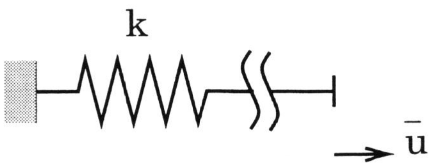

Figure 3: A model with a spring and a single crack element.

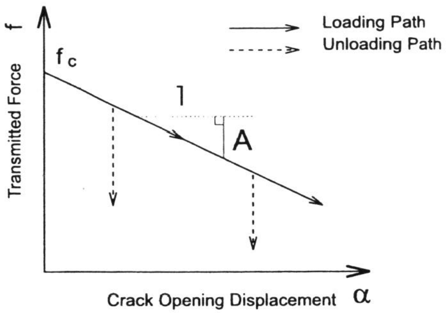

Figure 4: Behavior of crack element.

An equivalent formulation is possible with a potential function U,

$$U = \frac{1}{2}k(\bar{u} - \alpha)^2 + (f_c\alpha - \frac{1}{2}A\alpha^2). \tag{2}$$

The equilibrium condition is equivalent to the stationary condition:

$$\frac{dU}{d\alpha} = 0. \tag{3}$$

Two formulations are equivalent, but the variational formulation provides benefit for judgment of the stability of the solution. The stability of the solution is defined by the sign of the work done by the external system for any disturbance given to the equilibrium solution. It can be shown that the stability of the equilibrium solution is judged by the sign of the second derivative of the potential function U as,

$$\frac{d^2U}{d\alpha^2} \begin{array}{lll} > 0 & : & \text{stable} \quad (k > A) \\ < 0 & : & \text{unstable} \quad (k < A). \end{array} \tag{4}$$

When $A < k$, the solution in Eqn.(1) is stable, and the loading path in Fig.4 is followed with the increasing value of $\bar{u}$. On the other hand, when $A > k$, the solution in Eqn.(1) is unstable. Hence when $\bar{u}$ is increased from 0, the crack element starts to open at $\bar{u} = f_c/k$ and the crack opening displacement increases in an unstable manner.

The potential function in Eqn.(2) can be physically interpreted as a total energy, sum of mechanical energy and dissipated energy. The first term in the right hand side of Eqn.(2) is the strain energy stored in the spring. If the applied force is controlled, instead of the displacement $\bar{u}$, the potential for the external force must be added to the mechanical energy. The second term in the right hand side of Eqn.(2) is the energy dissipated at the crack element. The energy is understood to be dissipated and transformed into thermal energy.

Model with Two Crack Elements

A model shown in Fig.5 consists of two crack elements with no spring. The potential function or total energy is given as a sum of dissipated energy of two crack elements by,

$$U = f_c\alpha - \frac{1}{2}A\alpha^2 + f_c(\bar{u} - \alpha) - \frac{1}{2}A(\bar{u} - \alpha)^2, \tag{5}$$

where α and $\bar{u} - \alpha$ are crack opening displacement of two crack elements, respectively. The stationary condition leads to the fundamental solution:

$$\frac{dU}{d\alpha} = A(\bar{u} - 2\alpha) = 0 \rightarrow \alpha = \frac{1}{2}\bar{u}. \tag{6}$$

The stability of this solution can be checked by taking second derivative of the potential function U.

$$\frac{d^2U}{d\alpha^2} = -2A \begin{array}{lll} > 0 & : & \textit{stable} \\ < 0 & : & \textit{unstable} \end{array} \tag{7}$$

When the fundamental solution is stable with $A < 0$, α is $\bar{u}/2$ and the crack opening displacements of two crack elements are equal. When the stationary solution is unstable with $A > 0$, the crack opening displacement can no longer be equal.

When the fundamental solution is unstable, the "localized" solution is determined by seeking for a minimum point of total energy under the condition that the crack opening displacement of two crack elements are non-negative. The "localized" solution in this problem is either $\alpha = 0$ or $\alpha = \bar{u}$. Both of them make the total energy U minimum within the admissible values for α, $0 \leq \alpha \leq \bar{u}$, as shown in Fig.6.

