COMPUTATIONAL MECHANICS OF INTERFACIAL FRACTURE FOR COMPOSITE DEBONDING; AND OF ELASTOPLASTIC FRACTURE WITH MULTI-SITE-DAMAGE

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ABSTRACT
This paper deals with some recent work of the authors in the areas of integrity of composite structures and aging metallic aircraft. The first topic deals with a fracture criterion for predicting the onset of propagation of a delamination crack between two dissimilar laminae of arbitrary ply angles. This criterion assumes that the fracture of a delamination crack is dependent upon the criticality of the stress concentration near a damage radius ahead of the delamination crack tip. It is shown that by introducing the damage radius as the characteristic length in the definition of the stress intensity factors, the comparisons of the stress concentration near the damage radius can be made based on the stress intensity factors. To calculate the stress intensity factors for an interfacial crack in an anisotropic bimaterial continuum, a numerical method based on the virtual crack closure integral approach has been developed. In the second topic, methodologies for elastoplastic analyses of stability of multiple cracks are presented: (i) a superposition method to construct solutions for multiple cracks (as in a multiple-site-damage situation in aging metallic aircraft), subjected to arbitrary crack surface tractions, in an infinite domain; (ii) a procedure for elastoplastic analyses of multiple cracks in a finite body, using the finite element alternating method and an initial stress type iteration method; and (iii) a procedure for elastoplastic crack growth analyses.

KEYWORDS
Fracture criterion, interfacial crack, anisotropic bimaterial continuum, virtual crack closure, finite element alternating method, elastic-plastic, crack growth, multiple site damage.

1 Interfacial Fracture Mechanics for Composite Debonding

The mixed-mode stress intensity factors for an interfacial crack between dissimilar isotropic media were first defined by Williams (1959), and Rice and Sih (1965). They have found that, when the linear-elastic-fracture-mechanics (LEFM) theory is employed, the stress oscillates and the crack surfaces overlap near the interfacial crack tip. Rice (1988), however, has justified that this solution still makes engineering sense for small scale nonlinear material behavior and a small scale contact zone at the crack tip. Furthermore, Rice (1988) has also presented a fracture criterion based on the total energy release rate and the phase angle of the ratio of stress intensity factors. Since then, this fracture criterion has been well accepted in the area of interfacial fracture mechanics for an isotropic bimaterial continuum. However, this criterion cannot be easily applied to a delamination crack between two dissimilar laminae of arbitrary ply-angles. In this research, a new fracture criterion is introduced. This criterion assumes that the onset of propagation of a delamination crack in a composite laminate is dependent upon the criticality of the stress concentration near a damage radius ahead of the delamination crack tip. Furthermore, by introducing the damage radius as the characteristic length in the definition of the stress intensity factors, the asymptotic stress fields near the damage radius would decouple with their respective stress intensity factors. Henceforth, this fracture criterion based upon the comparisons of the stress concentration near the damage radius can be simplified using the stress intensity factors instead.

In order for this fracture criterion to be accepted for general applications, it would be essential that the individual components of the mixed-mode stress intensity factors for an interfacial crack be calculated efficiently and accurately. The stress intensity factor, used in the present research, for an interfacial crack in an anisotropic bimaterial continuum is based on the definition proposed by Wu(1989), Qu and Li(1991) and Qu and Bassani(1993). Based on this definition, Chow, Beom and Atluri (1995) have developed the hybrid stress element method and the mutual integral method to compute the stress intensity factors for an interfacial crack between dissimilar anisotropic media. However, these methods required the knowledge of the asymptotic stress and displacement fields around the interfacial crack tip. Since the asymptotic stress and displacement fields involve complex numbers, the calculation of the stress intensity factors using these methods require a considerable programming effort. To significantly reduce the complexity of calculating the stress intensity factors for an interfacial crack, Chow and Atluri (1995) have developed a numerical method based on the virtual crack closure integral approach. This approach first calculates the mixed-mode energy release rates, and then relates the energy release rates to the mixed-mode stress intensity factors. To determine if mixed-mode stress intensity factors can accurately predict the onset of propagation of delamination cracks, fracture analysis has been performed on various laminates under uniaxial tension load. In addition, an analytical study has been performed to predict the post-buckling strength of stiffened laminated composite panels. This study is meant to complement the experimental study by Starnes, Knight, and Rouse (1985).

