NEW DEVELOPMENTS OF BOUNDARY ELEMENT METHODS IN FRACTURE MECHANICS

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ABSTRACT

In this paper a review of various boundary element formulations for fracture mechanics problems is presented. The topics include the modelling strategies, boundary integral equations and advanced contour integral formulations for evaluation of fracture parameters (stress intensity factors or J-integral and T-stresses) that have been developed in elastostatics, elastodynamics, thermoelasticity and inelastic fracture problems. The boundary element method formulation of general crack problems in terms of displacement discontinuities appears to be the most effective among all existing formulations. The overall accuracy is dependent on the precision of the evaluation of boundary integrals. Therefore a great attention is devoted to find nonsingular boundary integral formulations.

KEYWORDS

Nonsingular boundary integral formulations, elastostatics, elastodynamics, thermoelasticity.

INTRODUCTION

The boundary element method (BEM) is now established in many engineering disciplines as an alternative numerical technique to the finite element method (FEM). The attraction of BEM can be largely attributed to the reduction in the dimensionality of the problem. On the other hand this basic BEM behaviour brings the loss of generality in comparison with the FEM. Higher generality is achieved at the expense of the significant approximation requirements. The loss of partial generality of the BEM is balanced by its high accuracy of results especially for stress concentration problems. Namely, the solution at an internal point of analysed domain is exactly expressed through the boundary values and no discretization of domain is required. This is the main reason why the BEM is the most accurate computational method for solution of crack problems. Straightforward application of the ordinary (standard) boundary integral equations (BIE) doesn't lead to a unique formulation of a general crack problem. In order to avoid this difficulty there are suggested various modelling strategies and integral formulations. The first and only one systematic explanation of various strategies can be found in Cruse's book (Cruse, 1988). He presents the Green function formulation and displacement discontinuity modelling for general crack problems in elastostatics and elastoplasticity. The BEM formulation of general crack problems in terms of displacement discontinuities can be found also in the book by Balas *et al.* (1989) and Sladek *et al.* (1986) who deals with static, dynamic and thermal effects of linear fracture mechanics. It is known that the overall accuracy of the BEM is largely dependent on the precision with which various integrals are evaluated. No doubt, the evaluation of singular integrals requires much more sensitive treatment than that of regular integrals. Therefore a proper consideration and evaluation of singular integrals belongs to the most frequently discussed topics in the boundary element research.

In this paper the displacement discontinuity formulation is overviewed for elastostatic, elastodynamic and thermoelastic crack problems with using a unique regularization approach (Tanaka *et al.*, 1994). The singular integrals are regularized in the global coordinate system independently of the discretization. Once the boundary unknowns are computed from the ordinary (displacements) and derivative (traction) BIE, the first fracture parameters (stress intensity factors or J-integral) and the second fracture parameter (T-stresses) can be evaluated. There are given advanced contour integral formulations for their evaluation. All the above mentioned discussion concerns the brittle fracture. Sometimes, the plastic response of the material in the near vicinity of the crack tip cannot be omitted. Therefore, the unique boundary integral formulation for general crack problems in elastoplasticity and thermoelastoplasticity is discussed too. Both two- and three-dimensional problems are analysed simultaneously.

LINEAR FRACTURE MECHANICS

Elasticity

Conventional theories of fracture assume that the state of stresses and strains in the vicinity of a crack tip is characterized by a single parameter. Under predominantly elastic conditions the controlling parameter is the stress intensity factor (SIF), $K_a (a = \mathrm{I, II, III}$ mode). When yielding is more widespread, deformation at the crack tip is characterised by either the J-integral parameter or the crack opening displacement (COD).

Over the last forty years many methods of obtaining the SIF have been developed. For simple geometrical configurations, or where a complex structure can be simply modelled, one can use the reference handbook of the SIF or the analytical method. If the geometry and loading of structure is complex, one of the numerical methods is required. These are: collocation method, body force method, edge function method, method of lines, finite element method, boundary element method and alternating technique. In general, the methods based on boundary elements or finite elements are the most widely used. In this paper, we present advanced boundary integral formulations for solution of boundary value problems with cracks.