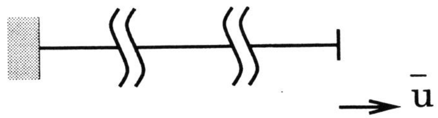

Figure 5: Model with two crack elements.

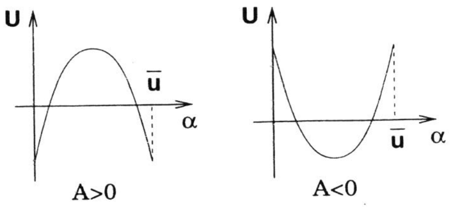

Figure 6: Variation of total energy.

Model with a Spring and Two Crack Elements

Next example is a model with a spring and two crack elements shown in Fig.7. The total energy of this system is given by,

$$U = f_c\alpha_1 - \frac{1}{2}A\alpha_1^2 + f_c\alpha_2 - \frac{1}{2}A\alpha_2^2 + \frac{1}{2}k(\bar{u} - \alpha_1 - \alpha_2)^2, \tag{8}$$

where α_1 and α_2 are crack opening displacements of two crack elements respectively and k is the spring constant. Stationary condition leads to

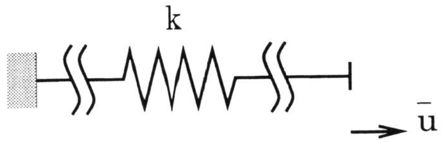

Figure 7: Model with a spring and two crack elements.

$$\begin{cases} \dfrac{\partial U}{\partial \alpha_1} = f_c - A\alpha_1 - k(\bar{u} - \alpha_1 - \alpha_2) = 0 \\ \dfrac{\partial U}{\partial \alpha_2} = f_c - A\alpha_2 - k(\bar{u} - \alpha_1 - \alpha_2) = 0. \end{cases} \quad (9)$$

And the stability can be examined by the sign of eigenvalues of Hessian matrix whose components are the second derivative of the total energy.

$$\left[\frac{\partial^2 U}{\partial \alpha_i \partial \alpha_j}\right] = \begin{bmatrix} -A+k & k \\ k & -A+k \end{bmatrix} \quad (10)$$

The eigenvalues for this Hessian matrix are $-A$ and $-A+2k$. The eigenvalues of the Hessian matrix are the principal curvature of the energy surface which represents the value of the total potential energy U as a function of α_i's.

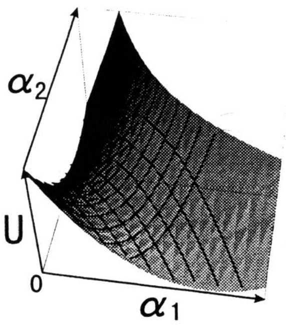

Figure 8: Energy surface ($2k > A > 0$).

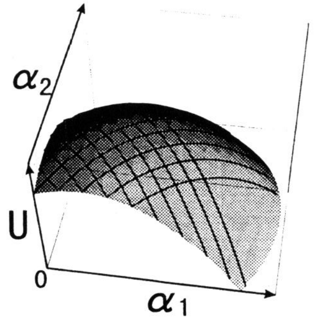

Figure 9: Energy surface ($A > 2k$).

If $A < 0$, the stationary solution is stable and the crack opening displacements of two crack elements are both $(f_c - k\bar{u})/(A-2k)$. If $2k > A > 0$, the stationary solution is unstable and the energy surface of this system is as shown in Fig.8. In this case, $(\alpha_1, \alpha_2) = (\frac{f_c - k\bar{u}}{A-k}, 0)$ and $(0, \frac{f_c - ku}{A-k})$ make energy surface minimum in the region where the crack opening displacement of the crack elements is non-negative. So, in the case of $2k > A > 0$, only one of two crack elements continues to open and the other crack element undergoes elastic unloading and remains closed. If $A > 2k$, the stationary solution is unstable and the energy surface is as shown in Fig.9. In this case, the crack opens in an unstable manner when the force reaches f_c.

