BRANCHING PROBLEM OF A CRACK AND A DEBONDING AT THE END OF A CLAMPED EDGE OF THIN PLATE

NORIO HASEBE[1] MASAHIRO MIWA[2] and MASAKI NAKASHIMA[3]

[1] Dept. of Civ. Engrg., Nagoya Inst. of Tech.
Gokiso-cho, Showa-ku, Nagoya, 466, Japan

[2] Tech. Research and Development Div., Central Japan Railway Co.
Yaesu 1-6-6, Chuo-ku, Tokyo 103, Japan

[3] Obayashi Co., Kandatsukasa-cho 2-3, Chiyoda-ku Tokyo, 101, Japan

ABSTRACT

In the case of out-of-plane loading on a thin plate under displacement constraint at a part of the boundary, the branching problem of a crack, generated at the end of a displacement constraint, and that of a debonding generated along the part of the displacement constraint are considered. Using a rational-mapping and complex-stress functions, a displacement constraint is considered in form of a clamped edge. The problem is solved for uniform bending and torsion. The stress intensity factor of a micro-crack generated from the end of the clamped edge in an arbitrary direction, and stress intensity of debonding along the clamped edge are calculated. Then the strain energy release rates of a crack and a debonding generation are obtained, and the direction of crack initiation, and whether a crack or a debonding is produced, are investigated.

KEY WORDS

Debonding, crack initiation, thin plate, clamped edge, branch, strain energy release rate, fracture criterion, stress intensity factor

INTRODUCTION

One of factors resulting in material fracture is stress concentration at the end of a displacement constraint. For example, in steel structure such as bridges etc., plates under out-of-plane deformation constraint with a stiffener, rib, flange or so, have been observed to generate fatigue cracks at these ends of displacement constraints, resulting from vibration and rolling (Fisher 1984). In the present study, fracture resulting from the displacement constraint at a part of the plate edge is analyzed.

As a structural model, a thin plate occupying a half plane with a clamped part of the boundary is under out-of-plane deformation (see Fig. 1). The deflection angles of the clamped part are zero and the part is called the clamped edge in the present paper. This problem can be solved as a mixed-boundary-value problem consisting of the displacement boundary where two

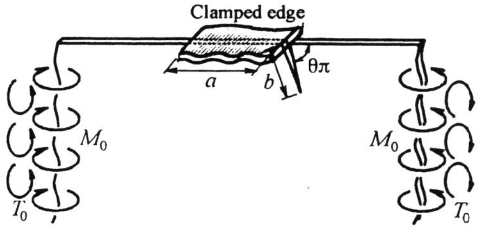

Fig. 1 Half-Plane with a Crack at the End of a Clamped Edge.

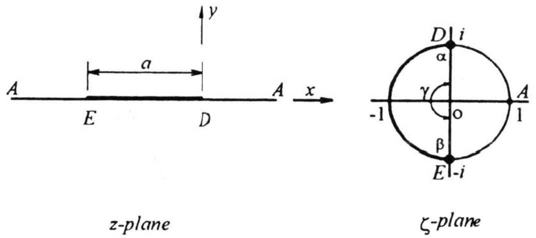

Fig. 2 Half-Plane and Unit Circle before Crack Initiation

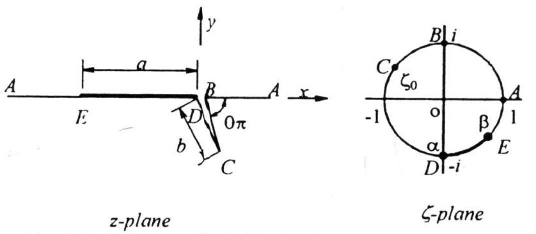

Fig. 3 Half-Plane and Unit Circle after Crack Initiation

components of displacement are given at a part of the boundary as a clamped edge, and the external force boundary where two components of external force are given at the remaining boundary. The clamped edge is not only the typical condition of plate supporting, but also appears in those cases of a part of a plate on which a stiffener is welded or adhered, or an intersecting part between the plate and another plate. Then the deflection angles of x and y directions of the part are assumed being zero.

