ASYMPTOTIC ANALYSIS IN FRACTURE

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ABSTRACT

The wide applicability of asymptotics and singular perturbations in fracture is illustrated by reference to three examples: small scale yielding and its extensions which show the influence of constraint, crack bridging of a brittle matrix reinforced by fibres in the long crack limit, and the three-dimensional dynamic perturbation of a propagating crack. Possibilities for further research are indicated.

KEYWORDS

Perturbation theory, scale effects, crack bridging, dynamic fracture, crack stability.

1. INTRODUCTION

Asymptotic theory abounds throughout analytical or computational studies of fracture. Sometimes it is explicit; very frequently, it is implicit, in that the model being analyzed has some asymptotic validity. Almost invariably, for example, stress analysis is performed under the assumption that the material is homogeneous, and a decision on whether a crack will extend will be based upon examination of features of the calculated stresses and strains. The actual material, however, is heterogeneous, and can only be regarded as homogeneous in first approximation, valid when the calculated fields vary only slowly. Even then, the stress that is calculated is a 'homogenized' stress, upon which will be superimposed possibly large fluctuations, varying on the scale of the microstructure. Also, even if this feature is disregarded, there is usually a 'fracture process zone' adjacent to the crack edge, in which local events (void formation, microcracking, etc.) will invalidate the constitutive description of the material that is employed in the stress analysis. Understanding the fracture behaviour — with the probable intention of improving performance by material selection or design — requires recognition and analysis of events in the process zone, and their interaction with the 'macroscopic' field.

Three examples are selected, for illustration. The first relates to the 'small-scale yielding' approximation which underlies linear elastic fracture mechanics, and various generalizations. The need to analyze 'inner' problems, focussing attention on the detail of events near the crack edge, is emphasised, if the fracture process and its sensitivity to scale is to be understood. A significant amount of analysis is already available for two-dimensional problems - at least sufficient to demonstrate the methodology. Real cracks exist in three dimensions, however, and the two-dimensional approximation itself is only valid asymptotically, for large radius of curvature of the crack edge. Three-dimensional analysis is needed, and almost none has yet been performed. Some of the features that it is bound to display are discussed.

The next example relates to brittle material, reinforced by aligned fibres. The matrix material may crack, and separation of the crack faces may be restrained by intact fibres. This is the phenomenon of crack bridging. It is usually modelled by applying to the crack faces a continuous distribution of cohesive traction (itself an asymptotic representation of the influence of the discrete fibres), which depends upon the relative separation of the crack faces. This model leads to a description involving an integral equation whose kernel is hypersingular. In general, it can be — and needs to be — solved numerically. One interesting case, however, occurs when the crack is long, and yet remains bridged over all (or most of) its length. This 'long crack' limit poses an interesting mathematical challenge: singular perturbation of hypersingular integral equations, for which a new twist on existing asymptotic technique had to be developed. The new technique is applicable more generally and has expanded the range of mathematical problems susceptible to singular perturbation treatment.

Finally, a recent example relating to the dynamic perturbation of a crack propagating in a linearly elastic medium is discussed. The solution will facilitate a full three-dimensional dynamic stability study. The underlying reasoning relies on notions of matched asymptotic expansions but the manner of implementation is novel and distinctive. The solution has already demonstrated how pulse markings on a fracture surface, called Wallner lines, may form. Further possible applications, and generalizations, are indicated.

2. SMALL-SCALE YIELDING AND GENERALIZATIONS

Linear elastic fracture mechanics requires the calculation of stresses within a body, under the assumption that the material is linearly elastic. This is a good approximation, if the body in fact is elastoplastic, subject to two provisos. The first is that the loading that is applied should be sufficiently low; that is, if the stress remote from the crack tip is of the order of σ^A, and the yield stress is characterized by a stress σ_0, then $\epsilon := \sigma^A/\sigma_0 \ll 1$. The other proviso is that the calculated stresses should not be taken literally near the crack tip. They are square-root singular, so that, for example, for Mode I loading conditions,

$$\sigma_{ij} \sim K_I f_{ij}(\theta)/\sqrt{2\pi r} \quad \text{as} \quad r \to 0, \tag{2.1}$$