GENERAL THEORY

In this section, the findings of the previous section with simple models are generalized to cover localization phenomena in various materials which undergo irreversible processes. The result follows the framework presented by Nguyen (1987).

First we introduce irreversible variable α_i, $(i = 1, \cdots, N)$ which characterizes inelastic behaviors of materials, such as crack opening or sliding of shear failure planes. Here a set of scalar values (vector of finite dimension) is considered for the irreversible variable. The theory can be extended to cover, for example, plastic deformation or damage evolution by introducing the irreversible variables which are field quantities defined as a function of position (vector of infinite dimension). For crack problems, the irreversible variable is a vector whose components are the crack extension length at each crack tips.

The total energy $U(u_k, \alpha_l)$, which is the sum of the mechanical potential energy U^M and the dissipated energy U^D, is defined as a function of a reversible variable u_i, $(i = 1, \cdots, M)$ and the irreversible variable α_i:

$$U(u_i,\ \alpha_j) = U^M(u_k,\ \alpha_l) + U^D(\alpha_m). \quad (11)$$

Note that Eqn.(11) is not a tensor equation. $U^D(\alpha_m)$, for example, is a function of all components of α_m. The mechanical potential energy is the sum of strain energy and external potential energy, and the dissipated energy is the energy transformed from mechanical energy to thermal energy through irreversible processes.

The fundamental solution is obtained from the stationary condition,

$$\frac{\partial U}{\partial u_i} = 0, \quad \frac{\partial U}{\partial \alpha_j} = 0, \quad i = (1, \cdots, M),\ j = (1, \cdots, N). \quad (12)$$

When the reversible variable is a field variable, the first equation of Eqns.(12) becomes variational derivative.

By solving the first equation of Eqns.(12), the reversible variable is expressed in terms of the irreversible as $u_i = u_i(\alpha_k)$. Substituting it into the expression of the total energy, the total energy is expressed as a function only of the irreversible variable as,

$$U^*(\alpha_l) = U^{*M}(\alpha_m) + U^D(\alpha_n), \quad U^{*M}(\alpha_l) = U^M(u_k(\alpha_m), \alpha_n). \quad (13)$$

We consider cases where the loading starts in the elastic region with $\alpha_i = 0$. Irreversible processces become possible when the ith generalized force σ_i associated with α_i reaches the generalized resistance σ_i^* where

$$\sigma_i = -\frac{\partial U^{*M}}{\partial \alpha_i}, \quad \sigma_i^* = \frac{\partial U^D}{\partial \alpha_i}. \quad (14)$$

When irreversible processes are possible at more than one system, the following procedure is necessary to judge the possibility of localization and to find the localized solution.

The sign of eigenvalues of the Hessian matrix, $\left[\frac{\partial^2 U^*}{\partial \alpha_i \partial \alpha_j}\right]$ are investigated to check the stability of the fundamental solution obtained from Eqns.(12). If all eigenvalues are positive, the fundamental solution is stable. Otherwise, the fundamental solution is unstable and the localization occurs.

When the localization takes place, the solution that nature chooses is obtained so that the total energy $U^*(\alpha_i)$ is minimized under the constraint condition,

$$\sigma_i^* \cdot \alpha_i \geq 0, \quad i = (1, \cdots, N) \tag{15}$$

where the summation over i is not taken. Equation (15) represents nothing but the second law of thermodynamics. In the next section, the theory presented in this section is applied for the analysis of crack growth in continuum body.

FEM ANALYSIS OF CRACKING LOCALIZATION

We consider a material like concrete. For simplicity, nonlinear behaviors in compression is ignored and elasticity is assumed. The tensile strength is σ_c and the tension-softening curve shown in Fig.10 is assumed. Crack closure in elastic unloading is ignored for simplicity.