Two failure modes at the end of the clamped edge are expected. One is initiation and propagation of a crack into the plate and another is initiation of a debonding along the clamped edge. The latter case is called "debonding" and is distinguished from a crack of the former case in the present paper. It is discussed whether a crack or a debonding will occur. If a debonding initiates first, it is also discussed whether or not a crack can branch into the plate during the extension of debonding. In these cases, the criterion of the initiation of the crack and the debonding is the strain energy release rate. The crack is supposed to generate or branch in the direction with the maximum strain energy release rate.

A rational mapping function and complex stress functions are used for the analysis. Using the general solution, the stress intensity factor at the tip of a crack in an arbitrary direction and stress intensity of debonding are calculated. Then the values of strain energy release rate of a crack and a debonding generation are obtained.

GEOMETRICAL SHAPE AND MAPPING FUNCTION

Fig. 2 shows a half plane with a clamped edge of length a at a part of the straight boundary on the x-axis before crack initiation. Fig. 3 indicates the case of crack initiation with an angle of $\theta\pi$ to the x-axis at an end of the clamped edge with the length a.

A mapping function which maps the semi-infinite region with a crack length b and an angle $\theta\pi$ $(0<\theta<1)$ to the x-axis, into the inside of a unit circle shown in Fig. 3, is formed as the following rational function obtained from the irrational function derived from the transformation formula of Schwarz-Christoffel (Hasebe and Inohara 1980):

$$z=\omega(\zeta)=\mathrm{k}\frac{(1+i\zeta)^{\theta}(1-i\zeta)^{1-\theta}}{1-\zeta}\fallingdotseq\frac{E_0}{1-\zeta}+\sum_{k=1}^{n}\frac{E_k}{\zeta_k-\zeta}+E_c \qquad (1)$$

where k is a constant relating to the length b of the crack, and is given by, $k=b(1-i)i^{-\theta}/\left[2\theta^{\theta}(1-\theta)^{1-\theta}\right]$. In addition, E_0, E_k, E_c, and ζ_k are complex constants, and n is the total number of fractional-expression terms which is $n=24$ in this study. Also, ζ_0 on the unit circle corresponding to the crack tip C is given by $\zeta_0=(1-2\theta+i)/(1-2\theta-i)$. Since the rational mapping function (1) is used, the crack tip C is not a strictly sharp corner, but has very small roundness. The ratio ρ/b of radius ρ of the curvature at the crack tip to the crack length b, has a minimum at $\theta=0.5$, and increases gradually as θ approaches to 0 or 1, but the ratio is very small $\rho/b=10^{-7}\sim10^{-12}$, so that the crack tip is quite sharp. This study treats the problem with a condition of $b/a\ll1$, which means that the length a of the boundary DE is taken larger enough than the crack length b. Then, the accuracy of the stress intensity factor calculated from this mapping function, increases relatively still more. While, length a does not appear explicitly in (1), it shows up as a parameter in the complex stress functions (Hasebe 1984). In addition, the mapping function which maps the shape before crack initiation, corresponds to the special case of $E_k=0$ $(k=1, 2,\cdots, n)$ in (1).

STRAIN ENERGY RELEASE RATES FOR A CRACK AND A DEBONDING

The stress intensity factor at the tip of a crack and the stress intensity of debonding are deduced from complex stress functions, and by further using these values, strain energy release rates for a crack and a debonding are obtained.

When two orthogonal components of deflection angles are given as $\partial w/\partial x=\partial w/\partial y=0$ along DE shown in Fig. 2 and 3, DE is called clamped edge, and junctures of D and E between the clamped edge and the free boundary are called clamped ends. Uniform bending moment M_0 and uniform torsional moment T_0 at infinity are considered as loading (Fig. 1).

Using the mapping function of (1), the complex stress functions $\phi(\zeta)$ for loads M_0 and T_0, respectively, were given by Hasebe (1984). Setting $E_0=-ia$, complex stress functions before crack initiation (Fig. 2) are obtained.

Stress components can be expressed in terms of the first derivative of complex stress function $\phi(\zeta)$ and the first derivative of the Plemelj function in $\phi'(\zeta)$ shows singularity at D and E, the clamped ends, so that the stresses become infinitely large there. Therefore, there is possibility to generate a crack into the plate from clamped ends as the starting points, or to produce a debonding along the interface between the clamped edge and plate.

