CIRCULAR RIGID PUNCH WITH ONE SMOOTH AND ANOTHER SHARP ENDS ON AN EDGE CRACKED MATRIX ACTED BY CONCENTRATED FORCES

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ABSTRACT

A vertical circular rigid punch with friction on a semi-infinite plane acted by arbitrarily located concentrated forces with one end smoothly contacting the matrix and another end of sharp corner to initiate an oblique edge crack in the matrix is studied in an analytical way in the present paper. The edge cracked semi-infinite matrix is mapped into a unit circle by a rational mapping function. The problem is classified into the Riemann-Hilbert type, and solved by dividing the whole problem into two parts, one is the edge cracked semi-infinite plane acted by the concentrated forces, the other is the problem solved by substituting the solution of the first part into the R-H equation of the whole problem. The position of the end with corner is fixed while the position of the sliding end is changed with other conditions. After the position of the sliding end is determined by the condition that the stresses at this end are not singular, the stress intensity factors of the crack and resultant moment on the contact region are calculated with different positions of the concentrated forces in the matrix.

KEYWORDS

Circular rigid punch, incomplete contact, oblique edge crack, Riemann-Hilbert problem, rational mapping function, complex function method, Plemelj function

INTRODUCTION

Punch problem is an important branch in contact mechanics. Many classical problems have been solved (Muskhelishvili, 1963, England, 1971, Gladwell, 1980). It is noticed that crack problem is often connected with punch problem owing to the existence of stress concentrations on the contact region and defects in the matrix (Hills and Nowell, 1994). There generally exist two types of contact for a punch with curve shape, one is called complete contact, i.e. the length of the contact region is given, and there exist two corners for the punch coming into the matrix; the other is called incomplete contact, i.e. one or both ends of the punch slide on the matrix, and the length of the contact region is undetermined. When a punch on a matrix with edge defect is referred, some inherent difficulties will appear in computation. Though the problem is possible to be dealt with by some numerical methods such as FEM and BEM, the contact interface and semi-infinite property of the matrix as well as the singularity of edge crack will result in much inconvenience in computation by FEM, and owing to the lack of the fundamental solution of punch problem, it is also not convenient to analyze the problem by BEM. Therefore it is important to present an analytical or semi-analytical method to solve the punch problem with edge crack, and it is also important to derive the fundamental solution of punch problem so that the boundary element method or boundary integral method can be used effectively to analyze the punch on a matrix with edge and inner defects.

A circular rigid punch in complete contact with an edge cracked half plane has been studied (Hasebe and Qian, 1995), and the fundamental solution of the problem has also been derived (Qian and Hasebe, 1996). Since there exists incomplete contact for circular punch problem, the problem of circular rigid punch with one end in smooth contact and the other end with a sharp corner on an edge cracked half plane acted by concentrated forces at an arbitrary point in the matrix is considered in the present paper. Coulomb's frictional force is assumed to exist on the interface which is balanced by a horizontal force on the punch. The vertical load on the punch is usually eccentric to keep the punch vertical, and the eccentric distance of the vertical load on the punch is determined by the resultant moment on the contact region. The position of the end with sharp corner is fixed while the sliding end is changed with the environments such as the magnitude, direction and location of the concentrated forces in the matrix, the frictional coefficient on the contact region, the length, inclined angle and position of the edge crack, the material properties of the matrix and radius of curvature of the circular punch as well as the magnitude of the vertical load on the punch. To solve the problem in an analytical way, the semi-infinite matrix with the edge oblique crack is mapped into a unit circle by a rational mapping function so that the Riemann-Hilbert equation of the problem can be solved explicitly by making use of the solution of the semi-infinite plane with the edge oblique crack acted by the concentrated forces. With the explicit expressions of complex stress functions of the problem, the position of the sliding end of the punch can be decided by satisfying the finite stress condition at the sliding end, and the stress intensity factors of the crack and resultant moment on the contact region can be calculated with different positions of the concentrated forces.