1.1 Fracture Criterion for Delamination Crack

Consider an interfacial crack between two laminae, $[\theta, \phi]$, where the ply angles of the upper and lower plies are θ and ϕ respectively. For a crack in a homogeneous material, in which θ is equal to ϕ, the singular stress field along the interface can be decoupled into three individual modes: σ_{22} relates to K_I, σ_{12} relates to K_{II}, and σ_{23} relates to K_{III}, where x_1 is along the crack, x_2 is normal to the crack plane, and x_3 is along the crack front. However, when θ is not equal to ϕ, there exists an oscillation index, ϵ, in the singular stress solution. This oscillation index, ϵ, is a function of the material properties of the plies with angles θ and ϕ. Because of this oscillation index, the singular stress field cannot be decoupled into the three unique individual modes. Each of the singular stress fields along the interface is a function of the three stress intensity factors coupled by the bimaterial matrix function $Y(r^{i\epsilon})$. Since the matrix $Y(r^{i\epsilon})$ is dependent upon the material properties of the bimaterial continuum, the relation between the stress intensity factors and the singular stress field differs for each set of $[\theta, \phi]$. Hence, the comparisons of singular stress field of a delamination crack in different ply lay-ups cannot be made based on the mixed-mode stress intensity factors.

In the present study, the fracture criticality of a delamination crack between two dissimilar laminae is assumed to be determined by the stress magnitudes near a critical damage radius. This damage radius defines a region with considerable fiber/matrix damage as well as nonlinear material behavior. By redefining the stress intensity factors with a characteristic length, Chow and Atluri (1996) have shown that the relationships between the asymptotic singular stress fields along the interface and the stress intensity factors would decouple near the characteristic length. As a result of this decoupling effect, along the interface near the damage region, the stress intensity factor of an interfacial crack has a physical interpretation similar to the stress intensity factor of a homogeneous crack. Henceforth, this fracture criterion based upon the comparisons of the stress concentration near the damage radius can be simplified using the stress intensity factors instead.

Consider an example of a delamination crack between the [0/90] laminate where the 0^o lamina denotes that the fiber direction is along the out-of-plane direction, while the 90^o lamina denotes that the fiber direction lies parallel to the crack. Since all of the experimental fracture data available for the critical stress intensity factor, K_C, are based on a delamination crack in a unidirectional laminate, a question arises on whether the fracture data for the [0/0] or [90/90] laminate could be used to predict the onset of a delamination crack in the [0/90] laminate. Using Irwin's approximation and the experimental data for T300-5208 unidirectional laminate, the damage radius can be approximated with the ply thickness of a single lamina. By defining the characteristic length as the ply thickness, it can be shown that the stress distributions along the interface of a [0/90] laminate and a homogeneous laminate ([0/0] or [90/90]), for any stress intensity factors $\hat{\boldsymbol{k}}_{r_o}$, are very similar except for a small region where there is considerable material nonlinearity and damage. Since the fracture of the delamination crack is assumed to be based on the stress concentration near the damage radius, the critical stress intensity factor for [0/90] laminate, $\hat{K}_{[0/90]C}$, should be related to either the value of $\hat{K}_{[0/0]C}$ or $\hat{K}_{[90/90]C}$. In addition, Lucas (1992) has performed some experiments on the fracture of a delamination crack in unidirectional laminates of different ply angle and found that the fracture toughness of a [90/90] laminate is less than that for a [0/0] laminate, $\hat{K}_{[90/90]C} < \hat{K}_{[0/0]C}$. Since a delamination crack between two dissimilar laminae can fracture at either the upper or lower lamina, it is reasonable to assume that the delamination crack would fracture at the material of lower fracture toughness. Hence, using this assumption, the critical stress intensity factor for [0/90] laminate, $\hat{K}_{[0/90]C}$, would be postulated as $\hat{K}_{[90/90]C}$.

Since the nature of the stress field near an interfacial crack is often defined by mixed-mode stress intensity factors, the fracture criterion to predict the onset of a delamination crack growth can be slightly modified as:

$$\left[\frac{\hat{K}_I}{0.85K_{IC}}\right]^2 + \left[\frac{\hat{K}_{II}}{K_{IIC}}\right]^2 + \left[\frac{\hat{K}_{III}}{K_{IIIC}}\right]^2 = \left(\theta_{fac}\hat{K}_0\right)^2 \tag{1}$$

$$\hat{K}_0 \geq \hat{K}_C \rightarrow \text{interfacial crack failure}$$

where θ_{fac} is a non-dimensional constant dependent on the angle between the crack front and the fiber's direction, and $\hat{K}_0$ is the normalized stress intensity factor. The critical stress intensity factors, $(K_{IC}, K_{IIC}, K_{IIIC})$, are defined as the critical values for a delamination crack in a [90/90] laminate. For this fracture criterion, the delamination crack would fracture when the normalized stress intensity factor, $\hat{K}_o$, exceeds a critical normalized stress intensity factor $\hat{K}_C$. Chow and Atluri (1995, 1996b) have demonstrated that the normalized stress intensity factor, $\hat{K}_o$, does not change significantly when the choice of the characteristic length is reduced by an order of a mag-