where the crack is situated on the plane $x_2 = 0$, and r, θ are polar coordinates with origin at a crack edge. Evidently, therefore, the calculated stresses violate the yield condition within a distance of order $d := (K_I/\sigma_0)^2$ from the crack edge, and 'small-scale yielding' prevails so long as this distance is much smaller than macroscopic dimensions of the body, characterized by a length L. This condition is bound to be satisfied when ϵ is small enough. As discussed by Rice (1968), the problem as described admits a 'boundary layer' formulation. The linear elastic field is a good approximation, except within some radius R of the crack edge. For sufficiently small ϵ, the order relations $d \ll R \ll L$ will apply. In that case, relative to the length scale d, the crack appears to be semi-infinite, in an infinite body, subject to *remote* loading given by (2.1) as $r/d \to \infty$. Evidently, whatever occurs near the crack edge is governed by the value of K_I in this situation, and the macroscopic fracture criterion is bound to be "$K_I = K_{Ic}$", where K_{Ic} is the fracture toughness. This sort of reasoning applies, even for other models of material behaviour near the crack edge, including the possibility of damage growth. However, any attempt to explain or predict the observed value of K_{Ic} requires an analysis of the 'inner' boundary value problem, close to the crack edge. One of its interests is that it is *universal*, and describes the near-tip response of a crack to any loading, in any body, subject to the restriction to small-scale yielding.

The elementary boundary-layer theory, as described, delivers just the first terms in asymptotic series, which may be developed, formally, in powers of ϵ. The linear elastic stress field is the leading-order term in the outer series, proportional to $\sigma_0\epsilon$ since it is linear in the applied loads. The boundary-layer solution can be expressed as σ_0 times a function of r/d and θ, independent of ϵ. Following terms in the series yield corrections, and so extend the range of values of ϵ for which the solution is a good approximation. This was done, for elastoplastic material behaviour, by Edmunds and Willis (1976a,b; 1977). The mathematical procedure was to use the formalism of matched asymptotic expansions (Van Dyke, 1964). Qualitatively, the reasoning is as follows.

First, assume that the 'inner' problem has been solved (except in special cases, this requires a finite-element computation). Now revert to the 'macroscopic' scale, with $r \sim R$. In the limit as $\epsilon \to 0$, the domain of the inner solution shrinks to a small neighbourhood of the crack tip. Outside of this neighbourhood, it has the effect of a force dipole, concentrated at the crack tip. Although, very close to the crack tip, the field becomes the inner field, the formal limit of the dipole field as $r \to 0$ has a singularity of order $r^{-3/2}$. Because boundary conditions on the crack faces are satisfied, this field is one of the eigenfunctions given by Williams (1957). The singularity is too strong to be physical, but this is of no consequence because the inner field takes over close to the crack tip. The strength of the singularity has to be, on dimensional grounds, proportional to $\sigma_0 d^{3/2} \equiv \sigma_0(K_I/\sigma_0)^3$, which is proportional to ϵ^3. This 'Williams eigenfunction' violates boundary conditions at the specimen surface, and so induces a correction to the linear elastic stress field which is of order ϵ^3. This, in turn, induces a correction of order ϵ^3 to K_I, whose leading term is of order ϵ.

Next, a correction to the inner solution can be considered. To the next order in accuracy, as $r \to 0$, the stress field has the form given in (2.1), *plus* a term $T\delta_{i1}\delta_{j1}$ — the so-called T-stress. It induces a correction of order ϵ to the boundary-layer solution because $T \propto \sigma_0\epsilon$, and the factor ϵ does not scale out as it did in the case of the K_I-term.

Iteration to higher order is now possible: the correction to the boundary layer solution induces a correction of order higher than $\epsilon^3 r^{-3/2}$ in the outer field, which reflects back to give a further correction to the inner field. The entire process was pursued systematically in the papers of Edmunds and Willis. A detailed account would be out of place here. It is appropriate, however, to indicate some deductions that follow from the analysis.