Straightforward application of the boundary element method to crack problems leads to a mathematical degeneration in the numerical formulation (singular matrix) if the two crack surfaces are considered co-planar. Consider a finite body containing an internal crack. Let, S_B represents the outer body surface, while S_{cr}^+ and S_{cr}^- the upper and lower crack surfaces, and V be the region bounded by $S = S_B \cup S_{cr}^+ \cup S_{cr}^-$. The two crack surfaces in the undeformed state are formed by the same set of points, $\{\forall \eta,\ \eta \in S_{cr}^+\} = \{\forall \eta,\ \eta \in S_{cr}^-\}$, and all the differences between S_{cr}^+ and S_{cr}^- are due to the opposite normal directions. The region V is assumed to be filled in with a homogeneous, isotropic, linear elastic medium. The Somigliana identity for the displacement is given by

$$\Delta(y)u_k(y) = \int_S [t_i(\eta)U_{ik}(\eta - y) - u_i(\eta)T_{ik}(\eta, y)]dS_\eta \tag{1}$$

in which

$$\Delta(y) = \begin{cases} 1\ , & y \in V \\ 0\ , & y \notin (V \cup S) \end{cases}$$

The kernels U_{ik} and T_{ik} are the Kelvin point load solution for displacement and tractions, repectively. It can be seen that the fundamental displacements are symmetric while the fundamental tractions are antisymmetric with respect to the choice of the field point η on S_{cr}^+ or S_{cr}^-. In general, we will consider the problems where the crack is loaded by equal and opposite tractions, hence $\Sigma t_i(\eta) = t_i^+(\eta) + t_i^-(\eta) = 0$. Then, the integral representation of displacement (1) can be rewritten as

$$\Delta(y)u_k(y) = \int_{S_B} [t_i(\eta)U_{ik}(\eta - y) - u_i(\eta)T_{ik}(\eta, y)]dS_\eta - \int_{S_{cr}^+} \Delta u_i(\eta)T_{ik}(\eta, y)dS_\eta \tag{2}$$

where $\Delta u_i(\eta) = u_i^+(\eta) - u_i^-(\eta)$.

We assume the surface S_{cr}^+ to be smooth. Then, the limit properties of the single and double layer potentials can be employed as $y \to \zeta^+ \in S_{cr}^+$ and eq. (2) becomes

$$\frac{1}{2}\Sigma u_k(\zeta^+) = \int_{S_B} [t_i(\eta)U_{ik}(\eta - \zeta^+) - u_i(\eta)T_{ik}(\eta, \zeta^+)]dS_\eta - \int_{S_{cr}^+} \Delta u_i(\eta)T_{ik}(\eta, \zeta^+)dS_\eta \tag{3}$$

where $\Sigma u_i(\eta) = u_i^+(\eta) + u_i^-(\eta)$.

Equation (3) has two deficiencies as a mathematical model for crack geometries. First, consider a problem in which the outer boundary S_B is free of tractions and the only loading is applied to the crack surface. Since any set of equal and opposite boundary tractions gives the same boundary integral equation (BIE), the formulation given by eq. (3) is non-unique. Second, while a single surface S_{cr}^+ is now being treated, two variables are unknown: Δu_i and Σu_i. Thus, the system of equations (3) and BIE if $y \to \zeta^B \in S_B$ underdetermines the mathematical basis for a solution of the unknowns on S_B and the unknowns on $S_{cr}^+, \Sigma u_i$ and Δu_i. In literature there are familiar four ways how to overcome these difficulties: a) multi-domain formulation b) Green's function technique c) dual boundary integral formulation and d) displacement discontinuity technique. In the first method (Blandford *et al.*, 1981) the cracked body is divided into two regions which were joined together such that equilibrium of tractions and compability of displacements were enforced. On both surfaces, the standard BIEs are written. The main disadvantage of this method is that the artificial cut surfaces increase substantially the number of unknowns. Moreover, in this case of crack growth problems, remeshing on cut surfaces is required.

In the second method, the Green's function can be found for two dimensional flat crack problems (Cruse, 1988). The analytical solution corresponding to the flat crack in an infinite plane with prescribed traction vectors on crack surfaces is superimposed with the Kelvin solution to satisfy stress-free condition on crack surfaces. In such an approach the crack surface need not to be discretized. However, this approach is restricted to two-dimensional straight crack configurations.

The dual boundary element method (DBEM) was developed by Portela *et al.* (1992). The DBEM overcomes the problem of the degeneration of the ordinary BIE on the crack surfaces by using two different equations for collocation points on opposite crack faces. For elastic problems, it uses the displacement equation for points on one crack surface and the traction equation for points on the other. Thus, full information (displacements on both crack faces) is received on the crack surface by solving these independent BIE.

Recall that the solution at any point of a cracked body can be expressed via integral representations in terms of the boundary displacements and tractions on S_B, and the displacement discontinuities on the crack (Sladek *et al.*, 1986; Balas *et al.*, 1989). Thus, the relevant unknowns at a BEM formulation for any crack problem are the densities of the displacement discontinuities on the crack and the unprescribed boundary densities on the outer boundary S_B. Hence, the use of the displacement discontinuity approach results in a substantial reduction of the size of discretized BIE which are to be solved for boundary unknowns. The set of totally regularized boundary integral equations consists of the displacement BIE, applied on the outer boundary S_B,