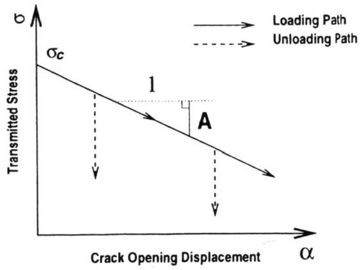

Figure 10: Tension softening curve

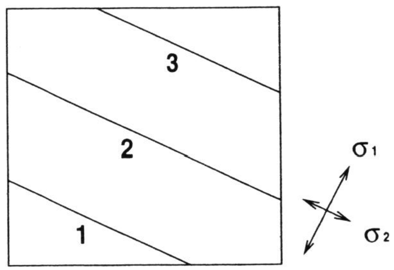

Figure 11: Cracked element

For FEM analysis of crack initiation and propagation, we introduced an element with three displacement continuities with independent displacement jumps, named cracked element, shown in Fig.11. Along each straight line, the displacement jump is assumed to be linear. When the maximum tensile stress reaches the tensile strength, the discontinuity is embedded in the direction normal to the maximum tensile stress. After the crack initiation condition is satisfied, the tension softening relationship shown in Fig.10 is assumed for normal stress and normal opening displacement. Shear stress along the crack is ignored.

All degrees of freedom corresponding to the crack opening are treated as the components of the irreversible variable, α_i, while all other degrees of freedom at nodal points as u_i in the previous theory. The solution of the minimization problem is obtained by Simplex method.

To check the performance of the cracked element an unnotched square region under tension and bending shown in Fig.12 is analyzed. Linear displacements are prescribed along the side boundaries. Three different meshes shown in Fig.13 are employed.

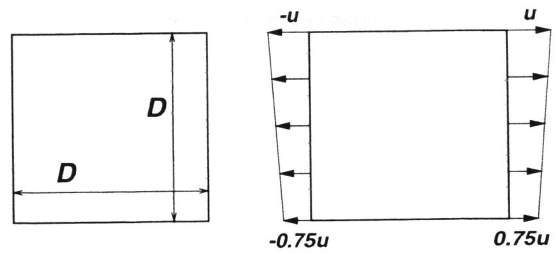

Figure 12: Unnotched square region under tension and bending: $DA/\sigma_c = 1200$, $E/\sigma_c = 10000$.

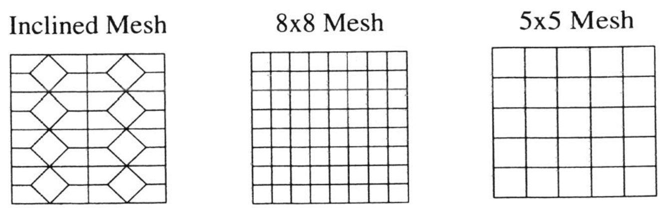

Figure 13: Meshes used in the analysis

Fig.14 shows the load-displacement curves when the localization judgment explained in the previous sections is carried out. It is seen that the results do not depend much on the meshing. The crack pattern is also insensitive to the meshing. Initially distributed cracks tend to localize and at close to the peak load all cracks outside of the central line undergoes elastic unloading. The central crack continues to open with the decreasing load after the peak.

Next example is a simulation of a four-point bending test of a concrete beam. Test configuration is shown in Fig.15 and the mesh used in this example is shown in Fig.16. Three types of calculation were conducted; 1) with localization judgment, 2) without localization judgment, and 3) allowing cracking only along the single line of central elements.

Fig.17 and Fig.18 show cracking patterns at different stages of loading in the analyses with and without localization judgment. Lines and arrows in each elements represent the orientation and the amount of crack opening displacement increments. It is seen that the distributed cracks are localized into a single crack when the localization judgment is carried out, while the cracking is distributed more or less until the end of the analysis without the localization judgment.

Fig.19 shows the associated load displacement curves for three cases. When a single crack is assumed (case 3), the behavior is very brittle. the pre-peak nonlinearity is small and the load decreases suddenly after the peak load is attained. When the localization judgment

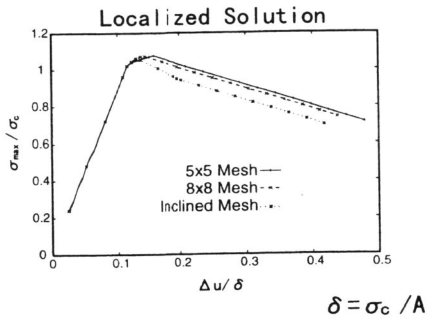

Figure 14: Flexural stress vs. applied displacement.

is not carried out (case 2), the pre-peak nonlinearity is remarkable and decrease in load after the peak is slow. The result of analysis with the localization judgment (case 1) is in-between the two cases.