The asymptotic analysis provides expressions in the form of series in powers of ϵ for features of the stress and deformation fields away from the crack tip, and in its vicinity. One interesting question is to comprehend what happens when the assumptions underlying LEFM do not apply. It is also possible that the plastic region may be too small for a region of 'J-dominance' to have physical significance. Such considerations require detailed study of the field close to the crack tip, in conjunction with explicit local fracture criteria.

The most general way to study near-tip features is probably to construct a complete finite-element model for the component or specimen, containing enough detail to allow the representation of near-tip events. The main point, though, is to establish trends and so contribute towards the development of criteria for analysis or design. One possible approach is to employ low-order asymptotic expansions. The simplest of these is provided by boundary-layer analysis as described above, but with the 'inner' problem solved to second order, by applying the boundary condition

$$\sigma_{ij}(r,\theta) \sim K_I f_{ij}(\theta)/\sqrt{2\pi r} + T_{ij} \quad \text{as} \quad r \to \infty, \tag{2.2}$$

where the constant term T_{ij} depends upon the component or specimen in question. The accuracy of this approximation, in comparison with full finite-element computations, has been discussed by Bilby *et al.* (1986), and Betegón and Hancock (1991), under the assumption of plane strain so that only the component $T_{11} = T$ was relevant. It was first considered *in conjunction with local fracture criteria* by Harlin and Willis (1989). They postulated that fracture would occur *either* when a critical stress condition was reached (Ritchie, Knott and Rice, 1973) *or* when a criterion for ductile fracture by void growth was satisfied (Rice and Johnson, 1970), the actual mechanism of fracture depending on which criterion was reached first as loading increased. If the former, the fracture would be brittle, while if the latter, the failure would be ductile. The novelty of the Harlin–Willis analysis was the observation that, since these criteria involved a microscopic length scale, it could be that the response of one specimen might be ductile, while the response of a geometrically-similar specimen of different dimensions might be brittle. Furthermore, because the competition between the two mechanisms is yield-stress and therefore temperature dependent, the *ductile/brittle transition temperature should be scale-dependent.* Harlin and Willis demonstrated this by full finite-element computations *and* corresponding 'modified boundary layer' calculations. The trends predicted for the variation of transition temperature with absolute size were the same, even quantitatively for the range of configurations investigated. The variations of transition temperature with size, as well as the absolute level of fracture resistance, varied with the specimen configuration and were thus demonstrated to correlate with the value of the T-stress. In words, the T-stress serves as an indicator of the degree of constraint and so, equally, the ductile/brittle transition depends on constraint. It is *not* implied that the T-stress provides an accurate characterization under large-scale yielding (for which O'Dowd and Shih (1991) advocate the use of a modified near-tip field, augmenting J by a non-singular stress Q). The easily-calculated T-stress does offer, however, a simple means of ranking specimens with respect to their sensitivity to scale. (It is remarked, too, that the Harlin–Willis analysis suggested the possibility that a surface-breaking crack of depth 1cm in a typical pressure vessel steel might display a transition temperature perhaps 40°K higher than a crack of depth 2cm. Naturally such an indication has to be checked experimentally.)

It seems to the writer that a three-dimensional boundary layer theory ought to be developed. In part, this is simply because real cracks are three-dimensional. A further legitimate observation is that full three-dimensional computations are at least inconvenient, and likely to remain so, and therefore some simple parametrization of constraint in three dimensions should be useful. The simplest boundary-layer calculation in this class would be to address the modified small-scale yielding boundary layer, in the context of 'generalized plane strain', allowing for the remote boundary condition (2.2), with

$$T_{ij} = T\delta_{i1}\delta_{j1} + S\delta_{i3}\delta_{j3}. \tag{2.3}$$

The 'S-stress' corresponds to tension along the edge of the crack. It seems evident that, for example, a positive value for S could work against a negative value for T, and seriously compromise predictions based on the assumption of plane strain. More complete boundary-layer analysis definitely is feasible, because the elastic Green's function for a semi-infinite crack is available, essentially from the work of Bueckner (1987).