$$\int_{S_B}\left[u_i(\eta)-u_i(\zeta^B)\right]T_{ik}(\eta,\zeta^B)dS_\eta+\int_{S_{cr}^+}\Delta u_i(\eta)T_{ik}(\eta,\zeta^B)dS_\eta = \int_{S_B} t_i(\eta)U_{ik}(\eta-\zeta^B)dS_\eta \tag{4}$$

and the traction BIE (Sladek *et al.*, 1993a,b; Sladek *et al.*, 1992) applied on S_{cr}^+

$$\left[\frac{\partial\Delta u_m}{\partial\tau}(\zeta^+)b_{ikm}(\zeta^+)+\frac{\partial\Delta u_m}{\partial\rho}(\zeta^+)c_{ikm}(\zeta^+)\right]n_p(\zeta^+)B_{lpik}(\zeta^+,\zeta^+)+$$
$$+n_p(\zeta^+)\int_{S_{cr}^+}\left[\hat{D}_k\Delta u_i(\eta)-\hat{\mathcal{D}}_k\Delta u_i(\zeta^+)\right]T_{lpik}(\eta-\zeta^+)dS_\eta+$$
$$+n_p(\zeta^+)\int_{S_B}\left[t_i(\eta)D_{lpi}(\eta-\zeta^+)+T_{lpik}(\eta-\zeta^+)\hat{D}_k u_i(\eta)\right]dS_\eta=0 \tag{5}$$

where

$$\hat{D}_k=\rho_k(\eta)\frac{\partial}{\partial\tau(\eta)}-\tau_k(\eta)\frac{\partial}{\partial\rho(\eta)}\ ,\qquad \hat{\mathcal{D}}_k=\rho_k(\zeta^+)\frac{\partial}{\partial\tau(\zeta^+)}-\tau_k(\zeta^+)\frac{\partial}{\partial\rho(\zeta^+)}$$

The tensor B_{lpik} can be expressed by the nonsingular integrals on S_{cr}^+. This definition and the expressions of all kernel are presented by Sladek *et al.*, (1993a,b). Symbol $\tau(\eta)$ in the differential operator $\hat{D}_k$ denotes the unit tangent vector to the boundary S at the point η. In three-dimensional problems, ρ is the other orthogonal unit tangent vector, while in two dimensions, $\rho_i=\delta_{i3}$ and $\partial(.)/\partial\rho\equiv 0$.

It is well known that the standard BEM formulations include singular integrals. The leading singularities of the integral kernels are given as

$$U_{ik}\approx r^{2-d}\ \ (\ln r \text{ in two dimensions})\ ,\qquad T_{ik},D_{lji},T_{ljik}\approx r^{1-d}\ ,$$

where d is the dimensionality of the problem. In view of this asymptotic behaviour of the integral kernels as well as the position of collocation points ζ^+ on smooth S_{cr}^+ one can see that all the terms are bounded at any of the BIE given by eqs. (4) and (5) provided that the displacements are Holder continuous on S_B ($u_i\in C^{0,\alpha}$) and $\Delta u_i\in C^{1,\alpha}$ on S_{cr}^+. The leading singularity of the present traction BIE is the strong singularity. Some authors object to the application of the traction BIE with strongly singular kernels to crack problems owing to the $r^{-1/2}$ singularity of tangential derivatives near the crack front. Since the collocation points are away from the crack front, the kernel is finite on the crack front and the weak singularity $r^{-1/2}$ can be easily removed by a polynomial transformation. If the traction BIE with hypersingular kernels ($\approx r^{-d}$) are employed, the use of C^1-continuos elements is necesssary for approximation of displacements (or displacement discontinuities). The present set of the BIE makes us possible to use only standard C^0-continuos conforming elements, if the latter are applied also to the approximation of densities of crack dislocations (i.e., tangent derivatives of the displacement discontinuities). In such a case, however, the set of the BIE is to be supplemented with a tangent derivative BIE resulting from the regularized integral representation of the displacement gradients (Tanaka *et al.*, 1994). The compatibility of the approximation of displacement discontinuities by C^0 conforming elements with the approximation of crack dislocations can be satisfied at nodal points in the least-squares sense (Polch *et al.*, 1987; Bonnet, 1989).

In linear fracture mechanics, it is well known the relationship between the J-integral and the stress intensity factors. In order to obtain stress intensity factors from a mixed mode case, the J-integral, stresses and displacement gradients are decomposed into symmetric and antisymmetric components (Aliabadi and Rooke, 1991). The integration path Γ_0 has to be symmetric with respect to the crack plane. The particular stress intensity factors can be computed also from the integral representation based on the use of the Betti's reciprocity theorem and the Bueckner's singular fields (Sladek and Sladek, 1993b). We can write

$$K_{I,II}=\int_{\Gamma_0}\left(t_i U_i^{I,II}-u_i T_i^{I,II}\right)d\Gamma \tag{6}$$

where U_i^I and U_i^{II} represent Bueckner's singular displacement fields as a result of two equal, opposite normal and sliding forces at the crack tip, respectively. The traction vectors $T_i^{I,II}$ are associated with these displacements. Note that Γ_0 can be chosen as the outer boundary S_B of cracked body.

