FINITE DEFORMATION ANALYSIS OF CRACK GROWTH IN ELASTIC BODY

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ABSTRACT

A model of crack growth in elastic material is developed. It is assumed that crack initiation leads to nucleation of micropores near a crack tip. These micropores are coalesced with one another and with the crack that results in crack growth. Crack is modelled by strongly elongated elliptic cavity. It is supposed that opening of new micropore takes place at the point where a strength criterion exceeds maximum. Analyses are carried out for two-dimensional case with allowance for finiteness of strains using Signorini's expansion and Muskhelishvili's technique.

KEYWORDS

Crack growth, nucleation of micropores, finite deformation.

INTRODUCTION

A conventional approach in fracture mechanics is based on consideration of a crack as a "mathematical cut" or a hole bounded by a contour with singular points. However, in some recent works devoted to detailed analysis of crack growth mechanism and interaction between crack and microdamages crack is considered as a hole with a smooth boundary. Thus, model of crack growth proposed by Bolotin (1993), Bolotin and Lebedev (1996) is based on the assumption that curvature radius at the crack tip is finite and may be changed in the process of crack propagation. In this model microdamages are simulated by by introduction of a special scalar parameter (microdamage measure) into constitutive equations. A similar approach is proposed by Paas *et al.* (1993).

In the present work crack is simulated by a narrow slot with non-zero width, assuming finite curvature of slot contour at the tip. It is supposed, that nucleation of a crack induces successive opening of micropores coalescing with one another and with the crack that leads to crack growth. We assume also that opening of the next micropore takes place at a point where a strength criterion exceeds maximum. Unlike Bolotin (1993) and Paas *et al.* (1993), we consider microdamages as a stress concentrators disposed near the crack tip and interacting with a crack. It should be noted that interaction between elliptical holes that are sufficiently differ in size (the smaller hole in this case may be considered as a micropore) and impact of configurations of these holes on fracturing patterns are investigated by Tsukrov and Kachanov (1994) within the scope of linear elasticity. In the present work we take into account of nonlinear effects caused by finiteness of deformations and nonlinearity of constitutive equations. We consider two-dimensional problems and use analytical technique for solution, (Muskhelishvili, 1963, and Savin, 1970).

CRACK GROWTH MODEL

In this section we consider mechanism of crack growth in previously loaded body undergoing finite deformation. This mechanism is founded on the theory of repeatedly superimposed large deformations developed by Levin and Taras'ev (1980), Levin(1988). Nucleation of initial crack is not considered in detail. The generalized statement of problem is the following.

Let the large plane static strains and stresses are brought about by the external forces in the nonlinear elastic body that was in the initial (unstressed) state . Body passes to the first intermediate state. Then a closed surface (boundary of a crack) is imagined in a body and the part of a body bounded by this surface is removed, and the effect of removed part on the remained one is replaced by the forces distributed over this surface (on the principle of releasing of braces). It is clear that this transformation doesn't change the state of stress and strain in a body. Then these forces passed to the category of the external forces are reduced to zero quasistatically (for example, isothermally). It raises a large (at the vicinity of formed surface) strains and stresses that are superimposed on the large initial strains and stresses already existed in a body. Body passes to the second intermediate state. Naturally, the shape of arisen boundary surface is changed, and one can prescribe this surface either in the first intermediate state or in the second intermediate state. In the second case the shape of newly formed boundary surface in the first intermediate state is unknown. An approach to solution of this kind of problems is proposed by Levin and Taras'ev (1980) and Levin (1988).

If the strength criterion K_S is exceeded by loading then the opening of micropore takes place according with the aforecited scheme (the micropore is considered here as a narrow slot). Body passes to the third intermediate state. Then the procedure of micropore opening continues while the strength criterion satisfies in a some part of a body.

To facilitate particular analysis in the work it is considered that all the narrow slots are elliptical either at instant of their formation or in transition to the state following after the state of formation. Either generally known strength criteria (see, for example, Cherepanov (1974)) or combinations of theirs are used in this work to study as the cases when the first micropore opens at some distance from the narrow slot as the cases when this micropore opens next to the surface of narrow slot. For the particular analysis the following models (problems) are used:

1. The first micropore opens at a distance more than b from the surface of narrow slot (b is the major axis of ellipse simulating micropore shape). It is considered that the micropores open sequentially and if the distance between the point where K_S is at a maximum and the nearest tip of the before nucleated hole (narrow slot or micropore) exceeds $b/2$ then the centre of newly opened micropore is at the point of maximum else the centre of newly opened micropore is at a distance of $b/2$ from this tip.

When the point of maximum lies between two before nucleated holes and the distance between these holes doesn't exceed b it is considered these holes coalesce. In this case, for the particular analysis, newly produced hole is approximated by ellipse and the strength criterion is tested. Slot growth may be either continued or stopped.

2. The first micropore opens at the boundary of narrow slot (a distance between the micropore opening centre and the boundary of narrow slot is $b/2$). In this case, for the particular analysis, at every stage of loading (that is, the new micropore opening) the approximation of newly produced slot by ellipse is used.