PRESENTATION OF THE PROBLEM

As shown in Fig.1, a circular rigid punch is supposed to be vertical on a semi-infinite elastic matrix with an oblique edge crack. Coulombs's frictional force is assumed to exist on the contact region. The punch is acted by a vertical load P, and the frictional force on the contact region is equilibrium by a horizontal load μP on the punch, where μ denotes the Coulombs's frictional coefficient. To keep the punch not to incline, the location of the vertical load on the punch is usually eccentric from the origin of the x-y coordinates to balance the moment produced by the stresses on the contact region about the origin. The right end with sharp corner has a distance of $a/2$ from the origin, while the left sliding end is undetermined. Besides the loads on the punch, it is assumed that a pair of concentrated forces acts at point z_0 in the matrix. To solve the problem in an analytical way, the semi-infinite matrix with the edge oblique crack is mapped into a unit circle by the following rational mapping function (Hasebe and Inohara, 1980):

$$z = \omega(\zeta) = \frac{E_0}{1-\zeta} + \sum_{k=1}^{N} \frac{E_k}{\zeta_k - \zeta} + E_c \tag{1}$$

where E_0, E_k and ζ_k are known coefficients, and E_c is decided by the distance from the edge crack to the punch.

The loading and displacement conditions of the problem can be described as

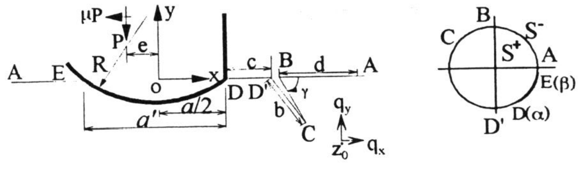

Fig.1 The punch and the unit circle

$$p_x = p_y = 0 \qquad \text{on} \quad L = L_1 + L_2 \tag{2a}$$

$$p_x = \mu p_y\,, \quad \int p_y ds = P \qquad \text{on} \quad M \tag{2b}$$

$$V = x^2/2R \qquad \text{on} \quad M \tag{2c}$$

$$Q(x,y) = (q_x + iq_y)\delta(z, z_0) \qquad \text{in the half plane} \tag{2d}$$

where $L_1 = ABCD'D$, $L_2 = EA$, $M = DE$ in Fig.1; p_x and p_y represent the components of traction in x and y directions on the surface of the half plane; $Q(x,y)$ represents the concentrated forces on the boundary or in the body of the half plane; $\delta(z,z_0) = 1$ when $z = z_0$ and 0 when $z \neq z_0$. V is the displacement of the punch, and R is the radius of curvature.

METHOD OF ANALYSIS

According to the above loading and displacement conditions, the problem can be transformed into the Riemann-Hilbert problem as follows (Hasebe et al., 1989):

$$\phi^+(\sigma) - \phi^-(\sigma) = f_L \qquad \text{on} \quad L = L_1 + L_2 \tag{3a}$$

$$\phi^+(\sigma) + \frac{1}{g}\phi^-(\sigma) = f_M \qquad \text{on} \quad M \tag{3b}$$

where

$$f_L = i\int (p_x + ip_y) ds \tag{3c}$$

$$f_M = \frac{4(1-i\mu)GiV + (1+i\mu)(1+\kappa)R(\sigma)}{(\kappa+1) - i\mu(\kappa-1)} \tag{3d}$$

$$R(\zeta) = \phi(\zeta) + \frac{1-i\mu}{1+i\mu}\overline{\phi}\left(\frac{1}{\zeta}\right) \tag{3e}$$

$$\frac{1}{g} = \frac{(\kappa+1) + i\mu(\kappa-1)}{(\kappa+1) - i\mu(\kappa-1)} \tag{3f}$$

$R(\zeta)$ is a function to be determined so as to satisfy (3e), G is the shear modulus of the half plane, $\kappa = 3 - 4\nu$ for plane strain state and $(3-\nu)/(1+\nu)$ for plane stress state, and ν represents the Poisson's ratio of the matrix.

The complex stress functions to be obtained are divided into two parts:

$$\phi(\zeta) = \phi_1(\zeta) + \phi_2(\zeta) \tag{4a}$$

$$\psi(\zeta) = \psi_1(\zeta) + \psi_2(\zeta) \tag{4b}$$

where $\phi_1(\zeta)$ and $\psi_1(\zeta)$ are the complex stress functions of the half plane with the edge oblique crack acted by the concentrated forces (Hasebe et al., 1996a), and $\phi_2(\zeta)$ and $\psi_2(\zeta)$ are the holomorphic parts of $\phi(\zeta)$ and $\psi(\zeta)$, which will be derived by the following procedures. Substituting (4a) into (3a, b), and introducing the Plemelj function $\chi(\zeta)$, the general solution of $\phi_2(\zeta)$ can be derived as (Hasebe et al. 1989)