Now, we present a computational method for evaluation of the T-stresses (Rice, 1974). Consider two independent equilibrium states $(\sigma_{ij}^A,u_i^A,\varepsilon_{ij}^A)$ and $(\sigma_{ij}^B,u_i^B,\varepsilon_{ij}^B)$. Then, one can define a new path independent integral, the so called mutual M-integral, by using the J-integrals for particular states (A), (B), and their superposition (A+B) by $M\equiv J^{(A+B)}-J^{(A)}-J^{(B)}$. The expression for the M-integral can be rearranged as follows (Kfouri, 1986; Sladek and Sladek, 1996a)

$$M=\int_{\Gamma_0}\left(\sigma_{ij}^A\varepsilon_{ij}^B n_1-t_i^A u_{i,1}^B-t_i^B u_{i,1}^A\right)d\Gamma \tag{7}$$

Let the first state (A) correspond to the analysed boundary value problem with unknown T-stress and the second state (B) be an auxiliary solution. Due to the path independence of individual J-integrals the M-integral is path independent too. Then, the integration contour can be chosen arbitrarily, say as a circle Γ_ε which is shrunk to zero radius. If the auxiliary fields are selected as solution for the problem of a semiinfinite crack loaded by a point force f applied to the crack tip in the direction parallel to the plane of a crack, it is possible to find relationship between the M-integral and the T-stress (Kfouri, 1986)

$$M=-\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon}\sigma_{1j}^B n_j\varepsilon_{11}^T d\Gamma=\frac{1-\nu^2}{E}Tf \tag{8}$$

The analytical solution of auxiliary fields is given in (Kfouri, 1986; Sladek and Sladek, 1996a). It is seen that the mutual M-integral provides sufficient information for determining the T-stresses.

Elastodynamics

Dynamic fracture mechanics can be broadly classified into i) impact fracture mechanics and ii) fast fracture mechanics. In impact fracture mechanics stationary cracks subjected to dynamic loads are analysed. The aim of the fast fracture mechanics is to analyse the growth, arrest and branching of moving cracks. Due to the highly transient features in dynamic fracture analytical solutions in dynamic fracture mechanics are more limited than in elasticity. Structures with arbitrary shape and time-dependent boundary conditions need to be analysed by numerical methods. The boundary element method has been successfully applied to stationary and growing cracks in infinite and finite domains (Sladek and Sladek, 1986, 1987; Nishimura *et al.*, 1988; Hirose and Achenbach, 1989; Dominguez and Gallego, 1992; Fedelinski *et al.*, 1995). The BEM solutions in elastodynamics are usually obtained by using one of the following approaches: the time-domain method, combination with the Laplace or Fourier transforms, and the dual reciprocity method. Computational modelling of dynamic crack propagation is described by Nishioka (1994) in his review paper. Stationary element and moving element procedures for finite and boundary element methods are analysed separately.

Similar to elastostatics, the general crack problem in finite bodies can be uniquely formulated using the displacement equations on the outer boundary S_B and the traction equations on one of the crack surfaces, S_{cr}^{+}. The displacement equation for the Laplace transforms can be written in a nonsingular form (Balas *et al.*, 1989) as

$$\int_{S_B} \left[\bar{u}_i(\eta,p)-\bar{u}_i(\zeta^B,p)\right]\bar{T}_{ik}(\eta,\zeta^B,p)dS_\eta + \bar{u}_i(\zeta^B,p)\int_{S_B}[\bar{T}_{ik}(\eta,\zeta^B,p)-T_{ik}^s(\eta,\zeta^B)]dS_\eta - \\ -\int_{S_B} \bar{t}_i(\eta,p)\bar{U}_{ik}(\eta-\zeta^B,p)dS_\eta + \int_{S_{cr}^+}\Delta\bar{u}_i(\eta,p)\bar{T}_{ik}(\eta,\zeta^B,p)dS_\eta = \bar{V}_k(\zeta^B,p) \tag{9}$$

and the nonsingular traction equation becomes (Sladek and Sladek, 1996b)