First, the case of crack initiation is considered. The stress intensity factor $K = k_B + ik_S$ at the crack tip is calculated from the complex stress function (Hasebe and Inohara 1980, Hasebe 1984). Using the stress intensity factor $K(=k_B+ik_S)$, the strain energy release rate G_{crack} is given as follows:

$$G_{crack} = \frac{\pi\kappa}{D(1+\nu)^2} K\overline{K} \tag{2}$$

Next the case of a debonding from clamped end D along clamped edge DE is considered. Corresponding to stress intensity factor K at the crack tip, the stress intensity factor at the tip of a debonding is called stress intensity of debonding in the present paper, and expressed by $\tilde{\alpha}_0$, which is given at $\zeta=\alpha$ (see Appendix).

When both M_0 and T_0 apply, the stress intensity of debonding is obtained as follows:

$$\tilde{\alpha}_0 = \{(1+\nu)T_0 - iM_0\}\frac{(1-\lambda)a^{\lambda}}{\sqrt{2}} \tag{3}$$

Then, using $\overline{1-\lambda} = \lambda$, $a^{\overline{\lambda}} = a^{1-\lambda}$ and (3), the strain energy release rate G_{deb} of generating a debonding is obtained finally as the following strict expression :

$$G_{deb} = \frac{\pi\kappa}{2D(1+\nu)^2}\tilde{\alpha}_0\overline{\tilde{\alpha}_0} = \frac{\kappa a}{16\pi D}\left\{\tau^2 + (\ln\kappa)^2\right\}\left\{\left(\frac{M_0}{1+\nu}\right)^2 + T_0^2\right\} \tag{4}$$

DEBONDING AND CRACK INITIATION

Using the strain energy release rates G_{crack} and G_{deb}, the direction of the crack initiation from the clamped end, and the branching problem of a crack during the extension of a debonding are investigated.

When the crack length b becomes zero, i.e. $b \to 0$, the strain energy release rate is defined as the strain energy release rate for the crack initiation. Strain energy release rate at generating a micro-crack is derived from the stress intensity factor of the micro-crack. However, the stress components near the clamped end after crack initiation have a singularity determined by the angle $(1-\theta)\pi$ between the clamped edge and the crack, but the stress components at the crack tip have the order of the power of -0.5 for the distance from the crack tip. Thus being affected by each different singularity of the clamped end and the crack tip, the stress intensity factor for $b \to 0$ does not converge to a constant value (Hasebe 1984). Therefore, to obtain the strain energy release rate for the crack initiation, the stress intensity factors of two crack lengths are used, i.e. $b/a = 0.001$ and $b/a = 0.0005$ which are small.

Stress intensity factors K are non-dimensionalized in the following equations with suffix (M) and (T) which show cases of bending moment M_0 and torsional moment T_0, respectively:

$$F_B^{(M)} + iF_S^{(M)} = \frac{3+\nu}{1+\nu}\cdot\frac{k_B^{(M)} + ik_S^{(M)}}{M_0\sqrt{a}}; \quad F_B^{(T)} + iF_S^{(T)} = \frac{3+\nu}{1+\nu}\cdot\frac{k_B^{(T)} + ik_S^{(T)}}{T_0\sqrt{a}} \tag{5a, b}$$

The stress intensity factor for both M_0 and T_0 is expressed by $k_B = k_B^{(M)} + k_B^{(T)}$ and $k_S = k_S^{(M)} + k_S^{(T)}$ by superposition. Therefore, using (2) and (5), the strain energy release rate for the crack initiation from the clamped end is obtained as follows:

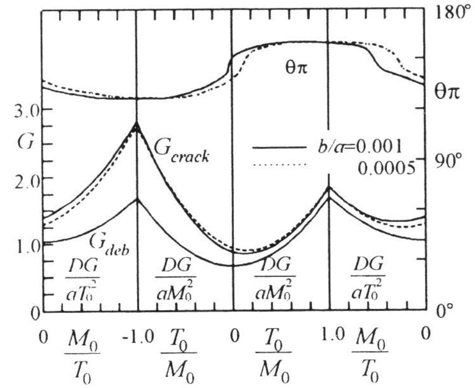

Fig. 4 Nondimensional Strain Energy Release Rates G_{crack} and G_{deb}, and Crack Angle $\theta\pi$ with Clamped Edge for b/a=0.001, 0.0005 and ν=0.25

$$G_{crack} = \frac{\pi\kappa a}{D(3+\nu)^2}\left\{\left(M_0F_B^{(M)} + T_0F_B^{(T)}\right)^2 + \left(M_0F_S^{(M)} + T_0F_S^{(T)}\right)^2\right\} \tag{6}$$