$$\phi_2(\zeta) = H_1(\zeta) + H_2(\zeta) + H_3(\zeta) + \frac{1+i\mu}{2} J(\zeta) + Q(\zeta)\chi(\zeta) \tag{5a}$$

where

$$H_1(\zeta) = P(1-i\mu)\frac{\chi(\zeta)}{2\pi i}\int_{\beta}^{1} \frac{d\sigma}{\chi(\sigma)(\sigma-\zeta)} \tag{5b}$$

$$H_2(\zeta) = \frac{Gi(1-i\mu)}{R(\kappa+1)}\frac{\chi(\zeta)}{2\pi i}\int_{M} \frac{\{\omega(\sigma)\}^2}{\chi(\sigma)(\sigma-\zeta)}d\sigma \tag{5c}$$

$$H_3(\zeta) = \frac{1-i\mu}{4\pi}\left[(\bar{q}-\kappa q)F_1 + (\kappa\bar{q}-q)F_2 + qG_1 + \bar{q}G_2 + 2\pi G_3\right] \tag{5d}$$

$$Q(\zeta)\chi(\zeta) = -\sum_{k=1}^{N} \frac{\chi(\zeta)\overline{A_k}B_k}{\chi(\zeta_k)(\zeta_k-\zeta)} \tag{5e}$$

$$J(\zeta) = -\sum_{k=1}^{N}\left[1-\frac{\chi(\zeta)}{\chi(\zeta_k)}\right]\frac{\overline{A_k}B_k}{\zeta_k-\zeta} + \frac{1-i\mu}{1+i\mu}\sum_{k=1}^{N}\left[1-\frac{\chi(\zeta)}{\chi(\zeta_k')}\right]\frac{A_k\overline{B_k}\zeta_k'^2}{\zeta_k'-\zeta} + \text{const} \tag{5f}$$

and $\chi(\zeta) = (\zeta-\alpha)^m(\zeta-\beta)^{1-m}$, $m = 0.5 - i\ln g/2\pi$, $q = -(q_x+iq_y)/(1+\kappa)$.

$H_1(\zeta)$ is related to the vertical load on the punch. Though it is in integral form, its first derivative can be expressed in the form without integration (Hasebe et al., 1989); $H_2(\zeta)$ is related to the displacement on the contact region induced by the radius of curvature of the punch. Owing to the use of the rational mapping function (1), the integration of $H_2(\zeta)$ can be carried out explicitly (Hasebe and Qian, 1995); A_k and $\overline{A_k}$ are determined by solving 2N linear simultaneous equations for real and imaginary parts of $A_k = \phi_2'(\zeta_k')$ $(k = 1,2,\dots,N)$. $H_3(\zeta)$ is related to the concentrated forces in the half plane which apply at $\zeta = \zeta_0$ and $\zeta = 1$ by the following expressions:

$$F_1 = \log(\sigma - 1/\bar{\zeta}_0) - \log(\sigma-1) + \chi(\zeta)\int_{1/\bar{\zeta}_0}^{1}\frac{d\sigma}{\chi(\sigma)(\sigma-\zeta)} \tag{6a}$$

$$F_2 = \log(\sigma - \zeta_0) - \log(\sigma-1) + \chi(\zeta)\int_{\zeta_0}^{1}\frac{d\sigma}{\chi(\sigma)(\sigma-\zeta)} \tag{6b}$$

$$G_1 = \frac{\overline{\omega(\zeta_0)} - \overline{\omega(1/\bar{\zeta}_0)}}{\omega'(\zeta_0)}\left[1-\frac{\chi(\zeta)}{\chi(\zeta_0)}\right]\frac{1}{\zeta-\zeta_0} - \sum_{k=1}^{N}\left(\frac{1}{\zeta_k'-\zeta_0} - \frac{1}{\zeta_k'-1}\right)\left[1-\frac{\chi(\zeta)}{\chi(\zeta_k')}\right]\frac{\overline{B_k}\zeta_k'^2}{\zeta-\zeta_k'} \tag{6c}$$