3. CRACK BRIDGING

A brittle matrix reinforced by stronger fibres (which may or may not be subject to plastic yielding) may display the phenomenon that the matrix cracks, leaving the crack faces restrained by fibres that have remained intact. This is the phenomenon of crack bridging. It is commonly modelled by regarding the crack as being present in an elastic matrix, whose elastic moduli are those of intact composite, with the crack faces subject to cohesive forces whose magnitude depends on the crack face separation. Calling the relative separation ϕ, and the cohesive force γ per unit area, a conventional linear elastic formulation shows that ϕ must satisfy the hypersingular integral equation

$$-A\int_{-a}^{a}\frac{\phi(y)}{(y-x)^2}\mathrm{d}x + \gamma(\phi(x)) = \sigma(x), \quad -a < x < a. \tag{3.1}$$

Here, either plane stress or plane strain conditions are assumed, the crack occupies the segment $(-a, a)$ of the x_1-axis, A is the appropriate combination of elastic constants and $\sigma(x)$ is the 22 component of stress that would prevail at the location of the crack, if the crack were not present. The integral is interpreted as a Hadamard finite part or, equivalently, in the sense of generalized functions. More detailed discussion of this equation is provided by Nemat-Nasser and Hori (1987). Dimensionless variables are introduced by measuring x, y in units of a, and ϕ in multiples of some microscopic length scale δ. Stress can be measured relative to some unit σ_0 representative of the cohesive stress. Equation (3.1) then becomes

$$-\epsilon\int_{-1}^{1}\frac{\phi(y)}{(y-x)^2}\mathrm{d}y + \gamma(\phi(x)) = \sigma(x), \quad -1 < x < 1, \tag{3.2}$$

where

$$\epsilon = A\delta/(\sigma_0 a). \tag{3.3}$$

The 'long crack' limit is that for which $a \gg A\delta/\sigma_0$, so that $\epsilon \to 0$. This poses a singular perturbation problem because, in first approximation, setting $\epsilon = 0$ reduces (3.2) to the algebraic form

$$\gamma(\phi(x)) = \sigma(x). \tag{3.4}$$

This has unique solution

$$\phi(x) = \phi_0(x) := \gamma^{-1}(\sigma(x)), \tag{3.5}$$

if γ is an increasing function and $\max\{\sigma(x)\} < \max\{\gamma\}$. This solution cannot be valid near the crack edges $x = \pm 1$, because it is a physical requirement that $\phi = 0$ there. It is therefore valid only at points on the crack away from the ends, and so represents the first term in an 'outer' expansion. An equation valid in an 'inner' region adjacent to the crack tip at $x = -1$, say, can be constructed by scaling the variables, so that

$$X = (x+1)/\epsilon, \quad Y = (y+1)/\epsilon. \tag{3.6}$$

Then, equation (3.2) becomes

$$-\int_{0}^{2/\epsilon}\frac{\tilde{\phi}(Y)}{(Y-X)^2}\mathrm{d}Y + \gamma(\tilde{\phi}(X)) = \sigma(-1+\epsilon X), \quad 0 < X < 2/\epsilon, \tag{3.7}$$

where $\tilde{\phi}(X) = \phi(-1 + \epsilon X)$. Taking the limit of this equation, as $\epsilon \to 0$ with X fixed, gives the following equation for $\tilde{\phi}_0$, the lowest-order approximation to $\tilde{\phi}$:

$$-\int_0^\infty \frac{\tilde{\phi}_0(Y)}{(Y-X)^2} \mathrm{d}Y + \gamma(\tilde{\phi}_0(X)) = \sigma(-1), \quad 0 < X < \infty. \tag{3.8}$$

It can be verified that, as $X \to \infty$, the integral in (3.8) becomes insignificant, so that $\tilde{\phi}_0 \sim \phi_0(-1)$, which is exactly the limiting value of $\phi(x)$, calculated from the outer approximation (3.4). Furthermore, an approximation which interpolates smoothly between the inner and outer approximations is

$$\phi(x) \sim \phi_0(x) + \{\tilde{\phi}_0[(x+1)/\epsilon] - \phi_0(-1)\} + \cdots, \tag{3.9}$$

where $\cdots$ represents a corresponding contribution from the vicinity of $x = 1$.