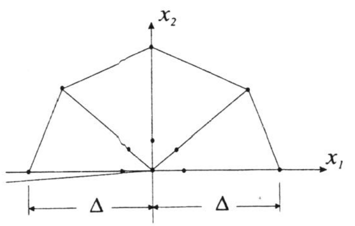

Figure 1: Quarter-point singular elements surrounding the crack tip

nitude or two. Based on the experiments performed by Lucas (1992), θ_{fac} for [90/90] laminate is 1.00, θ_{fac} for [45/45] is 1.15, and θ_{fac} for [0/0] laminate is 1.61. This experimental result indicates that the [90/90] laminate has the lowest fracture toughness followed by the [45/45] laminate and then the [0/0] laminate. For a delamination crack between dissimilar laminae, the constant θ_{fac} can be obtained using the assumptions discussed above, in which the bimaterial interfacial crack would fail at the material with a lower fracture toughness. For example, the constant for [45/90] and [45/0] would have the same values as [90/90] and [45/45] respectively.

1.2 Virtual Crack Closure Integral Method

According to Irwin (1958), the work required to extend a crack by an infinitesimal distance Δ is equal to the work required to close the crack to its original length. Thus, the energy release rate for mode I and mode II deformations can be expressed as

$$G_I = \frac{1}{2\Delta}\int_0^{\Delta}[\sigma_{22}(r)\,\delta_2(\Delta-r)]\,dr\,, \qquad G_{II} = \frac{1}{2\Delta}\int_0^{\Delta}[\sigma_{12}(r)\,\delta_1(\Delta-r)]\,dr \tag{2}$$

where σ is the stress distribution ahead of the crack tip, and δ is the crack opening displacement behind the crack tip. Here, the crack-axis is along the x_1 axis and normal to the x_2 axis as indicated in Fig. 1. For a crack in a homogeneous domain, there exists no coupling between $\sigma_{22}(r)$ and $\delta_1(\Delta-r)$, and between $\sigma_{12}(r)$ and $\delta_2(\Delta-r)$. However, these couplings do exist for an interfacial crack between two dissimilar media. As a result, a coupled energy release rate is introduced here as

$$G_{I-II} = \frac{1}{2\Delta}\int_0^{\Delta}[\sigma_{12}(r)\,\delta_2(\Delta-r)+\sigma_{22}(r)\,\delta_1(\Delta-r)]\,dr \tag{3}$$

The procedure to obtain these energy release rates from finite element solutions were given by Rybicki and Kanninen (1977) and Raju (1987).

For an orthotropic bimaterial aligned with the interface crack coordinate system (e.g., a [0/90] laminate where the fibers are parallel to the crack front in one material and perpendicular in the other), the stress field along the interface at a distance r ahead of a crack tip is given by Qu and Bassani (1993) as:

$$\beta_2^{1/2}\sigma_{22} + i\beta_1^{1/2}\sigma_{12} = \frac{\beta_2^{1/2}K_I + i\beta_1^{1/2}K_{II}}{\sqrt{2\pi r}}\,r^{i\epsilon} \tag{4}$$

and the crack opening displacements at a distance r behind the crack tip are

$$\beta_1^{1/2}\delta_2 + i\beta_2^{1/2}\delta_1 = \frac{(s_{\#1}-s_{\#2})\sqrt{2\pi r}}{\pi\beta(1+2i\epsilon)\cosh(\pi\epsilon)}\left(\beta_2^{1/2}K_I + i\beta_1^{1/2}K_{II}\right)r^{i\epsilon} \tag{5}$$

The bimaterial constants

$$\epsilon = \frac{1}{2\pi}\ln\left[\frac{1+\beta}{1-\beta}\right], \qquad \beta = \sqrt{\beta_1\beta_2}\,, \tag{6}$$

$$\beta_1 = \frac{s_{\#1}-s_{\#2}}{s_{\#1}\rho_{\#1}/\eta_{\#1} + s_{\#2}\rho_{\#2}/\eta_{\#2}}, \quad \text{and} \quad \beta_2 = \frac{s_{\#1}-s_{\#2}}{s_{\#1}/\eta_{\#1} + s_{\#2}/\eta_{\#2}} \tag{7}$$