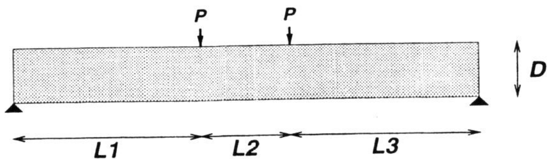

Figure 15: Configuration of four-point bending test: $DA/\sigma_c = 2400$, $E/\sigma_c = 10000$.

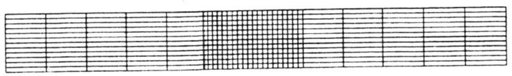

Figure 16: Mesh used in the analysis

CONCLUSION

In the present paper, a theory and analysis method for cracking localization is presented. The problem is reduced to a minimization problem under constraint conditions. Although only results of cracking localization in bending of beams are presented in this paper, the proposed method can predict the shear strength of RC columns and other phenomena. The method can be easily extended to treat propagation of shear failure planes in geological materials.

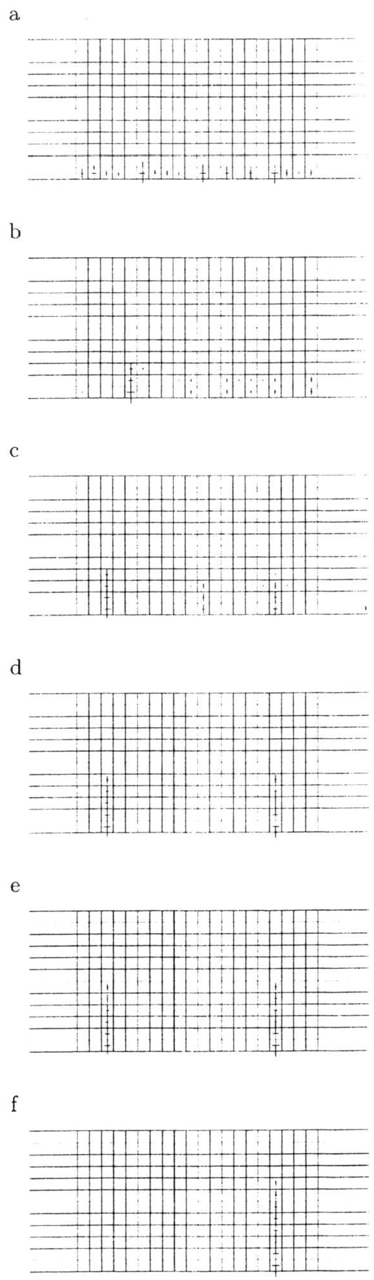

Figure 17: Cracking pattern with the judgment of localization.

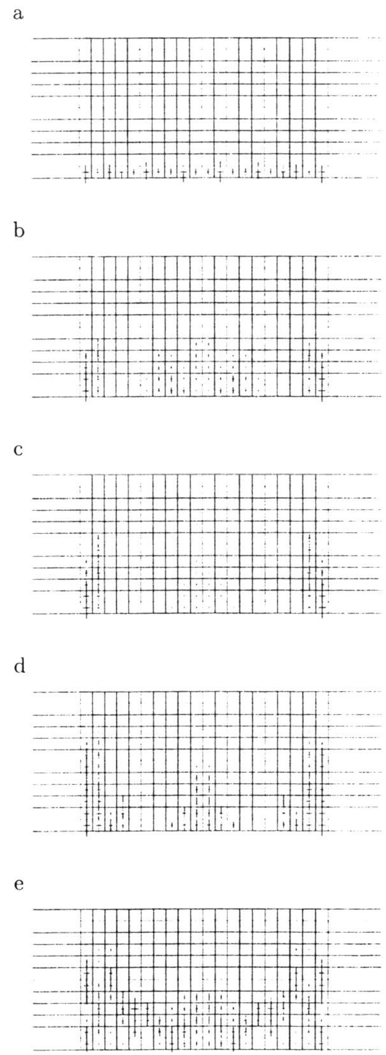

Figure 18: Cracking pattern without the judgment of localization.