The value of G_{crack}, can be calculated by substituting values F_B and F_S into (6) and changing angle $\theta\pi$ for a crack initiation. The angle $\theta\pi$ to give the maximum value of G_{crack} and maximum value of G_{crack} are shown as well as for G_{deb} expressed by (4) in Fig. 4 for $\nu = 0.25$. Solid and broken lines show cases of $b/a = 0.001$ and $b/a = 0.0005$, respectively. The left-hand side of vertical axis G shows non-dimensional G_{deb} or G_{crack} (for example, $DG_{deb}/[aT_0^2]$, $DG_{crack}/[aT_0^2]$ etc.) for each ratio of loading, and D is the flexural rigidity. The loading ratio M_0/T_0 or T_0/M_0 of the horizontal axis corresponds to the range from 0 to $\pm\infty$ for both M_0 and T_0. Direction angle $\theta\pi$ for the crack initiation, shown in the right-hand side of vertical axis, varies in the range from 127° to 160° by changing M_0 and T_0. There is a little difference of $\theta\pi$ depending on b/a. There is not so large difference of G_{crack} depending on b/a. However, in the range of $|M_0| > |T_0|$, the value of G_{crack} for $b/a = 0.0005$ is larger than that of $b/a = 0.001$, while in the range of $|M_0| < |T_0|$ there is the inverse state.

Now, the conditions of generating a debonding and a crack are investigated. Fracture toughness values of generating a debonding and a crack, expressed by strain energy release rates, are defined as $(G_{deb})_{CR}$ and $(G_{crack})_{CR}$, respectively. Four cases of fracture phenomena produced by relative magnitude between G_{deb} and $(G_{deb})_{CR}$, and by that between G_{crack} and $(G_{crack})_{CR}$ are considered as follows:

(i) In the case of $G_{deb} < (G_{deb})_{CR}$ and $G_{crack} < (G_{crack})_{CR}$, no debonding and no crack generate. When the value of $(G_{deb})_{CR}$ is larger than that of G_{deb} determined by given loading shown in Fig. 4, and the value of $(G_{crack})_{CR}$ is also larger than that of G_{crack}, neither fracture produce.

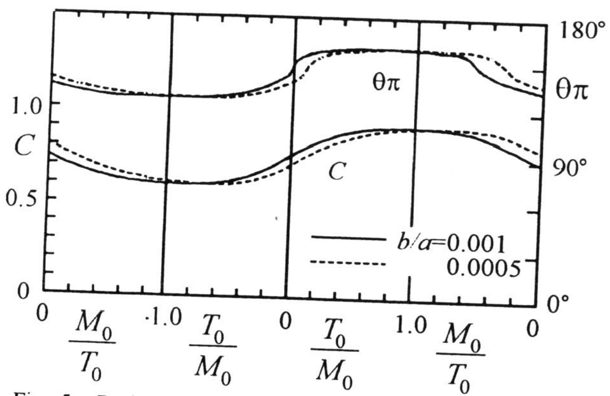

Fig. 5 Ratio C of Eq.(7) and Crack Angle $\theta\pi$ with Clamped Edge for b/a=0.001, 0.0005 and ν=0.25

(ii) In the case of $G_{deb} > (G_{deb})_{CR}$ and $G_{crack} < (G_{crack})_{CR}$, a debonding generates, but no crack generates.

(iii) In the case of $G_{deb} < (G_{deb})_{CR}$ and $G_{crack} > (G_{crack})_{CR}$, no debonding generates, but a crack generates.

(iv) In the case of $G_{deb} > (G_{deb})_{CR}$ and $G_{crack} > (G_{crack})_{CR}$, there is possibility of generating both a debonding and a crack. However in practice, it is considered that either a debonding or a crack is produced and it must be determined. Therefore, with comparing relative magnitude of ratio $C = G_{deb}/G_{crack}$ and ratio $C_0 = (G_{deb})_{CR}/(G_{crack})_{CR}$ of fracture toughness values, it can be determined which phenomenon produces. Namely when $C > C_0$, i.e. $G_{deb}/(G_{deb})_{CR} > G_{crack}/(G_{crack})_{CR}$, a debonding produces, while when $C < C_0$, a crack produces. When $C = C_0$, a debonding and a crack may generate simultaneously.