$$G_2 = \frac{\omega(\zeta_0) - \omega(1/\bar{\zeta}_0)}{\overline{\omega'(\zeta_0)}}\left[1-\frac{\chi(\zeta)}{\chi(1/\bar{\zeta}_0)}\right]\frac{(1/\bar{\zeta}_0)^2}{\zeta-1/\bar{\zeta}_0} - \sum_{k=1}^{N}\left(\frac{1}{\overline{\zeta_k'}-\bar{\zeta}_0} - \frac{1}{\overline{\zeta_k'}-1}\right)\left[1-\frac{\chi(\zeta)}{\chi(\zeta_k)}\right]\frac{B_k}{\zeta-\zeta_k} \tag{6d}$$

$$G_3 = -\sum_{k=1}^{N}\frac{A_{qk}\overline{B_k}\zeta_k'^2}{\zeta-\zeta_k'}\left[1-\frac{\chi(\zeta)}{\chi(\zeta_k')}\right] - \sum_{k=1}^{N}\frac{\overline{A_{qk}}B_k}{\zeta-\zeta_k}\left[1-\frac{\chi(\zeta)}{\chi(\zeta_k)}\right] \tag{6e}$$

where A_{qk} is decided by the fundamental solution of the half plane with the oblique edge crack (Hasebe et. al, 1996a), $B_k = E_k/\overline{\omega'(\zeta_k')}$ and $\zeta_k' = 1/\overline{\zeta_k}$.

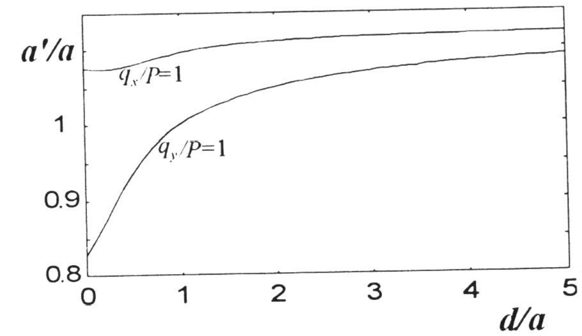

Fig.2 The length of the contact region

LENGTH OF THE CONTACT REGION

The stress components on the boundary can be expressed as (Hasebe and Qian, 1996b)

$$\sigma_r + i\tau_{r\theta} = \frac{1}{\omega'(\sigma)}\left\{\phi'^{+}(\sigma) - \phi'^{-}(\sigma)\right\} \tag{7}$$

Substituting (4a) into (7), it is obtained that

$$\sigma_r + i\tau_{r\theta} = \frac{1}{\omega'(\sigma)}\left\{\frac{i(1-i\mu)P}{2\pi}\frac{(1-\alpha)(1-\beta)}{\chi(1)(1-\sigma)(\sigma-\alpha)(\sigma-\beta)} + \frac{e_1(\sigma)}{(\sigma-\alpha)(\sigma-\beta)} + \frac{mf_1(\sigma)}{\sigma-\alpha} + \frac{(1-m)f_1(\sigma)}{\sigma-\beta} + g_1(\sigma)\right\}\left[\chi^{+}(\sigma) - \chi^{-}(\sigma)\right] \tag{8}$$

where $e_1(\sigma)$, $f_1(\sigma)$ and $g_1(\sigma)$ are listed in Appendix I.

It is found from (8) that the stress components become infinite when σ tends to α or β. In order to satisfy the finite stress condition at the sliding end $(\sigma = \beta)$, the coefficient of $1\ (\sigma-\beta)$ in (8) must be vanished. Therefore the restraining condition is established as follows:

$$\frac{i(1-i\mu)(1-\alpha)}{2\pi\chi(1)(\beta-\alpha)}P + \frac{e_1(\beta)}{\beta-\alpha} + (1-m)f_1(\beta) = 0 \tag{9}$$

If the values of $Ga^2/(PR)$ and α are given, β satisfying (9) can be determined by iterative calculation. After β is determined, a' can be simply decided.

Ga^2/PR is a parameter which gives the relation among shear modulus G, punch width a (see Fig.1), radius of the circular punch R and vertical force P, here R is supposed to be sufficiently larger than the length of the contact region.

In the following calculation, the concentrated forces are typically assumed to be acted on the surface of the half plane in x and y directions, $\kappa = 2$, $\mu = 0.5$, $b/a = 0.5$, $c/a = 0$, $\gamma = 60^0$ and $Ga^2/PR = 1$ are selected.

Fig.2 shows a' with different d, where d denotes the distance from the crack to the concentrated force on the surface of the half plane. It is shown that a' increases and tends to a stable value with the increase of d.

