HOLONOMIC ANALYSIS OF QUASIBRITTLE FRACTURE WITH NONLINEAR SOFTENING

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ABSTRACT

The holonomic (single-step) analysis of quasibrittle fracture processes is formulated as a nonlinear complementarity problem under the assumptions of proportional loads, small displacements, and a fairly general cohesive crack model which exhibits a nonlinear softening behaviour. These nonlinear softening constitutive laws are expressed in a complementarity format, a feature of which is the orthogonality of two sign-constrained vectors. The underlying features of a recent numerical algorithm with the potential of capturing all possible solutions for any given load level is presented. A well-known example, the three-point bending test, is used to illustrate numerical application of the method, its potentialities and limitations.

KEYWORDS

Boundary elements, cohesive crack, complementarity, holonomy, mathematical programming, nonlinear softening, quasibrittle fracture.

INTRODUCTION

The problem considered herein concerns the numerical simulation of quasibrittle fracture processes in concrete-like structures subjected to proportionally applied quasistatic loads, using the well-known and established cohesive crack model popularized primarily by Barenblatt (1962), Dugdale (1960) and Hillerborg *et al.* (1976). In addition to the usual assumptions of small displacements and linear elasticity outside the cohesive-softening crack, we adopt a holonomy hypothesis for an *a priori* known crack itinerary. A holonomic constitutive law (in the spirit of the deformation theory of plasticity) implies essentially nonlinear elasticity and hence reversibility. Such an assumption may not be correct in view of unloading, but can still provide a simple means of capturing essential features of the structural behaviour.

The main thrust of the work described in this paper is to propose a mathematical programming method capable of capturing multiplicity of solutions, or showing that none exists, due to such critical phenomena as bifurcation and loss of overall stability, in a single-step nonevolutive analysis. At variance with previous work in this direction (Bolzon *et al.*, 1995, 1996) which uses piecewise linear softening laws, we consider nonlinear softening.

The organization of this paper is as follows. In the next section, we briefly formulate the space-discretized problem as a nonlinear complementarity problem (NCP), the key feature of which is the orthogonality of two sign-constrained vectors. This formulation is of course only possible if the nonlinear softening models are amenable to a complementarity format, which we present for a fairly general and typical softening holonomic constitutive law. The space-discretization is carried out within a conventional direct collocation boundary element framework, and hence is not elaborated upon. We then describe in some detail a recently developed nonlinear complementarity algorithm which has the potential of efficiently capturing multiplicity of solutions, albeit at present via an *ad hoc* search scheme. Finally, application of this algorithm is illustrated using a classical example.

PROBLEM FORMULATION

Let Ω denote the domain occupied by the structure (Fig. 1a) with boundary Γ consisting of a constrained part Γ_u and an unconstrained part Γ_t for which displacements and tractions are prescribed, respectively. The known crack locus Γ_p of possible displacement discontinuity w is characterized by a generally nonlinear law (Fig. 1b) relating tractions p across its faces to w. The particular case shown in Fig. 1b simulates the mode I crack propagation in cementitious materials and will be assumed, in its holonomic form, throughout this work without undue loss of generality.

The assumption of linear elasticity in Ω allows us to construct, using the geometric and elastic properties of the structure, a Green function $G(x,\xi)$, $x,\xi \in \Gamma_p$ ($x \equiv$ field point, $\xi \equiv$ source point) which relates tractions p to displacement discontinuities w on Γ_p in the unloaded body.

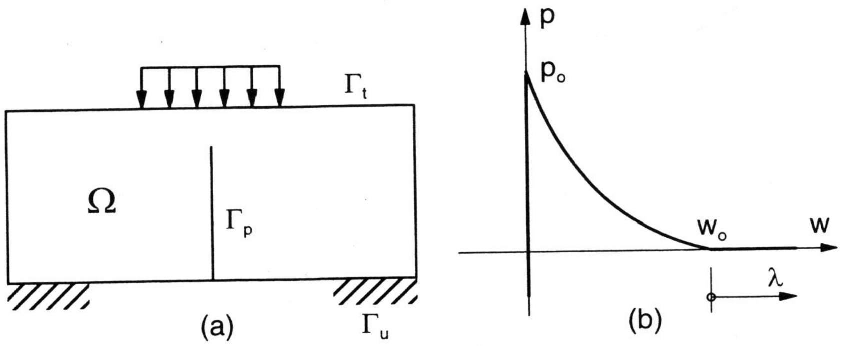

Fig. 1. Problem definition: (a) structure, (b) softening law.

Thus,

$$p(x) = \int_{\Gamma_p} G(x,\xi)w(\xi)d\Gamma + p^e(x), \quad x,\xi \in \Gamma_p, \tag{1}$$

where p^e represents tractions at the interface location, generated by the external actions on the structure assumed to be purely elastic and homogeneous.