are obtained from the orthotropic material constants

$$s = \left[C_{1122} + \sqrt{C_{1111}C_{2222}}\right]^{-1}, \quad \rho = \sqrt{\frac{C_{1111}}{C_{2222}}}\,, \quad \eta = \left[\frac{C_{1212}\left(\sqrt{C_{1111}C_{2222}} - C_{1122}\right)}{C_{2222}\left(2C_{1212} + C_{1122} + \sqrt{C_{1111}C_{2222}}\right)}\right]^{1/2}$$

where C_{ijkl} is the stiffness matrix that relates the stresses, σ_{ij}, to strains, ϵ_{jk}. The subscripts #1 and #2 refer to the materials above and below the interfacial crack. By substituting Eq. (2) into the mixed-mode energy release rates, Chow and Atluri (1995) have obtained the simple relationships between the stress intensity factors and the energy release rates as:

$$\begin{bmatrix} a_{11}a_{21}+a_{12}^2 & a_{22}a_{21}-a_{12}^2 \\ a_{22}a_{21}-a_{12}^2 & a_{11}a_{21}+a_{12}^2 \end{bmatrix}\begin{Bmatrix} \beta_2 K_I^2 \\ \beta_1 K_{II}^2 \end{Bmatrix} = \lambda\begin{Bmatrix} a_{21}\lambda_1 G_I - a_{12}\lambda_2 G_{I-II} \\ a_{21}\lambda_1 G_{II} + a_{12}\lambda_2 G_{I-II} \end{Bmatrix} \tag{8}$$

where $\lambda_1 = \dfrac{8\cosh^2(\pi\epsilon)\,(1+4\epsilon^2)\,\beta^2}{s_{\#1}-s_{\#2}}$ and $\lambda_2 = \dfrac{16\cosh^2(\pi\epsilon)\,(1+4\epsilon^2)\,\beta^3}{(s_{\#1}-s_{\#2})(\beta_1+\beta_2)}$

and the constants in the matrix for Eq.(8) are

$$a_{11} = \left(1+4\epsilon^2\right) + \left(1-\frac{134}{21}\epsilon^2\right)\cos\left[2\epsilon\ln\Delta\right] + \left(\frac{34}{7}\epsilon - \frac{4}{3}\epsilon^3\right)\sin\left[2\epsilon\ln\Delta\right] \tag{9}$$

$$a_{12} = 2\left\{\left(\frac{34}{7}\epsilon - \frac{4}{3}\epsilon^3\right)\cos\left[2\epsilon\ln\Delta\right] - \left(1-\frac{134}{21}\epsilon^2\right)\sin\left[2\epsilon\ln\Delta\right]\right\} \tag{10}$$

$$a_{22} = \left(1+4\epsilon^2\right) - \left(1-\frac{134}{21}\epsilon^2\right)\cos\left[2\epsilon\ln\Delta\right] - \left(\frac{34}{7}\epsilon - \frac{4}{3}\epsilon^3\right)\sin\left[2\epsilon\ln\Delta\right] \tag{11}$$

$$a_{12} = 4\left\{\left(1-\frac{134}{21}\epsilon^2\right)\cos\left[2\epsilon\ln\Delta\right] + \left(\frac{34}{7}\epsilon - \frac{4}{3}\epsilon^3\right)\sin\left[2\epsilon\ln\Delta\right]\right\} \tag{12}$$

The constant Δ is the length of the crack tip element shown in Fig. 1. The squares of the stress intensity factors, K_I^2 and K_{II}^2, can be easily solved from the linear equation in Eq.(8). The signs for the stress intensity factors, K_I and K_{II}, can be ascertained from the crack opening displacements.

1.3 Validation of the Fracture Criterion for a Delamination Crack

To validate the fracture criterion based on the mixed-mode stress intensity factors, Chow and Atluri (1996a) have performed a fracture analysis on laminates under uniaxial tensile load. The mixed-mode stress intensity factors in the fracture criterion are normalized with the critical values obtained from the experimental data of double cantilever beam specimens. By comparing the numerical results with some experimental observations, it has been shown that the calculated stress intensity factors can be effectively used as the fracture parameters to predict the onset of propagation of delamination cracks in composite laminates when the crack propagates in a 2-D fashion as assumed in the numerical model. In this fracture analysis, the inability of the fracture criterion based on the total energy release rate to correctly predict the failure strain has also been demonstrated.