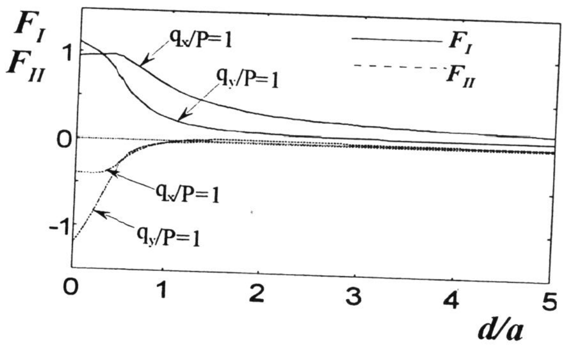

Fig.3 The stress intensity factors of the crack

STRESS INTENSITY FACTORS OF THE CRACK

The stress intensity factors of the crack are calculated by

$$K_I - iK_{II} = 2\sqrt{\pi}\exp\left(-\frac{\delta}{2}i\right)\frac{\phi'(\sigma)}{\sqrt{\omega''(\sigma)}} \tag{10}$$

where $\sigma = (1-2s+i)/(1-2s-i)$ is ζ on the unit circle corresponding to the tip C of the crack, $s = \gamma/180$ and $\delta = -i\gamma\pi/180$.

The non-dimensional stress intensity factors of the crack are defined as

$$F_I + iF_{II} = \frac{(K_I + iK_{II})}{P\sqrt{\pi}}\sqrt{a} \tag{11}$$

Fig.3 shows F_I and F_{II} with different d. F_I decreases with the increase of d, and F_{II} is on the contrary. Both F_I and F_{II} tend to stable values which are due to the punch without concentrated force with the increase of d.

RESULTANT MOMENT ON THE CONTACT REGION

The problem is analyzed on the condition that the punch does not incline. Thus the resultant moment R_m about the origin of the coordinates can be used to determine the position of the vertical load P on the punch which applies at the distance e away from the y-axis satisfying $Pe = R_m$ (see Fig.1).

The non-dimensional resultant moment is defined as

$$M_r = \frac{R_m}{aP} \tag{12}$$

where R_m is calculated by

$$R_m = -\mathrm{Re}\left[\int_\alpha^\beta \omega(\sigma)\overline{\phi'}\left(\frac{1}{\sigma}\right)\frac{d\sigma}{\sigma^2} + \int_\alpha^\beta \overline{\omega}\left(\frac{1}{\sigma}\right)\phi'(\sigma)d\sigma\right] \tag{13}$$

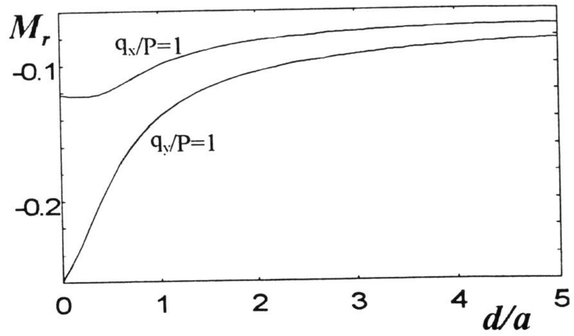

Fig.4 The resultant moment on the contact region

Fig.4 shows M_r with different d. M_r increases with the increase of d and tends to a stable value which corresponds to the moment of the punch without concentrated forces ($q_x = q_y = 0$) in the matrix. The negative value of M_r represents that the moment on the contact region is in clockwise direction with the load P on the right side of y-axis.

CONCLUSIONS

The fundamental solution of circular rigid punch in incomplete contact with a semi-infinite edge cracked matrix acted by a pair of concentrated forces was derived. By simply changing the coefficients of the mapping function (1), the problem with other types of edge defect, such as an edge triangular notch initiating a crack at the tip of the notch (Hasebe and Iida, 1979), can be studied in the same manner. If the radius of the circular punch tend to be infinite value, the solution of the corresponding flat-ended punch can be formed. Since the interface conditions on the contact region, boundary conditions on the surface of the matrix and edge crack are satisfied completely in the process of derivation, the present solution can be forwardly applied to calculate the punch problems with internal defects by boundary integral method or boundary element method without loss of the analytical properties.