Space-discretization of (1) can be easily and efficiently performed by using a boundary element approach. In the present work, we used a direct collocation nonsymmetric method (e.g. Banerjee, 1993) with quadratic elements. After condensation is carried out on a standard multiregion boundary element model to retain only interface variables, the discrete counterpart of the integral equation (1) is easily obtainable in the form

$$\boldsymbol{p} = \boldsymbol{Z}\boldsymbol{w} + \boldsymbol{p}^e, \tag{2}$$

with self-evident meanings for the symbols involved. Clearly, (2) expresses, in an identical form to Colonnetti's well-known imposed rotations approach for bending, the total tractions at the nodes as being the sum effects of discontinuities $\boldsymbol{w}$ (through the influence matrix $\boldsymbol{Z}$ which approximates the kernel G) and of a purely elastic component $\boldsymbol{p}^e$.

To complete the discrete problem formulation, we need to supplement (2) with the holonomic nonlinear softening constitutive law describing the cohesive crack model. Without loss of generality, assume that this is as shown in Fig. 1b, where the nonlinear portion can be described by a single function $f(w)$. An elegant description, based on work by Maier initially in the area of piecewise linear structural plasticity (e.g. Maier, 1970) and later extended to piecewise linear cohesive crack models (e.g. Bolzon *et al.*, 1994), is as follows:

$$\varphi_1 = p - f(w) + g(\lambda) \le 0, \quad w \ge 0, \quad \varphi_1 w = 0, \tag{3a}$$
$$\varphi_2 = -w_0 + w - \lambda \le 0, \quad \lambda \ge 0, \quad \varphi_2 \lambda = 0, \tag{3b}$$

where $g(\lambda) = f(w_0 + \lambda)$; note that $g(0) = 0$ since $f(w_0) = 0$. The linear case given by Bolzon *et al.* (1994) can be recovered by setting $f(w) = p_0 + hw$ and $g(\lambda) = h\lambda$, where h is the (negative) slope of the softening branch. Note that activation of the softening mode, namely $p = f(w)$, leads to the condition $w \le w_0$ (since $\lambda = 0$) which is in effect a restriction on the shape of the softening curve.

On close inspection, it can be seen that these relations describe fully the holonomic behaviour of a cohesive crack model with interface Γ_p which can be conceived as a union of cracks (where $p = 0$), process zone or craze (where $p \ne 0$ and $w \ne 0$) and undamaged material (where $w = 0$). A key feature of (3) is complementarity (nonlinear due to nonlinearity of $f(w)$).

Finally, assume that (3) applies directly to any interface node i so that combination of (2) with the collected nodal softening constitutive laws leads to an NCP in the standard form

$$\boldsymbol{F}(\boldsymbol{z}) \ge \boldsymbol{0}, \quad \boldsymbol{z} \ge \boldsymbol{0}, \quad \boldsymbol{z}^T\boldsymbol{F}(\boldsymbol{z}) = 0. \tag{4a}$$

The vectors F and z are defined as follows:

$$F(z) = \begin{bmatrix} -Zw - p^e + f(w) - g(\lambda) \\ w_0 - w + \lambda \end{bmatrix}, \quad z = \begin{bmatrix} w \\ \lambda \end{bmatrix}, \tag{4b}$$

where $(f(w))_i = f_i(w_i)$, and $(g(\lambda))_i = g_i(\lambda_i)$, for each node i on the interface.

Some remarks are appropriate at this stage.

(a) To date, with some exceptions (e.g. Tin-Loi and Pang, 1993), most complementarity formulations in structural mechanics are cast as linear complementarity problems (LCP) characterized by an affine $F(z) = Mz + q$, where both M and q are known.

(b) Solving NCP (4) is not a trivial task. In fact, as highlighted in Bolzon *et al.* (1995), even in the affine case, the resulting LCP involves a nonsymmetric and indefinite matrix M and hence belongs to the class of problems for which no polynomial time algorithm is known to exist. This implies that it can take exponential time to determine a solution or the fact that no solution exists.

(c) Even less is known about the solvability of the NCPs which arise in quasibrittle fracture nonlinear softening problems.

SOLUTION ALGORITHM

In view of the success we have had (Bolzon *et al.*, 1994, 1995) with the recently developed NCP solver PATH (Dirkse and Ferris, 1995a) in capturing multiple solutions for the LCP case, we also tried it on the NCP formulation. In the following, we give the basic ideas, first explored by Ralph (1994), underlying this solver; we consider specifically, for the sake of simplicity, the NCP case only, rather than the more general mixed NCP formulation.