In addition, Chow and Atluri (1996b) have also conducted an analytical investigation to predict the post-buckling strength of laminated composite stiffened panels under compressive loads. The results from this study are compared with an experimental investigation conducted by Starnes, Knight, and Rouse (1987). It is found that for the eight different specimens that are considered in this study, the calculated critical energy release rate for the propagation of the edge delamination crack in each specimen differs substantially from those for the others; hence it may be concluded that the total energy release rate would not be a suitable fracture parameter for predicting the post-buckling strength of the stiffened panels. On the other hand, using the fracture criterion based on the critical mixed-mode stress intensity factors, the predicted post-buckling strength of the stiffened panels compares quite favorably with the experimental results and the error of prediction is less than 10%. Furthermore, by applying the criterion of critical mixed-mode stress intensity factors on a simple damage model, the present analysis is able to predict the significant reduction in the post-buckling strength of stiffened panels with a damage due to a low-speed impact at the skin-stiffener interface region.

2 Finite Element Alternating Method

Fracture mechanics problems can be solved using a number of different methods, including finite element and boundary element methods, singular/hybrid finite elements, alternating method, and the use of domain-independent and path-independent integrals, etc [See Atluri(1986), Atluri and Nishioka(1989), Wang and Atluri(1996) for comprehensive discussions and detailed summaries]. Of these, the finite element (or boundary element) alternating method is considered to be a very efficient and accurate method.

The finite element alternating method (FEAM) solves for the cracks (including surface cracks) in finite bodies by iterating between the analytical solution for an embedded crack in an infinite domain, and the finite element (or boundary element) solution for the uncracked finite body. The cohesive tractions at the locations of the cracks in the finite element (or boundary element) model of the uncracked body, and the residuals at the far field boundaries in the analytical solution for the infinite body, are corrected through the iteration process. Essentially, the alternating method is a linear superposition method[see Fig. 2]. Fracture mechanics parameters can be found accurately

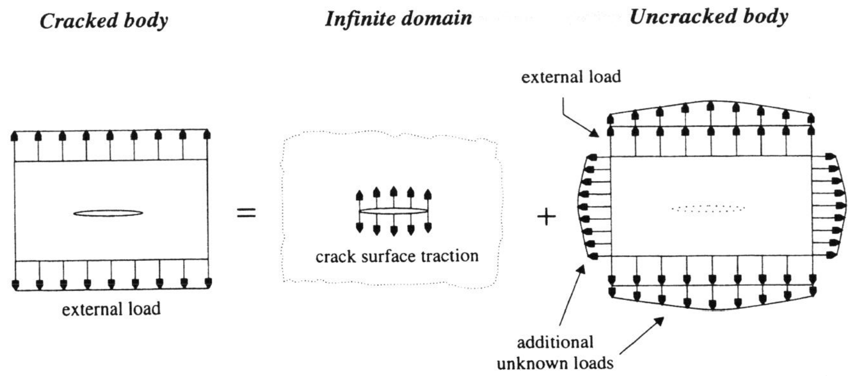

Figure 2: Superposition principle for the finite element alternating method

because the near crack tip fields are captured exactly by the analytical solution. Coarser meshes can be used in the finite element analysis because the cracks are not modeled explicitly. In a crack growth analysis, or in conducting a parametric analysis of various crack sizes, the stiffness of the uncracked body remains the same for all crack sizes. Thus, the global stiffness matrix of the finite element model is decomposed only once. In the most common finite element analysis for fracture problems, it is necessary to use very fine meshes (or adaptive mesh refinements) around the crack tips, and to decompose the global stiffness matrix every time a crack size changes. Thus, the alternating method is very efficient in saving both time in the computational analysis and human effort in the mesh generation.

The finite element alternating method can be applied to the elastic-plastic analysis of cracks when it is used in conjugation with the initial stress method, even though the alternating method itself is based on the superposition principle which is valid only for linear problems. The initial stress approach converts the elastic-plastic analysis into a series of linear elastic steps, in which the superposition principle holds. The elastic-plastic finite element method was first presented by Nikishkov and Atluri (1994). It was used in 2-D elastic-plastic analyses of wide-spread fatigue damage in ductile panels [Pyo, Okada and Atluri (1994,1995)]. The method was successfully used to simulate the stable crack growth in the presence of multiple site damage in Wang, Brust and Atluri(1995a,b,c).