This case of (iv) will be mentioned in detail. Using (4) and (6), the ratio C is expressed as follows:

$$C = \frac{G_{deb}}{G_{crack}} = \frac{(3+\nu)^2\left\{\tau^2 + (\ln\kappa)^2\right\}\left\{\left(\frac{M_0}{1+\nu}\right)^2 + T_0^2\right\}}{16\pi^2\left\{\left(M_0F_B^{(M)} + T_0F_B^{(T)}\right)^2 + \left(M_0F_S^{(M)} + T_0F_S^{(T)}\right)^2\right\}} \tag{7}$$

Fig. 5 shows (7) for Poisson's ratio ν=0.25. There are some difference for crack lengths b/a=0.001 and 0.0005. When the value of C_0 is larger than that of C in Fig.5 for given loads, a crack produces, while in the reversed case a debonding produces. As a whole, in the case that M_0 and T_0 have the same sign, Fig. 5 shows the tendency to produce a debonding more easily (C is larger value). However, when a crack generates at this condition ($C < C_0$), it generates with an angle near at the clamped edge (it means $\theta\pi$ is larger). On the contrary, in the case of the opposite signs of M_0 and T_0, it shows the tendency that a crack is likely to generate. When a crack generates at this condition, it generates with an angle far from the clamped edge (smaller $\theta\pi$).

Next, a debonding generates first and on the way of the debonding extension, the possibility of a crack initiation is investigated. When length "a" of the clamped edge decreases by the extension of the debonding, it is noticed that both G_{deb} and G_{crack} decrease in proportion to length a from (4) and (6). Therefore, the debonding is stopped at the time of $G_{deb} < (G_{deb})_{CR}$ by decreasing G_{deb}. With regard to the possibility of a crack initiation at this time, in the case of (ii), no crack generates since G_{crack} also decreases by decreasing a. Namely, there is no change from case (ii) to (iii). Moreover in the case of (iv), C value has no influence on a, as shown by (7), so that the relation of $C > C_0$ is always satisfied, and then G_{crack} becomes smaller than $(G_{crack})_{CR}$ before G_{deb} becomes smaller than $(G_{deb})_{CR}$. Therefore, after the debonding generates, it extends or stops on the way, but branching to a crack should never occur. The behavior of the crack after it has been generated is out of this study, so that it is not stated here.

CONCLUSIONS

As regarding the problem of a clamped edge, the stress intensity of debonding is given as the strict expression by (3). Also using the stress intensity factor or the stress intensity of debonding, the strain energy release rate to generate a crack or a debonding is given by (2) or (4) (Fig. 4). Supposing the criterion of a crack generated at an angle with the maximum strain energy release rate, in the case of a clamped edge, this angle is varied in the range of $\theta\pi$ = 127° ~ 160° due to the loading condition. By comparing the strain energy release rate with the fracture toughness value, it was investigated whether a crack or a debonding generate. In order that neither debonding or crack generate, $(G_{deb})_{CR}$ and $(G_{crack})_{CR}$ of the material must be chosen larger than G_{deb} and G_{crack} shown in Fig.4, respectively. In addition, when there is possibility to generate a crack and a debonding simultaneously, the fracture phenomenon was investigated by considering the magnitude between the ratio of strain energy release rates and the ratio of fracture toughness values. After a debonding generates, there are two possibilities that the debonding stops or not, according to conditions of the loading, but there is no possibility for a crack to generate during the extension of the debonding.

As a criterion of generating a crack and a debonding, the strain energy release rate criterion was used. In case that the materials disobey this criterion, an investigation similar to this study can be performed with a criterion that is appropriate to the materials instead of that of the strain energy release rate.