There is a well-established theory for solving nonlinear equations $H(x) = 0$ using Newton's method in combination with a linesearch damping step (e.g. Ortega and Rheinboldt, 1970) that increases the domain in which the basic method converges. In order to extend these ideas to complementarity problems, the NCP is recast as a zero-finding problem for a system of nonlinear equations. This can be achieved in several ways; the PATH scheme uses the normal map F_+ introduced by Robinson (1992). This map is formed by composing the projection map x_+ with F in the following manner:

$$F_+(x) = F(x_+) + (x - x_+) = 0. \tag{5}$$

It is well known that x_+ can be computed componentwise as $x_{+i} = \max\{x_i, 0\}$ and that F_+ is a piecewise smooth map; it is not differentiable whenever some $x_i = 0$. It is also clear that if x solves (5) then $z = x_+$ solves (4) and conversely if z solves (4) then $x = z - F(z)$ solves (5).

For differentiable systems of equations, Newton's method, at iteration k, determines the Newton point by finding a zero of the first-order (linear) Taylor series expansion:

$$H(x^k) + \nabla H(x^k)(x - x^k) = 0.$$

Since F_+ is nonsmooth (but is piecewise smooth), it is approximated by the piecewise linear normal map A_k defined by

$$A_k(x) = Mx_+ + q + x - x_+, \tag{6}$$

where $M = \nabla F(x_+^k)$ and $q = F(x_+^k) - Mx_+^k$. The Newton point x_N^k is a zero of A_k.

The path generation component of the scheme involves constructing a path between the current iterate x^k and the Newton point x_N^k. This piecewise linear path, parametrized by a variable t, is constructed by pivotal techniques using an extension of Lemke's well-known method (e.g. Cottle *et al.*, 1992). Each portion of the path is identified by a new pivot step in the manner now described.

If we define $z = x_+$, $v = (z - x)_+$ and $r = F_+(x^k)$ then

$$A_k(x) = (1 - t)r \tag{7}$$

can be rewritten as the following system:

$$[M \quad -I \quad r]\begin{bmatrix} z \\ v \\ t \end{bmatrix} = -q + r, \quad z \geq 0, \quad v \geq 0, \quad z^T v = 0, \quad 0 \leq t \leq 1. \tag{8}$$

Note that $z = x_+^k$, $v = (z - x^k)_+$, $t = 0$ satisfies (8). The first equation in (8) is used to relate the basic (essentially nonzero) variables to the nonbasic (zero) variables as is done in linear programming. Increasing t from 0 using a pivot in (8) generates a parametric solution (in t), namely $(z(t), v(t), t)$. Each ensuing pivot step forces a variable to leave the basis using the usual ratio test, whereupon an entering variable is chosen in accordance with the complementarity condition $z^T v = 0$. The Newton point is found when t leaves the basis at value 1. The parametric solution $x(t)$ to (7) can be recovered using $x(t) = z(t) - v(t)$. Each pivot thus represents a breakpoint in the piecewise linear path $x(t)$.

The path generation serves not only to find the Newton point, but also provides the necessary information to damp the method. Damping is the standard technique used to improve the convergence properties of an algorithm. For example, the linesearch damping step for systems of equations $H(x) = 0$ takes

$$x(t) = x^k + t(x_N^k - x^k)$$

and chooses a value of t between 0 and 1 to force an appropriate decrease in $\|H(x(t))\|$. Typically, an Armijo backtracking search is performed. That is, the points $x(1), x(\frac{1}{2}), x(\frac{1}{4}), \ldots$ are tried successively until a desired decrease in $\|H(x(t))\|$ is achieved.

In the NCP case, the damping step makes use of the fact that A_k decreases linearly in t on the path as indicated by (7). The piecewise linear path is traversed backwards, checking at each breakpoint whether

$$\|F_+(x(t))\| \leq (1-\sigma t)\|F_+(x^k)\|, \quad \sigma \in (0,1). \tag{9}$$

This condition will be satisfied for small t due to (7). The constant σ is chosen to be close to 0, so that almost any decrease in $\|F_+\|$ will be sufficient for acceptance. A standard Armijo search is carried out on the first segment of the path if necessary.

PATH also employs techniques to relax the acceptance criterion further to improve speed and robustness. These techniques include sparse updating schemes, nonmonotone stabilization and a watchdog technique (see Dirkse and Ferris, 1995a for further details). Also, since PATH starts each pivotal sequence at the current point, it is typically more efficient that Lemke's scheme, even for LCPs. Other enhancements are described in Dirkse and Ferris (1995b).

ILLUSTRATIVE EXAMPLE

The familiar 3-point bending (3PB) case is used for illustrative purposes. The softening laws assumed are not meant to be representative of actual experimental data; they merely serve to illustrate the capability of the solver in handling such types of constitutive relations.