This solution permits the calculation, to lowest order, of the stress intensity factors at the crack edges, and also of the relative separation of the crack faces. The problem occurs when an attempt is made to develop further terms. Reverting to the full equation, an outer expansion correct to order ϵ requires the integral to be estimated to order ϵ^0. Superficially, this can be done by substituting into the integrand the lowest-order outer approximation $\phi_0(x)$. The integral then has a singularity of order $(x+1)^{-1}$ near the crack end $x = -1$. This is of no consequence, because a boundary layer solution takes over there, in any case. The next problem is to find a two-term inner expansion. Two hints of trouble appear. One is that the singularity in the second term of the outer expansion is not integrable. Also, anticipating that the two-term inner expansion should agree, as $X \to \infty$, with the limit, as $x \to -1$, of the two-term expansion of the outer solution suggests that the inner solution will contain, in its asymptotic form as $X \to \infty$, the expression $\phi_0(-1) + \epsilon X \phi_0'(-1)$. This is linear in X and further complicates the treatment of the integral in (3.7) in the limit $\epsilon \to 0$. The resolution of these difficulties is conceptually very simple, though delicate in its implementation. At any stage, a *uniform approximation* to whatever order is required should be used for the integrand. For this purpose, the asymptotic matching principle of Van Dyke (1964) is invoked. To describe this, it is necessary to introduce notation. Denote the exact solution $\phi(x, \epsilon)$, and let its asymptotic expansion to m terms (in the present case, this means up to order ϵ^m, counting $\ln \epsilon$ as $O(1)$), as $\epsilon \to 0$ keeping x fixed, by $O_m(\phi)$. Similarly, denote the asymptotic expansion to n terms as $\epsilon \to 0$ of $\phi(-1 + \epsilon X, \epsilon)$, keeping X fixed, by $I_n(\phi)$. The matching principle proposes that

$$O_m[I_n(\phi)] = I_n[O_m(\phi)]; \tag{3.10}$$

it proposes further that the expression

$$\phi(x, \epsilon) \sim O_n(\phi) + I_n(\phi) - O_n[I_n(\phi)] + \cdots \tag{3.11}$$

provides a uniform approximation to $\phi(x, \epsilon)$, correct to $O(\epsilon^n)$ in the present case. The dots indicate similar terms, involving the inner expansion near the other end of the crack. The approximation (3.9) fits this pattern, with $n = 0$.

An integral first has to be treated in considering the lowest-order inner problem. This requires the integral in (3.7) to be estimated correct to order ϵ^0, and this can be achieved by replacing $\tilde{\phi}$ by its uniform approximation corresponding to (3.9), in which, so far, $\tilde{\phi}_0$ should be regarded as unknown. Elementary analysis then confirms that, to order ϵ^0, the integral takes the form given in (3.8), as already asserted. Equally, the first-order outer approximation requires the evaluation of the integral in (3.2) correct to order ϵ^0. This again can be accomplished by replacing ϕ by its uniform approximation (3.9), and elementary analysis confirms the simple form already given. To proceed further, knowledge of ϕ to order ϵ means that $O_1(\phi)$ is known, and hence that $O_1[I_1(\phi)] \equiv I_1[O_1(\phi)]$ can be determined to give, using (3.11) with $n = 1$, a uniform approximation valid to first order which can be employed to derive an equation which determines the inner expansion of $\tilde{\phi}$, to first order. After some analysis, the result is that

$$\tilde{\phi} \sim \tilde{\phi}_0 + \tilde{\phi}_1, \tag{3.12}$$

where $\tilde{\phi}_1$ satisfies the equation

$$-\int_0^\infty \frac{\mathrm{d}Y}{(Y-X)^2}[\tilde{\phi}_1(Y) - \epsilon Y \phi_0'(-1)] + \tilde{\phi}_1(X)\gamma'(\tilde{\phi}_0(X)) = \epsilon X \sigma'(-1) + \epsilon \phi_0'(-1) \ln\left(\frac{2}{\epsilon X}\right) + \epsilon C, \tag{3.13}$$

with

$$C = \int_{-1}^1 \frac{\mathrm{d}y}{(y-1)^2}[\phi_0(y) - \phi_0(-1) - (y+1)\phi_0'(-1)] - \tfrac{1}{2}\phi_0(-1) - \phi_0'(-1). \tag{3.14}$$