2.1 Solutions for multiple embedded cracks in an infinite domain

Solutions for multiple embedded cracks in an infinite body, subjected to arbitrary crack surface tractions, can be constructed using the solution[1] for a single crack in an infinite domain subjected

[1] Analytical solutions for a single embedded 2D crack in an infinite domain, subjected to point loads or piecewise linear/constant loads at the crack surface, may be found in Wang and Atluri(1996). Analytical solutions for an embedded elliptical crack, subjected to arbitrary polynomial distributed crack surface traction, can be found in Vijayakumar and Atluri (1981), Nishioka and Atluri (1983).

to arbitrary crack surface loading. Analytical solutions for multiple embedded cracks in an infinite domain are available only for some special configurations, such as multiple collinear cracks subjected to arbitrary crack surface tractions [Muskhelishvili (1953)]. However, it is in general easier to construct the solution of multiple embedded cracks in an infinite body using the solution for a single embedded crack. Solutions for arbitrarily located cracks can be obtained using this approach. Even when the analytical solution is available, such as for the multiple collinear cracks in an infinite domain, it can be more accurate and efficient to build the multiple crack solutions from that for a single crack.

In the context of the finite element alternating method, it seems natural to use the Schwartz-Neumann alternating method to obtain the analytical solution iteratively. This approach has been used by many authors, such as O'Donoghue, Nishioka and Atluri (1985), Chen and Chang (1990), etc. Using the alternating method and the solution for the single crack in the infinite domain, residuals induced by closing the other cracks are erased by reversing them and applying them as loads on the crack surfaces.

However, the solution can be obtained using a non-iterative approach in a simpler and more efficient fashion. Consider the superposition of n solutions of single cracks in the infinite body. Each of these n solutions involves only one crack. Denote the k'th solution as S_k, where the crack is at the same location as that of the k'th crack of the original multiple-crack problem. The crack surface traction $\boldsymbol{T}_k$ for the problem S_k ($k = 1, 2, \ldots, n$) is to be determined(see Fig. 3).

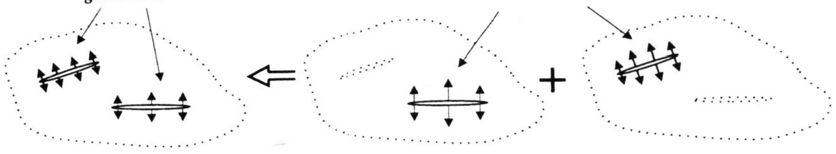

Figure 3: Superpose single crack solutions

The traction at the location of the j'th crack in the the problem S_k can be found for any load $\boldsymbol{T}_k$, i.e.

$$\boldsymbol{t}_{jk} = \boldsymbol{K}_j^{[k]}\boldsymbol{T}_k \qquad j,k = 1,2,\ldots,n. \tag{13}$$

It is noticed that $\boldsymbol{K}_k^{[k]} = \boldsymbol{I}$ ($k = 1, 2, \ldots, n$) are identity operators, because the tractions at the crack surfaces are the same as the applied loads.

The superposition of the n solutions should give back the original problem, i.e. the tractions at the locations of the crack surfaces should be the same as the given crack surface loads. Thus, the linear system to be solved is

$$\sum_{k=1}^{n} \boldsymbol{t}_{jk} = \sum_{k=1}^{n} \boldsymbol{K}_j^{[k]}\boldsymbol{T}_k = \boldsymbol{T}_j^o \qquad j = 1,2,\ldots,n. \tag{14}$$

Denote collectively the undetermined crack surface loads $\boldsymbol{T}_k$ ($k = 1, 2, \ldots, n$) as $\boldsymbol{T}$, and the given loads $\boldsymbol{T}_k^o$ ($k = 1, 2, \ldots, n$) as $\boldsymbol{T}^o$ so that Eq. (14) can be rewritten as $\boldsymbol{KT} = \boldsymbol{T}^o$. Once the linear operator $\boldsymbol{K}$ is evaluated numerically, we can solve the linear system directly instead of using alternating method.

Let the undetermined load $\boldsymbol{T}$ and the given load $\boldsymbol{T}^o$ be approximated by N basis functions $\boldsymbol{B}_j$ ($j = 1, 2, \ldots, N$).

$$\boldsymbol{T} \approx \sum_{j=1}^{N} T_j \boldsymbol{B}_j \quad \text{and} \quad \boldsymbol{T}^o \approx \sum_{i=1}^{N} T_i^o \boldsymbol{B}_i \tag{15}$$

We apply load $\boldsymbol{B}_i$ on the cracks. Close all other cracks except the *single* crack on which $\boldsymbol{B}_i$ has non-zero values. Find the tractions at the locations of the n cracks of the original problem (see[2] Fig. 4), using the analytical solution for a single crack in an infinite domain. The tractions $\boldsymbol{t}_j = \boldsymbol{KB}_j$ ($j = 1, 2, \ldots, N$) are also approximated by these basis functions.

$$\boldsymbol{t}_j \approx \sum_{i=1}^{N} t_{ji} \boldsymbol{B}_i \qquad j = 1,2,\ldots,N \tag{16}$$

Once the magnitude t_{ij} are evaluated, the linear system Eq. (14) leads to the following linear system of equations for the magnitudes T_j ($j = 1, 2, \ldots, n$).

$$\sum_{j=1}^{N} t_{ji} T_j = T_i^o \qquad i = 1,2,\ldots,N \tag{17}$$

The alternating method essentially solves the same approximated linear system with a fixed point iteration scheme.