3. Two commensurable narrow slots were nucleated sequentially (or simultaneously) and micropores open between the slots until the coalescence of theirs.

BASIC RELATIONS

We use notation presented by Levin and Taras'ev (1980) and Levin (1988).

Equilibrium equation:

$$\overset{m}{\nabla} \cdot \overset{m}{\Sigma}_{0,n} - \overset{m}{\Sigma}_{0,n} \cdot \overset{m}{\nabla} \ln(1+\Delta_{0,n}) + \overset{m}{\Sigma}_{0,n} \cdot\cdot \overset{m}{\nabla} \Psi_{m,n} \cdot {\Psi_{m,n}}^{-1} - \left[\overset{m}{\nabla} \cdot {\Psi^T_{m,n}}^{-1} \cdot \Psi^T_{m,n}\right] \cdot \overset{m}{\Sigma}_{0,n} + \overset{m}{F}_n \rho_m (1+\Delta_{0,m}) = 0. \quad (1)$$

Boundary conditions:

$$\overset{(m)}{N}_n \cdot \overset{m}{T}_{0,n}\Big|_{\Gamma_n^{(m)}} = 0. \quad (2)$$

Here $\Gamma_n^{(m)}$ is the contour of body in the n-th state in coordinates of the m-th state; $\overset{(m)}{N}_n$ is the unit normal to $\Gamma_n^{(m)}$.

Conditions on infinity:

$$T_{0,n}\big|_\infty = \sigma^\infty. \quad (3)$$

Hereafter the upper index over a quantity denotes the number of state this quantity is given in or referred to.

$$\overset{m}{\Sigma}_{0,n} = (1+\Delta_{0,n})\, \Psi^T_{n,m} \cdot T_{0,n} \cdot \Psi_{n,m}, \quad (4)$$

is the total (for the n-th state) stress tensor referred to the basic set of the m-th state; $T_{0,n}$ is true total (for the n-th state) stress tensor ($T_{0,1}$ is the Cauchy stress tensor); $\Delta_{0,n}$ is the relative volume change in transition from the initial to the n-th state; $\Psi_{q,p}$ is the deformation gradient: $\Psi_{q,p} = \Psi_{q,f} \cdot \Psi_{f,p}$.

We use constitutive equations as for compressible as for incompressible materials. For compressible materials, Murnaghan-type constitutive equation is used (cf. Murnaghan, 1951):

$$\overset{0}{\mathbf{\Sigma}}_{0,n} = \lambda(\overset{0}{\mathbf{E}}_{0,n} : \mathbf{I})\mathbf{I} + 2\mu\overset{0}{\mathbf{E}}_{0,n} + 3C_3\left(\overset{0}{\mathbf{E}}_{0,n} : \mathbf{I}\right)^2\mathbf{I} + C_4\left[\left(\overset{0}{\mathbf{E}}_{0,n}\right)^2 : \mathbf{I}\right]\mathbf{I} + 2C_6\left(\overset{0}{\mathbf{E}}_{0,n} : \mathbf{I}\right)\overset{0}{\mathbf{E}}_{0,n} + 3C_5\left(\overset{0}{\mathbf{E}}_{0,n}\right)^2. \quad (5)$$

Here $\overset{m}{E}_{q,p}$ is the strain tensor, describing change of strains in transition from the q-th state to the p-th state and referred to the coordinate basis of the m-th state (Levin, 1988):

$$\overset{m}{E}_{q,p} = \frac{1}{2}\left(\Psi_{m,p} \cdot \Psi^T_{m,p} - \Psi_{m,q} \cdot \Psi^T_{m,q}\right) \quad (6)$$

For incompressible material, Mooney-type constitutive equations are used:

$$T_{0,n} = \mu F_{0,n} + p_{0,n} I \quad (7)$$

Here $F_{0,n} = \Psi^T_{0,n} \cdot \Psi_{0,n}$ is the strain measure tensor (at $n = 1$ $F_{0,1}$ is Finger strain measure tensor), and μ is shear modulus.

Notation of constitutive equation in the space of correspondent state is derived with the help of "non-energetic transition" procedure considered by Levin and Taras'ev(1980) and Levin(1988).

STRENGTH CRITERIA

We utilize strength criteria considered by Gol'denblat and Kopnov (1968). The following relations are used:

- for the first type of problems

$$\frac{1}{\mu^2}\left(\alpha|T_1 + T_2|^2 + (1-\alpha)\,|T_1 T_2|\right) = K_S, \quad (8)$$

where T_i $(i = 1, 2)$ are the principal values of true stress tensor $T_{0,n}$;

- for the second type of problems

$$\frac{1}{\mu}|T_1 + T_2| = K_S, \quad (9)$$

- for the third type of problems - as (8), as (9).

TECHNIQUE OF SOLUTION

In order to apply strength criterion, it is necessary to solve stress concentration problem at each step of micropore opening procedure. This problem is solved analitically using Signorini's expansion (cf., for example, Truesdell(1972); second-order effects are taken into account). Savin (1970) used this technique for two-dimensional problems of stress concentration near holes.