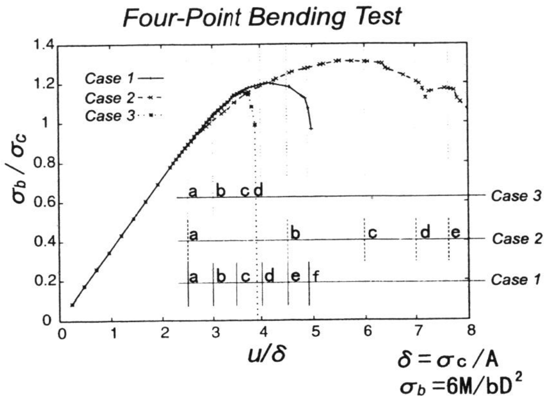

Figure 19: Load displacement curvs; the points indicated with a-f correspond to the cracking patters in Figs.17 and 18.

Acknowledgment. This work has been supported in part by the Kajima Foundation's Research Grant and the Grant-in-Aid for Scientific Research from the Japanese Ministry of Education, Science and Culture to the University of Tokyo.

REFERENCES

An, X. (1996). *Failure analysis and evaluation of seismic performance for reinforced concrete in shear.* Doctoral Thesis, University of Tokyo.

Hillerborg, A., M. Modeer and P. E. Peterson (1976). Analysis of crack formation and crack growth in concrete by means of fracture mechanics and finite elements, *Cement and Concrete Research*, 6, 773-782.

Kabele, P. and H. Horii (1996). Analytical Model for Fracture Analysis of Pseudo Strain-Hardening Cementitious Composites. *J. Materials, Conc. Struct., Pavements*, JSCE, No. 532/V30, 209-219.

Karihaloo, B. (1995). *Fracture Mechanics and Structural Concrete.* Longman Scientific & Technical.

Li, V.C. (1996). Engineered Cementitious Composites Through Micromechanical Modeling. to appear in *Fiber Reinforced Concrete: Present and the Future* (N. Banthia, A. Bentur, and A. Mufti, ed.), Canadian Society of Civil Engineers.

Mihashi, H., H. Okamura and Z. P. Bazant, editors (1994). *Size Effect in Concrete Structures.* Proc. of Japan Concrete Institute Int. Workshop, 1993, Sendai, Japan, E & FN Spon, London.

Nanakorn, P. and H. Horii (1996a). A fracture mechanics-based design method for SFRC tunnel linings. *J. Materials, Conc. Struct., Pavements*, JSCE, No.532/V30, 221-233.

Nanakorn, P. and H. Horii (1996b). Back analysis of tension-softening relationship. *J. Materials, Conc. Struct., Pavements*, JSCE, No. 544/V32, 265-275.

Needleman, A. and V. Tvergaard (1992). Analyses of plastic flow localization in metals. Appl. Mech. Rev., 45, S3-S18.

Nguyen, Q. S. (1987). Bifurcation and post-bifurcation analysis in plasticity and brittle fracture. *J. Mech. Phys. Solids*, 35, 303-324.

Shah, S. P., S. E. Swartz and C. Ouyang (1995). *Fracture Mechanics of Concrete.* John Wiley & Sons.

Tomita, Y. (1994). Simulations of plastic instabilities in solid mechanics. *Appl. Mech. Rev.*, 47, 171-205.

Wittmann, F. H., editor (1995). *Fracture Mechanics of Concrete Structures.* Proc. of the Second Int. Conf. on Fracture Mechanics of Concrete Structures (FRAMCOS 2), ETH Zurich, 1995, AEDIFICATIO, Germany.