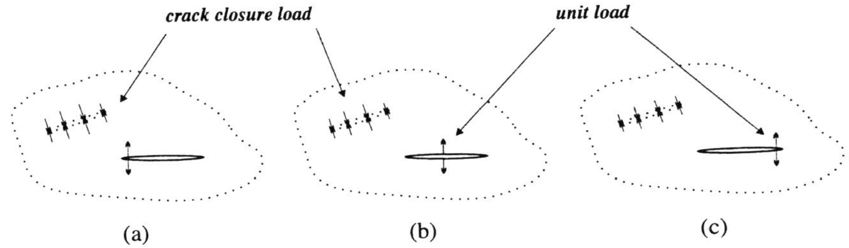

Figure 4: Evaluate the traction at the locations of cracks for each load in terms of unit basis functions

The coefficients of the linear system remain the same in the analysis of the same cracks under different loadings, because they depend only on the crack configuration and the basis functions. Thus, the linear system can be solved for different loads without recomputing the coefficients of the system. This feature is particularly useful when the constructed multiple crack solution is used in the finite element alternating method, where it is necessary to evaluate the solution for the same cracks under different loadings during the alternating procedure.

[2]Point loads (Delta functions) are used to illustrate the base functions in the figure. Only some of the loading cases are illustrated in the figure.

2.2 Elastoplastic analysis of multiple cracks in a finite body

Elastoplastic analysis can be carried out by the Initial Stress Method [Nayak and Zienkiewicz (1972)], which reduces the nonlinear analysis into a series of linear analyses, for which the principle of superposition holds. Thus, the finite element alternating method can be used to perform these linear analyses.

The initial stress method can be described as the following. Assuming no body forces, the virtual work principle is

$$\int_{\Omega} \boldsymbol{\sigma} : \delta \nabla \boldsymbol{u} \, d\Omega = \int_{\Gamma_t} \boldsymbol{t}^o \cdot \delta \boldsymbol{u} \, d\Gamma \tag{18}$$

where $\boldsymbol{\sigma}$ is the elastoplastic stress, $\boldsymbol{t}^o$ is the prescribed surface traction, Ω is the domain of the body, and Γ_t is the boundary with prescribed tractions.

First, the elastic prediction is found by assuming that the deformation is entirely elastic. The elastoplastic stress $\boldsymbol{\sigma}^p$ within the body is found by using the displacements obtained in the linear elastic analysis. But $\boldsymbol{\sigma}^p$ may not satisfy the equilibrium equations. Let $\boldsymbol{\sigma}^c$ be the undetermined correction for the stress, i.e. $\boldsymbol{\sigma} = \boldsymbol{\sigma}^p + \boldsymbol{\sigma}^c$. Substituting this into Eq. (18), we find that $\boldsymbol{\sigma}^c$ satisfies

$$\int_{\Omega} \boldsymbol{\sigma}^c : \delta \nabla \boldsymbol{u} \, d\Omega = \int_{\Gamma_t} \boldsymbol{t}^o \cdot \delta \boldsymbol{u} \, d\Gamma - \int_{\Omega} \boldsymbol{\sigma}^p : \delta \nabla \boldsymbol{u} \, d\Omega \tag{19}$$

The right hand side of the equation can be viewed as the virtual work done by the unbalanced force. The left hand side of Eq. (19) is the virtual work done by the correction stress. The elastic estimate of the correction stress $\boldsymbol{\sigma}^c$ can be solved by the alternating method for the linear elastic analysis. The new elastic prediction for the displacements is the sum of the old one and the correction term. This correction procedure is repeated until the unbalanced force becomes negligible.

Since the alternating method corrects the residuals at the boundary and the initial stress method corrects the error at in the plastic zone, we can combine them to form a single loop. The elastoplastic analysis of the cracked structures, using initial stress method and finite element alternating method, can be outlined as the following.