$$\left[\frac{\partial\Delta\bar{u}_m}{\partial\tau}(\zeta^+,p)b_{ikm}(\zeta^+)+\frac{\partial\Delta\bar{u}_m}{\partial\rho}(\zeta^+,p)c_{ikm}(\zeta^+)\right]n_j(\zeta^+)B_{ljik}(\zeta^+,\zeta^+)+ \\ +n_j(\zeta^+)\int_{S_{cr}^+}\left[\hat{D}_k\Delta\bar{u}_i(\eta,p)-\hat{D}_k\Delta\bar{u}_i(\zeta^+,p)\right]\bar{T}_{ljik}(\eta-\zeta^+,p)dS_\eta + \\ +\hat{D}_k\Delta\bar{u}_i(\eta,p)n_j(\zeta^+)\int_{S_{cr}^+}[\bar{T}_{ljik}(\eta,\zeta^+,p)-T_{ljik}^s(\eta,\zeta^+)]dS_\eta + \\ +n_j(\zeta^+)\int_{S_B}\left[\bar{t}_i(\eta,p)\bar{D}_{lji}(\eta,\zeta^+,p)+\rho p^2\bar{E}_{lji}(\eta,\zeta^+,p)+\bar{T}_{ljik}(\eta,\zeta^+,p)\hat{D}_k\bar{u}_i(\eta,p)\right]dS_\eta + \\ +\rho p^2\int_{S_{cr}^+}\Delta\bar{u}_i(\eta,p)\bar{E}_{lji}(\eta,\zeta^+,p)dS_\eta = -\bar{W}_l(\zeta^+,p) \tag{10}$$

where p is the Laplace parameter and Laplace transforms are denoted by an overbar. The integral kernels with the superscript *s* correspond to the static solutions, and the right hand sides of eqs. (9) and (10) represent the contribution from the body forces and initial conditions.

The BIE for the Laplace transforms are solved numerically by the same scheme as that in the elastostatic case. In order to obtain the time-dependent unique formulation for a general crack problem the inverse Laplace transformation of displacement and traction equations (9), (10) has

to be performed. In such an inversion, the algebraic products of Laplace transforms are converted to the convolution products. The dual reciprocity formulation can be derived from a reciprocal relation between an elastostatic fundamental solution state and the actual elastodynamic state in which the inertia term is treated approximately as body forces. The domain integral of the inertia term is converted into boundary ones and an ordinary differential equation is received for the unknown time-dependent expanding coefficients. This equation is of the second order and it is an alternative of the FEM equilibrium equation in elastodynamics.

For structures subjected to dynamic loads, the dynamic stress intensity factors can be computed from the $\hat{J}$-integral which represents an energy release rate for a stationary crack and has the physical meaning of a crack driving force. The $\hat{J}$-integral is defined (Kishimoto *et al.*, 1980a) as

$$\hat{J} = \int_{\Gamma_0+\Gamma_c}\left(Wn_1 - t_iu_{i,1}\right)d\Gamma + \lim_{\varepsilon\to 0}\int_{\Omega-\Omega_\varepsilon}\rho\ddot{u}_iu_{i,1}d\Omega \tag{11}$$

where $\Omega-\Omega_\varepsilon$ is the domain enclosed by $\Gamma_0+\Gamma_c-\Gamma_\varepsilon$.
In the Laplace transform domain formulation, one can define a path independent J-integral which is related to the Laplace transforms of the SIFs ($\bar{K}_I$ and $\bar{K}_{II}$) analogically as in the static case. The individual stress intensity factors can be obtained from the integral representations which utilize the Bueckner's singular fields

$$K_N = \int_{\Gamma_0}\left(t_iU_i^N - u_iT_i^N\right)d\Gamma - \lim_{\varepsilon\to 0}\int_{\Omega-\Omega_\varepsilon}\rho\ddot{u}_iU_i^Nd\Omega \tag{12}$$

for N= I, II fracture mode.

The spatial distribution of stresses at a crack tip vicinity of a stationary crack subjected to a dynamic load is the same as in the case of a static load (Sih *et al.*, 1972). Therefore, the T-stresses in elastodynamic crack analysis are as important as in the static case. The mutual M-integral is given by

$$M = \int_{\Gamma_0}\left(\sigma_{ij}^A\varepsilon_{ij}^Bn_1 - t_i^Au_{i,1}^B - t_i^Bu_{i,1}^A\right)d\Gamma + \lim_{\varepsilon\to 0}\int_{\Omega-\Omega_\varepsilon}\rho\ddot{u}_i^Au_{i,1}^Bd\Omega \tag{13}$$

Note that it is impossible to convert the domain integral into a contour one neither in the Laplace transform domain. It can be seen that the use of the auxiliary fields employed in elastostatics yields the same relatioship between the value of the M-integral and T-stresses as in the elastostatic case. The only difference is that now both the M-integral and T-stresses coupled by eq. (8) are time-dependent now.