APPENDIX I

$e_1(\sigma)$, $f_1(\sigma)$ and $g_1(\sigma)$ in equation (8)

$$e_1(\sigma) = \frac{1-i\mu}{4\pi}(\bar{q} - \kappa q)\left[\frac{(1/\overline{\zeta_0} - \alpha)(1/\overline{\zeta_0} - \beta)}{\chi(1/\overline{\zeta_0})(1/\overline{\zeta_0} - \sigma)} - \frac{(1-\alpha)(1-\beta)}{\chi(1)(1-\sigma)}\right]$$
$$+ \frac{1-i\mu}{4\pi}(\kappa\bar{q} - q)\left[\frac{(\zeta_0 - \alpha)(\zeta_0 - \beta)}{\chi(\zeta_0)(\zeta_0 - \sigma)} - \frac{(1-\alpha)(1-\beta)}{\chi(1)(1-\sigma)}\right]$$

$$f_1(\sigma) = f(\sigma) - \frac{1-i\mu}{4\pi} q \frac{\omega(\zeta_0) - \omega(1/\overline{\zeta_0})}{\omega'(\zeta_0)} \frac{1}{\chi(\zeta_0)(\sigma - \zeta_0)} - \frac{1-i\mu}{4\pi} \bar{q} \frac{\overline{\omega(\zeta_0) - \omega(1/\overline{\zeta_0})}}{\overline{\omega'(\zeta_0)}} \frac{(1/\overline{\zeta_0})^2}{\chi(\zeta_0)(\sigma - 1/\zeta_0)}$$
$$+ \frac{1-i\mu}{4\pi} q \sum_{k=1}^{N} \left(\frac{1}{\zeta_k' - \zeta_0} - \frac{1}{\zeta_k' - 1} \right) \frac{\overline{B_k} \zeta_k'^2}{\chi(\zeta_k')(\sigma - \zeta_k')} + \frac{1-i\mu}{4\pi} \bar{q} \sum_{k=1}^{N} \left(\frac{1}{\overline{\zeta_k'} - \overline{\zeta_0}} - \frac{1}{\overline{\zeta'}_k - 1} \right) \frac{B_k}{\chi(\zeta_k)(\sigma - \zeta_k)}$$
$$+ \frac{1-i\mu}{2} \sum_{k=1}^{N} \frac{A_{qk} \overline{B_k} \zeta_k'^2}{\sigma - \zeta_k'} \frac{1}{\chi(\zeta_k')} + \frac{1-i\mu}{2} \sum_{k=1}^{N} \frac{\overline{A_{qk}} B_k}{\sigma - \zeta_k} \frac{1}{\chi(\zeta_k)}$$

$$g_1(\sigma) = g(\sigma) + \frac{1-i\mu}{4\pi} q \frac{\overline{\omega(\zeta_0)} - \overline{\omega(1/\overline{\zeta_0})}}{\omega'(\zeta_0)} \frac{1}{\chi(\zeta_0)(\sigma - \zeta_0)^2} + \frac{1-i\mu}{4\pi} \bar{q} \frac{\omega(\zeta_0) - \omega(1/\overline{\zeta_0})}{\overline{\omega'(\zeta_0)}} \frac{(1/\overline{\zeta_0})^2}{\chi(\zeta_0)(\sigma - 1/\zeta_0)^2}$$
$$- \frac{1-i\mu}{4\pi} q \sum_{k=1}^{N} \left(\frac{1}{\zeta_k' - \zeta_0} - \frac{1}{\zeta_k' - 1} \right) \frac{\overline{B_k} \zeta_k'^2}{\chi(\zeta_k')(\sigma - \zeta_k')^2} - \frac{1-i\mu}{4\pi} \bar{q} \sum_{k=1}^{N} \left(\frac{1}{\overline{\zeta_k'} - \overline{\zeta_0}} - \frac{1}{\overline{\zeta'}_k - 1} \right) \frac{B_k}{\chi(\zeta_k)(\sigma - \zeta_k)^2}$$
$$- \frac{1-i\mu}{2} \sum_{k=1}^{N} \frac{A_{qk} \overline{B_k} \zeta_k'^2}{(\sigma - \zeta_k')^2} \frac{1}{\chi(\zeta_k')} - \frac{1-i\mu}{2} \sum_{k=1}^{N} \frac{\overline{A_{qk}} B_k}{(\sigma - \zeta_k)^2} \frac{1}{\chi(\zeta_k)}$$

where $f(\sigma)$ and $g(\sigma)$ can be found from previous paper(Hasebe and Qian, 1996b)

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