1. Solve the crack closure traction $T(1)$ using finite element method, assuming that the material is elastic. Denote the solution of displacement gradients as $\boldsymbol{F}_{(1)}^{FEM}$.
2. *Reverse* the traction obtained in the previous step and apply it as the load on the crack surfaces. Denote the analytical solution of displacement gradient as $\boldsymbol{F}_{(1)}^{ANA}$.
3. Compute the elastoplastic stress $\boldsymbol{\sigma}_{(1)}^p$ due to the displacement gradient $\boldsymbol{F}_{(1)}^{FEM} + \boldsymbol{F}_{(1)}^{ANA}$.
4. Compute the boundary load $\boldsymbol{u}^{(1)}$, $\boldsymbol{t}^{(1)}$ and the distributed load $\boldsymbol{f}^{(1)}$ due to the incorrectness of the stress $\boldsymbol{\sigma}_{(1)}^p$.
5. Apply the load $\boldsymbol{u}^{(1)}$, $\boldsymbol{t}^{(1)}$ and $f^{(1)}$ on the uncracked body, assuming that the material is elastic. Repeat the procedure of finding residuals until the process converges.

The above procedure can be applied using the deformation theory of plasticity, which is valid for a cracked structure undergoing monotonic proportional loading. For a plastic material undergoing loading/unloading, it is only valid for the first loading step using a J_2 flow theory of plasticity, i.e. loading the unstressed body to the given level of boundary load. But similar procedures can be applied to any loading/unloading precess: the deformation gradient in the above procedure should be replaced by its increment for the loading step. The stress is determined from the previous stress state and the increment of displacement gradient. In the analysis of crack growth, the newly created crack surfaces in general experience plastic deformation. To remove the crack closure stress, a step of evaluating elastoplastic stress at the crack surface must be added before the evaluation of the analytical solution for cracks in the infinite domain.

2.3 Elastic-plastic crack growth analyses

The alternating method can be used in a crack growth analysis. As shown in Fig. 5, the crack closure traction ahead of the crack tips can be removed using the analytical solution for a crack that has the same length as the new crack.

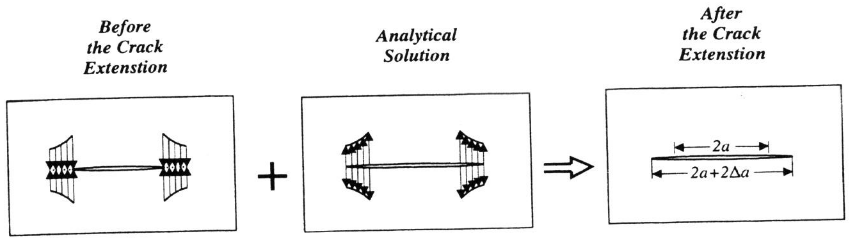

Figure 5: Remove the tractions at the newly created crack surface

Denote the original crack length as $2a$. The amount of crack extension is $2\Delta a$. The crack closure traction $\boldsymbol{T}$ ahead of the crack tip for the problem with crack length $2a$ is evaluated from the solution S^o obtained for the crack before the crack extension. In order to create new additional traction-free crack surfaces of additional length $2\Delta a$, we reverse the traction $\boldsymbol{T}$ and apply it on the surface of the crack of length $2a + 2\Delta a$. Boundary residuals can be computed from this analytical solution S_o^{ANA}. Then, the usual finite element alternating method can be used to remove these boundary residuals for the crack of length $2a + 2\Delta a$. We superpose all the solutions, including S^o and S_o^{ANA}, to obtain the solution for the extended crack.

It is noted that crack closure traction is distributed only at the recently generated crack surface. This traction is actually singular around the original crack tip. This type of localized and non-smooth crack surface traction can be captured by localized special basis functions. It can not be captured correctly by smooth basis functions, such as polynomials.

The new crack surface was in the plastic zone. Therefore, the elastoplastic crack closure traction must be evaluated before the analytical solution is applied. The alternating procedure for an elastoplastic crack extension step is thus outlined as the following.

1. Compute the *elastoplastic* crack closure traction $\boldsymbol{T}^o$ for the solution S^o obtained for the crack of length $2a$.

2. *Reverse* the traction obtained in the previous step and apply it as the load on the crack surfaces. Denote the analytical solution of displacement gradient as F_o^{ANA}.
3. Compute the elastoplastic stress $\sigma_{(1)}^p$ due to the increment of displacement gradient F_o^{ANA}.
4. Compute the boundary load $u^{(1)}$, $t^{(1)}$ and the distributed load $f^{(1)}$ due to the incorrectness of the stress $\sigma_{(1)}^p$.
5. Apply the load $u^{(1)}$, $t^{(1)}$ and $f^{(1)}$ on the uncracked body, assuming that the material is elastic. Repeat the procedure of finding residuals until the process converges for the crack of length $2a + 2\Delta a$.

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