Thermoelasticity

The general theory of thermoelasticity includes such branches as theory of thermal stresses, quasi-static coupled thermoelasticity, quasi-static uncoupled thermoelasticity and stationary thermoelasticity. For all special classes of thermoelasticity problems one can find advanced pure boundary formulations (Balas *et al.*, 1989). Usually, the coupling coefficient has a small influence on values of analysed quantities in a practice. Moreover, we don't want to complicate the analysis by taking into account the inertia term in the governing displacement equation, which has been treated in the previous section, elastodynamics. Thus, we are interested in stationary and quasi-static uncoupled thermoelasticity. In both the above mentioned thermoelastic problems the stress field equations are identical to those for isothermal problems with applied body forces. Since the presence of heat flow produces no additional singularities, according to Sih (1962) the local character of thermal stresses is of the same nature as for problems with mechanical stresses. Thus, the stress intensity factor (or alternatively the energy realese rate) is the only fracture characteristic in conventional fracture mechanics. In order to evaluate the SIF for a crack problem, it is sufficient to know the solution only on the boundary or at small number of internal points usually lying far from the crack tip. In view of this fact, the boundary element method appears to be very convenient for solution of these problems. In literature one can find a lot of papers devoted to thermoelastic crack analysis by the BEM (Balas *et al.*, 1989; Sladek and Sladek, 1984, 1987b, 1992; Sladek *et al.*, 1986; Tanaka *et al.*, 1984; Raveendra and Banerjee, 1992; Raveendra *et al.*, 1993; Lee and Cho 1990; Prasad et al., 1994). The aim of this section is to give a unique formulation of general crack problems. We suppose the crack to be loaded by equal and opposite tractions on both crack surfaces ($\Sigma t_i = 0$) . Moreover, neither heat source nor sink are supposed to be between the crack faces $\Sigma\bar{q}(\eta,p) = \bar{q}^+(\eta,p) + \bar{q}^-(\eta,p) = 0$, where $\bar{q}$ denotes the Laplace transform of the heat flux for a non-stationary state. The Laplace transform is used to elliminate the time variable. The time-domain formulation of the heat equation by the BEM is known too (Balas *et al.*, 1989; Brebbia *et al.*, 1984; Sladek and Sladek, 1990, 1992; Banerjee, 1994). In uncoupled theories, thermal fields are not influenced with mechanical ones. On the other hand displacements and stresses are influenced with thermal fields. Therefore, firstly in such theories it is necessary to analyse thermal fields (heat equation). Similarly to elastostatics or elastodynamics, if we take the limit of an internal point to a point lying on the crack surface in the standard representation of temperature the non-unique formulation is obtained for a general crack problem due to the existence of the sum of temperatures on both surfaces. Then, we can write the ordinary BIE (temperature BIE) on the outer surface S_B and the derivative BIE (flux BIE) on the crack surface S_{cr}^+ to give the unique formulation. Assuming S_{cr}^+ to be smooth and $\Sigma q = 0$, we may write the set of BIE as (Balas *et al.*, 1989; Tanaka *et al.*, 1994)

$$\kappa_0\bar{\theta}(\zeta^B,p)\int_{S_B}\left[\bar{F}(\eta,\zeta^B,p) - F^s(\eta,\zeta^B)\right]dS_\eta + \kappa_0\int_{S_B}\left[\bar{\theta}(\eta,p) - \bar{\theta}(\zeta^B,p)\right]\bar{F}(\eta,\zeta^B,p)dS_\eta -$$

$$- \kappa_0\int_{S_B}\bar{q}(\eta,p)\bar{T}(\left|\eta - \zeta^B\right|,p)dS_\eta + \kappa_0\int_{S_{cr}^+}\Delta\bar{\theta}(\eta,p)\bar{F}(\eta,\zeta^B,p)dS_\eta = \bar{Y}(\zeta^B,p) \tag{14}$$

and

$$\left[\frac{\partial\Delta\bar{\theta}}{\partial\tau}(\zeta^+,p)\tau_l(\zeta^+) + \frac{\partial\Delta\bar{\theta}}{\partial\rho}(\zeta^+,p)\rho_l(\zeta^+)\right]n_j(\zeta^+)M_{jl}(\zeta^+) +$$

$$+ \kappa_0 n_j(\zeta^+)\int_{S_{cr}^+}\overset{*}{\varepsilon}_{kji}\left[\hat{D}_k\Delta\bar{\theta}(\eta,p) - \hat{D}_k\Delta\bar{\theta}(\zeta^+,p)\right]T_i^s(\eta - \zeta^+)dS_\eta +$$

$$+ \kappa_0 n_j(\zeta^+)\int_{S_{cr}^+}\left\{\left[\bar{T}_{,i}(\eta - \zeta^+,p) - T_i^s(\eta - \zeta^+)\right]\overset{*}{\varepsilon}_{kji}\hat{D}_k + n_j(\eta)\bar{\Psi}(\eta - \zeta^+,p)\right\}\otimes$$

$$\otimes\Delta\bar{\theta}(\eta,p)dS_\eta + \kappa_0 n_j(\zeta^+)\int_{S_B}\left\{\left[n_j(\eta)\bar{\Psi}(\eta - \zeta^+,p) + \bar{T}_{,i}(\eta - \zeta^+,p)\overset{*}{\varepsilon}_{kji}\hat{D}_k\right]\bar{\theta}(\eta,p) -\right.$$

$$\left.-\bar{q}(\eta,p)\bar{T}_{,j}(\eta - \zeta^+,p)\right\}dS_\eta = \bar{q}(\zeta^+,p) - \bar{\Omega}(\zeta^+,p) \tag{15}$$