Further terms can be calculated. The method applies generally, and is applicable to other equations. Willis and Nemat-Nasser (1990) proposed the method and demonstrated it for the linear case $\gamma(\phi) = k\phi$. Nonlinear examples have also been worked out, including admitting the possibility of fibre breakage or pullout (see Movchan and Willis 1993, Movchan and Willis 1996a). The method has also been applied to the bridging of a circular crack, which satisfies an integral equation with a kernel more complicated than that in (3.2) (Movchan and Willis, 1996b).

The modelling leading to the equation (3.2) has some limitations, of which perhaps the most important is that the 'cohesive law' $\gamma(\phi)$ is applicable only where ϕ varies slowly relative to fibre separation. Better modelling, allowing for more rapid variation (which is bound to occur near a crack tip), is in progress.

4. DYNAMIC PERTURBATION OF A PROPAGATING CRACK

A crack propagates dynamically so that, at time t, it occupies the domain

$$S_\epsilon = \{-\infty < x_1 - Vt < \epsilon\phi(x_2, t),\ -\infty < x_2 < \infty,\ x_3 = \epsilon\psi(x_1, x_2)\}, \tag{4.1}$$

where $\epsilon \ll 1$. This represents a general dynamic perturbation of a plane crack in uniform motion defined by the surface S_0, realised when $\epsilon = 0$. Maintenance of such a uniform motion would require loading to be independent of x_2 and to depend on (x_1, t) only in the combination $X = x_1 - Vt$. This special loading yields no simplification in the analysis, so the boundary value problem may just as well be solved for general loading. The basic objective is to solve the boundary value problem asymptotically, to first order in ϵ, and then to impose a fracture criterion, which restricts the functions ϕ and ψ. Rice, Ben-Zion and Kim (1994) solved a problem of this type, but for the scalar wave equation, and with $\psi = 0$, so that the crack remained in the plane $x_3 = 0$. In particular, they calculated the stress intensity factor in terms of the function ϕ, and then invoked a fracture criterion in which the local toughness was specified as a function of (x_1, x_2) to derive an integral equation for ϕ. The full elastodynamic problem is technically much more difficult but it has been solved (Willis and Movchan 1995, Movchan and Willis 1995, Willis and Movchan 1996).

The technique devised by Willis and Movchan made use of an integral identity. Take $d \gg \epsilon$ but still small, and take $\mathbf{u}$ to be the *additional* displacement generated by introducing a crack over S_ϵ, into an infinite domain in which the displacement and stress fields in the absence of the crack would have components u_i^A, σ_{ij}^A. The displacement field $\mathbf{u}$ is therefore allowed to jump across the surface S_ϵ, while stresses are continuous except possibly at S_ϵ, and satisfy the boundary conditions

$$\sigma_{ij}n_j + \sigma_{ij}^A n_j \to 0 \tag{4.2}$$

as $\mathbf{x} \to S_\epsilon$ from either side, $\mathbf{n}$ being the normal (with $n_3 > 0$) to S_ϵ. Denote by $\mathbf{u}(d)$ the displacement $\mathbf{u}$ evaluated on $x_3 = d$, and let $\boldsymbol{\sigma}(d)$ be the corresponding traction vector, with components $\sigma_{i3}(d)$. These are related by the identity

$$\mathbf{u}(d) = -\mathbf{G} * \boldsymbol{\sigma}(d), \tag{4.3}$$

where $\mathbf{G}(x_1, x_2, t)$ represents the Green's tensor for surface loading of the half-space $x_3 > d$ evaluated on the surface $x_3 = d$, and $*$ represents the operation of convolution with respect to (x_1, x_2, t). It follows also, by considering the half-space $x_3 < -d$, that

$$\mathbf{u}(-d) = \mathbf{G}^{\mathrm{T}} * \boldsymbol{\sigma}(-d). \tag{4.4}$$