APPENDIX STRESS INTENSITY OF DEBONDING

The coordinates of points D and E in the physical plane, corresponding to points α and β on the unit circle shown in Fig. 2, are defined here as $z = z_A$ (point D) and $z = z_B$ (point E), so that complex stress function $\Phi'(z)$ in the physical plane is generally expressed, when $(z - z_A)$ is infinitesimal, i.e. z is a point near z_A, as follows:

$$\Phi'(z) = \frac{\tilde{\alpha}_0 \exp(\pi\delta)}{2\sqrt{2}D(1+\nu)}(z - z_A)^{-\lambda} \tag{A1-1}$$

where the coefficient $\exp(\pi\delta)/\left[2\sqrt{2}D(1+\nu)\right]$ has been taken for stress intensity factor K to agree with $\tilde{\alpha}_0$ of stress intensity of debonding in the case of a crack in homogenous material. Therefore, the definitions of K and $\tilde{\alpha}_0$ become equal at the tip of a crack and a debonding.

In this case, points $z = z_A$ and z_B of the displacement constraint are mapped into $\zeta=\alpha$ and β, respectively. Then, $\phi'(\zeta)$ is expressed as follows:

$$\phi'(\zeta) = (\zeta-\alpha)^{-\lambda}(\zeta-\beta)^{\lambda-1}g(\zeta)+g_0(\zeta) \qquad \text{(A1-2)}$$

Since $\Phi'(z)=\phi'(\zeta)/\omega'(\zeta)$, and considering the limit $z\to z_A$, $\tilde{\alpha}_0$ is expressed by the following expression derived from (A1-1) and (A1-2):

$$\tilde{\alpha}_0 = 2\sqrt{2}D(1+\nu)\exp(-\pi\delta)\cdot\lim_{z\to z_A}(z-z_A)^{\lambda}\Phi'(z)$$

$$=2\sqrt{2}D(1+\nu)\exp(-\pi\delta)\cdot\lim_{\zeta\to\alpha}\left\{\frac{\omega(\zeta)-\omega(\alpha)}{\zeta-\alpha}\right\}^{\lambda}\frac{(\zeta-\beta)^{\lambda-1}g(\zeta)}{\omega'(\zeta)}$$

$$=2\sqrt{2}D(1+\nu)\exp(-\pi\delta)\frac{|\omega'(\alpha)(\alpha-\beta)|^{\lambda}g(\alpha)}{\omega'(\alpha)(\alpha-\beta)}\cdot\exp\left\{i\lambda\left(\theta_A+\pi+\frac{\gamma}{2}\right)\right\} \qquad \text{(A1-3)}$$

where $g(\alpha)=\lim_{\zeta\to\alpha}\left[\phi'(\zeta)(\zeta-\alpha)^{\lambda}(\zeta-\beta)^{1-\lambda}\right]$, θ_A is an angle of the debonding surface to the x-axis, and γ shows $\angle\alpha o\beta$ of the central angle between α and β on a unit circle (see Fig. 2 in this case, $\theta_A=\pi$ and $\gamma=\pi$) (Hasebe et al. 1988).

$\tilde{\alpha}_0$ expressed by (A1-3) is a coefficient to show the strength of singularity at the tip of debonding, which is called stress intensity of debonding. Moreover, $\tilde{\alpha}_0$ is a complex constant, and expressed as $\tilde{\alpha}_0 \equiv A_B + iA_S$. Stresses near the tip of debonding, on the interface of the debonding, are expressed by the use of $\tilde{\alpha}_0$ as follows (Hasebe and Salama 1994):

$$M_x=\frac{1}{\sqrt{2r}(1+\nu)}\{(1-\nu)\cosh\pi\delta-2(1+\nu)\sinh\pi\delta\}|\tilde{\alpha}_0|\cos(\theta_0+\delta\ln r);$$

$$M_y=\frac{-(3+\nu)}{\sqrt{2r}(1+\nu)}\cosh\pi\delta|\tilde{\alpha}_0|\cos(\theta_0+\delta\ln r);$$

$$M_{xy}=\frac{1}{\sqrt{2r}(1+\nu)}\{(1+\nu)\cosh\pi\delta-2\sinh\pi\delta\}|\tilde{\alpha}_0|\sin(\theta_0+\delta\ln r) \qquad \text{(A1-4a, b, c)}$$

where r is the distance from the tip of debonding, and θ_0 is the argument of $\tilde{\alpha}_0$ given by $\theta_0=\tan^{-1}(A_S/A_B)$. Therefore, it is recognized that $|\tilde{\alpha}_0|$ can be used as the index of the stress intensity of debonding. As shown by (4), use of $|\tilde{\alpha}_0|$ as the index is the same as that of the strain energy release rate to evaluate the strength of the debonding.

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