Recall, the leading singularities of the integral kernels are given as

$\bar{\Psi} \approx r^{2-d}$, $\bar{T}, T^s \approx r^{2-d}$ (ln r in two dimensions)

$\bar{F}, F^s, \bar{T}_{,i}, T_i^s \approx r^{1-d}$

Similar to the case of elastostatics, the collocation point ζ^+ is out of the crack front. It can be seen that both the BIE given by eqs. (14) and (15) are nonsingular. The sufficient conditions for

the existence of such regularized BIE are (Tanaka et al., 1994): $\bar{\theta} \in C^{0,\alpha}$ on S_B and $\Delta\bar{\theta} \in C^{1,\alpha}$ on S_{cr}^+. In full analogy with the displacement discontinuities, the required smoothness for the temperature discontinuities can be achieved either by using C^1-continuous elements for approximation of $\Delta\bar{\theta}$ or C^0 conforming elements for both $\Delta\bar{\theta}$ and its tangent derivatives with satisfying the compability at nodal points in the least-squares sense. The flux BIE can be reconsidered also in the form with hypersingular kernels ($\approx r^{-d}$), but then C^1-elements are necessary for approximation of $\Delta\bar{\theta}$ on S_{cr}^+.

Having known the relevant thermal boundary densities, one can compute the mechanical unknowns on S_B and displacement discontinuities on S_{cr}^+ by solving the set of the displacement BIE and tractin BIE collocated on S_B and S_{cr}^+, respectively. As compared with the BIE employed in elastostatics, there are some new integrals of thermal terms on the *r.h.s.* (Sladek and Sladek, 1996b).

Wilson and Yu (1979) showed that the Rice's J-integral over a closed path is not zero in thermoelasticity. Therefore, Kishimoto *et al.* (1980b) modified the J-integral by the new path-independent $\hat{J}$-integral

$$\hat{J} = \int_{\Gamma_0+\Gamma_c}\left(W^e n_1 - t_i u_{i,1}\right)d\Gamma + \lim_{\varepsilon\to 0}\int_{\Omega-\Omega_\varepsilon}\sigma_{ii}\alpha\theta_{,1}d\Omega \tag{16}$$

A direct evaluation of the SIFs in stationary thermoelasticity by boundary integral representations has been proposed by the present authors (Sladek and Sladek, 1993b). Bueckner singular fields are utilized in the reciprocity theorem to derive such integral representations

$$K_N = \int_{\Gamma_0}\left(t_i U_i^N - u_i T_i^N\right)d\Gamma + \gamma\int_{\Gamma_0}\left(\theta\frac{\partial H^N}{\partial n} - H^N q\right)d\Gamma + 2\gamma\delta_{N1}\int_{\Gamma_{cr}^+}\theta\frac{\partial H^N}{\partial n}d\Gamma \tag{17}$$

Because the local character of thermal stresses at a crack tip vicinity is the same as in the problems with mechanical stresses, the reason for introduction of the T-stress as the second fracture parameter is the same as in elastostatics. The mutual M-integral

$$M = \int_{\Gamma_0}\left(\sigma_{ij}^A\varepsilon_{ij}^B n_1 - t_i^A u_{i,1}^B - t_i^B u_{i,1}^A\right)d\Gamma + \lim_{\varepsilon\to 0}\int_{\Omega-\Omega_\varepsilon}\alpha\sigma_{ii}^B\theta_{,1}^A d\Omega \tag{18}$$

provides sufficient information for determination of T-stresses because the following relation between the M-integral and T-stresses is valid (Sladek and Sladek, 1996c)

$$M = \frac{1-\nu^2}{E}Tf + \alpha\theta_0 f(1+\nu) \tag{19}$$

where θ_0 is the temperature at the crack tip. All the mechanical and thermal fields occuring in the integral representations of the stress intensity factors, the M-integral and $\hat{J}$-integral are obtained by the BEM. These fields are computed far away from the crack tip, so the accuracy of presented methods is sufficiently high.