The PATH solver was designed to compute a single solution. However, as in Bolzon *et al.* (1995), we attempted to capture all solutions by trying different starting vectors z. The crude scheme adopted to generate these vectors is as follows. The starting vector typically was divided into 6 subvectors of approximately equal lengths; in the case of an NCP of size 42, each subvector was thus of length 7x1. All elements of a subvector were assigned a value of either 0 or z^* (say 0.001). This gave rise to an obvious 64 (2^6) combinations to try for each problem. The main reasons for such a scheme were that 6 subvectors do not lead to an unduly large number of trials and it is expected that any cracking would occur in definite patterns with either nonzero values occurring in consecutive or alternate sequences. This scheme managed to capture all solutions (the numbers of which were known in advance) for all examples ran to date, with the exception of one example that had 11 distinct solutions. Computing times for all examples (maximum size = 146) were too small (a few seconds) to usefully report in detail.

Our data for the 3PB example are: length = 400 mm; height, depth = 100 mm; Young's modulus E = 14700 MPa; Poisson's ratio $\nu = 0.1$; $p_0 = 1.285$ MPa. Three cases were analyzed.

Case (i): a nonlinear softening law asymptotic to the w-axis and described by

$$\varphi = p - p_0(100^{-40w}) \leq 0, \quad w \geq 0, \quad \varphi w = 0,$$

Case (ii): a single linear softening branch law with the same p_0 and initial slope as case (i).

Case (iii): a nonlinear law obeying (3) and closely approximating (i) with $w_0 = 0.03$ mm and

$$f(w) = p_0(1 - w/w_0)^5.$$

The results for 10 quadratic boundary elements on the interface (21 nodes) are shown in Fig. 2. Solid lines represent the response for the asymptotic nonlinear case (i) softening law with the points on the response graph indicating the load levels used; dashed lines represent the linear case (ii) behaviour; and case (iii) response is shown by the dotted line. No computational difficulties were experienced, even in the presence of the observed snap-back behaviour.

As is well-known (e.g. Elices and Planas, 1995), the maximum tensile strength and the initial slope are sufficient to predict maximum load capacity. This is confirmed by the results shown in Fig. 2 for cases (i) and (ii). Also, as expected, the linear law with a smaller mode I fracture energy (per unit surface) leads to a more severe snap-back behaviour. The similarity of the responses for cases (i) and (iii) is obviously due to the closeness of the corresponding softening models.

CONCLUDING REMARKS

Holonomic analyses of quasibrittle fracture based on the cohesive crack model with nonlinear softening can be elegantly formulated and solved as NCPs. Investigation of a recent Newton-type algorithm, implemented as the general purpose mixed complementarity code PATH, has shown that it has the potential of capturing multiplicity of solutions. The fact that nonlinear laws can now be accommodated easily in the analysis obviates the computational burden of piecewise linearization and also means that other more complex, not easily linearized nonlinear fracture laws as arise, say, in mixed-mode processes, can be directly adopted. However, further research still needs to be carried out in improving the efficiency and robustness of the search procedure for obtaining multiple solutions, or determining that no solution exists.

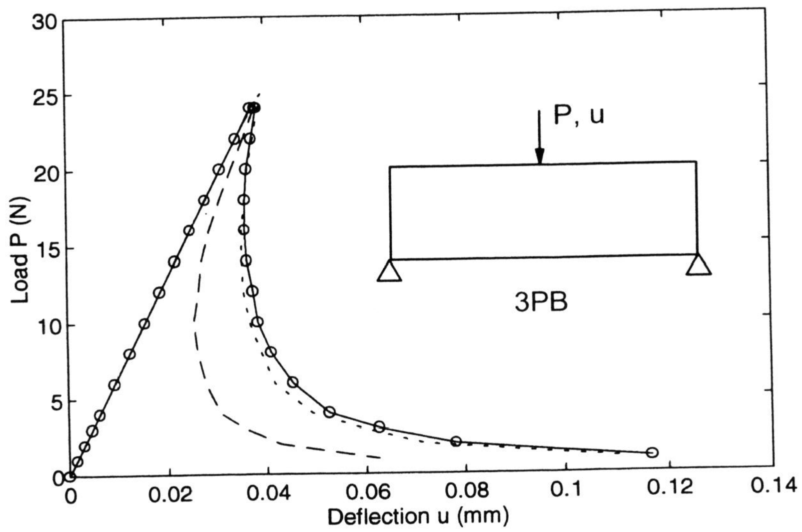

Fig. 2. Numerical simulations of 3PB example: — case (i); – – case (ii); - - case (iii).

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