Ratio $q = \max\limits_{i,j}\left|\sigma^{\infty}_{i,j}\right|/\mu$ is used as asmall parameter ($\sigma^{\infty}_{i,j}$ are components of σ^{∞}). Linearized problem is solved by Muskhelishvili's method with the use of Schwartz alternating technique (Muskhelishvili, 1963). At each step of Schwartz iterative procedure, the boundary-value problem is solved for infinite region, bounded by simple closed curve Γ_k, bounding one of the holes (k is changed from one step to another, $1 \le k \le n$, where n is total number of holes). For solution of this problem at each step of Schwartz procedure the conformal mapping $z = \omega_k(\xi_k)$ of the infinite region, bounded by unit circle $|\xi_k| \ge 1$, onto the infinite region, bounded by Γ_k, is used. In this paper, contours $\Gamma_k (k = 1, \ldots, n)$ are ellipses, and function $\omega_k(\xi_k)$ is represented as:

$$\omega_k(\xi_k) = R_k \exp(\imath\alpha_k)\left(\xi_k + \frac{\xi_k}{m_k}\right) + x_k + \imath y_k, \quad (10)$$

where $R_k = (a_k + b_k)/2$ $m_k = (a_k - b_k)/(a_k + b_k)$

a_k, b_k are semiaxes of the k-th ellipse $(a_k \le b_k)$;

α_k is the slope of the major axis of k-th ellipse; x_k,y_k are coordinates of centre of k-th ellipse.

The following class of functions is used for solution:

$$f(z,\bar{z}) = f\left(\xi_1, \ldots, \xi_n; \bar{\xi}_1, \ldots, \bar{\xi}_n\right) = \sum_{i=1}^{N} R_i\left(\xi_{m_{i,1}} - c_{i,1}\right)^{k_{i,1}}\left(\bar{\xi}_{m_{i,2}} - c_{i,2}\right)^{k_{i,2}} \quad (11)$$

where $k_{i,j}$, $m_{i,j}$ are integers $(1 \le m_{i,j} \le n)$.

Note that more particular class of functions

$$\sum_{i=1}^{N} R_i {\xi_{m_{i,1}}}^{k_{i,1}} {\bar{\xi}_{m_{i,2}}}^{k_{i,2}} \quad (12)$$

has been used by Kosmodamianskii and Chernic (1981), Xiwu *et al.* (1995) to solve plane problems of elasticity for multiply connected regions.

To implement Schwartz technique using functions in the form of (11), it is necessary to approximate expressions $(\xi_m - c)^k$ by rational functions of ξ_l $(l \ne m)$ at the contour $|\xi_l| = 1$. For this purpose, Faber expansion is used.

For implementation of computational procedure, outlined above, specialized computer algebra system have been developed. This system partially makes possible to perform the following operations on the functions of the form (11): addition, multiplication, differentiation with respect to z or $\bar{z}$, computation of Cauchy-type integrals over boundaries of holes, complex conjugation, computation of limit when $|z| \to \infty$, computation of a value at a given values of arguments (Levin and Zingerman, 1987).

Operations of tensor calculus on second-rank tensors which components are functions of the form (11) are also provided, namely: addition, multiplication, determination of invariants, conjugation, application of gradient operator.

When the particular problems was solved it was considered that the initial stressed state is uniform although the availabled computer algebra system permits to solve this problem in the general case.

NUMERICAL EXAMPLE

For the particular analysis (for the first type of problems, criterion (8)) the following values of parameters are used.

The shape of narrow slot is prescribed in the second intermediate state and is approximated by the ellipse with semiaxes b_1 and a_1 ($m_1 = (b_1 - a_1)\,/\,(b_1 + a_1) = 0.75$). A case of uniaxial initial loading is considered: $\sigma_{11}^{\infty} = \sigma_{12}^{\infty} = 0$; $\sigma_{22}^{\infty}/\mu = 0.1$. We consider the case when the major axis of ellipse is parallel to x_1 axes. For simplicity, we assume that the major axes of micropores are parallel to the major axis of the main crack.

For compressible material (5) with material constants $\lambda/\mu = 1.341$, $C_3/\mu = -0.272$, $C_4/\mu = -3.183$, $C_5/\mu = -2.715$ at $\alpha = 0.75$, $K_S = 0.54$ first micropore opens at a distance of $l \approx 0.007b_1$ from the tip of initial slot. In this case, we assume that micropores are circular with the radius $b = 0.002b_1$. For incompressible material (7) with $\beta = 1$ at $\alpha = 0.1$, $K_S = 0.42$ first micropore opens at a distance of $l \approx 0.002b_1$ from the tip of initial slot. In this case we assume that micropores are circular with the radius $b = 0.0005b_1$. As for compressible material as for incompressible one opening of the following micropores leads to the coalescence of micropores with narrow slot.

In the problem of the second type (strength criterion (9)) at the same values of parameters slot growth continues after the opening of some micropores.

In the problem of the third type two equal narrow slots are formed simultaneously, large principal axes of ellipses lie on the same line and the distance between their centres is $3b_0$. When the criterion (8) is used at a choosen values of parameters the coalescence of these slots takes place, and the first micropore opens in the middle of line joining the centers of ellipses.

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