INELASTIC FRACTURE MECHANICS

Elastoplasticity

Plasticity is defined as the property which enables a material to be deformed continuously and permanently without rupture during the application of stresses that exceed the elastic limit of the material. Thus, residual strains are expected to occur on removal of the load and the final deformation depends not only on the final stresses but also on the path-stress history from the beginning of yield. All these requirements put on a computational model are satisfied in this paper by the incremental theory of plasticity which provides the most general theory for inelastic deformations of a wide range of materials. Since material non-linearities may be introduced into an elastic analysis method as a set of initial strains, stresses or body forces for each increment in the same way that thermal looading, the boundary element method can be developed for inelastic analysis (Banerjee and Butterfield, 1981; Brebbia *et al.*, 1984; Balas *et al.*, 1989; Banerjee, 1994).

In elasticity, it is shown that standard boundary integral equations don't lead to a unique formulation of a general crack problem. One of the first attempts in applying BEM formulations to fracture mechanics was made by Mukherjee and co-authors (1982) by using Green's functions for ellipses to introduce crack. All three ways, multi-domain formulation (Sladek and Sladek, 1995a), Green's function technique (Cruse and Polch, 1986), displacement discontinuity method (Balas *et al.*, 1989; Sladek and Sladek, 1993a) or dual boundary integral formulation (Leitao *et al.*, 1995), known from elasticity have be extended into elastoplasticity too. Since the governing equations in the rate form and hence also the boundary integral equations are formally the same for elastic case expect the additional domain integral with initial stress rates (or strain), we don't present a complete unique BEM formulation for crack problems now. In eqs. (4) it is necessary to change all the state variables by their increments and add to the right hand side the domain integral

$$\int_{V_p} \dot{\sigma}_{ij}^{p}(x) U_{ijk}(x-\zeta^{B}) \mathrm{d}V_x$$

where V_p is a domain with non-zero plastic stress rates $\dot{\sigma}_{ij}^{p}$.

The traction equation (5) is to be supplemented on its right hand side by the following domain integral (Balas *et al.*, 1989; Sladek and Sladek, 1993a)

$$-v.p.\int_{V_p} \dot{\sigma}_{ki}^{p}(x) n_j(\zeta^{+}) E_{ljki}(x-\zeta^{+}) dV_x - g_{lj}(\zeta^{+}) n_j(\zeta^{+})$$

where the free term is a function of plastic stress rates and its explicit expression can be found in (Banerjee and Butterfield, 1981; Brebbia *et al.*, 1984; Balas *et al.*, 1989; Banerjee, 1994). The volume integral over V_p domain exists in the CPV sense. According to (Dallner and Kuhn, 1993; Sladek and Sladek, 1995a) it can be transformed into a regular domain and contour integrals. The integration path of the contour integral is identical with the boundary of the assumed plastic zone V_p .

In elastoplastic fracture mechanics the leading terms of stress and deformation field expansions are controlled by the J-integral according to the HRR-theory (Hutchinson, 1968; Rice and Rosengren, 1968). Similarly to an elastic case the dominance of the leading-order term is limited in elastoplasticity too. To obtain a more realistic asymptotic expansion of stresses, it is necessary to introduce the additional terms. Recently, Yang *et al.* (1993) have introduced three term asymptotic expansion formulation. The BEM, as a very accurate method, can be used to quantification of conditions under which the HRR solution reasonably represents realistic results.

Thermoelastoplasticity

The BEM is potentially very attractive for solution thermoelastoplastic problems due to its ability to produce accurate solutions in elastic cases. The earliest BEM formulations for thermoelastoplastic problems were due to Mukherjee (1977) and Bui (1978). Their formulations also include the domain integral of temperature gradients and so the discretization is required within the whole domain. Recently, Chopra and Dargush (1994) have presented the boundary formulation. The only domain-type integral in these formulations is the integral of initial plastic stresses. Its contribution is considered iteratively. During each iterative step the formulation has a pure boundary character.

The application of the BEM to thermoelastoplastic crack problems doesn't appear in literature frequently due to its complexity. Up to date only one paper (Sladek and Sladek, 1995b) has been devoted to unique formulation of a general crack problem. The displacement discontinuity method is used there. Because the uncoupled theory is considered, it is possible to analyse thermal and mechanical fields separately. First, the thermal problem is resolved, because this solution is not influenced by the mechanical fields. Nonlinear behaviour is assumed only for those material parameters which affect mechanical variables but don't thermal ones. Therefore, the ordinary and derivative BIE for thermal unknowns in thermoelastoplastic problem are the same as in the elastic case (see eqs. (14) and (15)). The only change consists in the replacement of state variables by their increments corresponding to the increments of thermal loading. The displacement and traction equations, which are valid for elastic case, have to be supplemented by the domain integrals involving plastic stress rates. These integrals, however, are identical with those presented in elastoplasticity.

The boundary element method seems to be the most accurate computational method for crack analysis and can be successfully utilized also in the determination of path independent J-integral and the additional fracture parameters in higher-order asymptotic theories. The present authors (Sladek and Sladek, 1996d) used the BEM for computation of the second amplitude parameters in the three term asymptotic theory.

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