Now associate with the unperturbed configuration displacement and stress vector fields $\mathbf{U}$ and $\boldsymbol{\Sigma}$. Their values on $x_3 = \pm d$ are similarly related as in (4.3) and (4.4). Next, note that

$$\mathbf{U}^{\mathrm{T}}(-d) * \boldsymbol{\sigma}(d) = \boldsymbol{\Sigma}^{\mathrm{T}}(-d) * \mathbf{G} * \boldsymbol{\sigma}(d) = -\boldsymbol{\Sigma}^{\mathrm{T}}(-d) * \mathbf{u}(d). \tag{4.5}$$

A similar identity applies with d replaced by $-d$. Subtracting the two gives

$$-\langle\mathbf{U}\rangle_d^{\mathrm{T}} * [\boldsymbol{\sigma}]_d + [\mathbf{U}]_d^{\mathrm{T}} * \langle\boldsymbol{\sigma}\rangle_d = \langle\boldsymbol{\Sigma}\rangle_d^{\mathrm{T}} * [\mathbf{u}]_d - [\boldsymbol{\Sigma}]_d^{\mathrm{T}} * \langle\mathbf{u}\rangle_d, \tag{4.6}$$

where

$$\langle f\rangle_d = f(d) + f(-d), \quad [f]_d = f(d) - f(-d). \tag{4.7}$$

Finally, by writing side-by-side three linearly independent solution pairs $\mathbf{U}$, $\boldsymbol{\Sigma}$, these may be regarded as 3×3 matrices in the fundamental identity (4.6) which applies for all ϵ and d, subject only to the restriction $d \gg \epsilon$. Willis and Movchan exploited the identity (4.6) by choosing $\mathbf{U}$ to be a *weight function matrix* for the unperturbed moving crack. This satisfies the equations of motion except on the plane $x_3 = 0$, $\boldsymbol{\Sigma}$ is continuous across $x_3 = 0$, $\mathbf{U}$ is allowed to be discontinuous across $x_3 = 0$ when $X = x_1 - Vt > 0$, and $\boldsymbol{\Sigma} = 0$ when $x_3 \to \pm 0$ and $X > 0$. The field $\mathbf{U}$ is not identically zero because it is chosen to have an 'unphysical' singular behaviour as $X \to 0$ with $x_3 = 0$. Specifically,

$$[\mathbf{U}] \sim \left(\frac{2}{\pi X}\right)^{1/2} H(X)\delta(x_2)\delta(t)\mathbf{I} \quad \text{as } X \to 0. \tag{4.8}$$

Here, $[.]$ denotes the limit as $d \to 0$ of the corresponding quantity with subscript d. Similar notation will be deployed below for $\langle . \rangle$. These conditions suffice to define $\mathbf{U}$ and $\boldsymbol{\Sigma}$ uniquely. Of course, finding them is a task of substance. It follows from symmetry that Mode I (for which only $[U_{33}] \neq 0$) uncouples from Modes II and III which, unlike the case of two

dimensions, remain coupled. The Mode I problem can be reduced to a scalar Wiener–Hopf problem for $[U_{33}]$ and is soluble by standard means (Willis and Movchan, 1995). By good fortune, the matrix Wiener–Hopf problem for the the coupled Modes II and III is also soluble (Movchan and Willis, 1995): this is one of very few examples of an explicit solution to such a problem.

The use of the identity (4.6) may be illustrated by applying it to the unperturbed crack. Then, $\epsilon = 0$ and it is possible to let $d \to 0$ immediately. Now set

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_+ + \boldsymbol{\sigma}_-, \tag{4.9}$$

where $\boldsymbol{\sigma}_+ = 0$ for all $X < 0$ (so that $\boldsymbol{\sigma}_+ = \boldsymbol{\sigma} H(X)$), and split other functions similarly. It should be noted that the given boundary conditions imply that

$$\langle\boldsymbol{\Sigma}\rangle = \langle\boldsymbol{\Sigma}\rangle_-, \ \ [\boldsymbol{\Sigma}] = 0, \ \ [\mathbf{U}] = [\mathbf{U}]_+, \ \text{ and } \ [\mathbf{u}] = [\mathbf{u}]_-, \ \ [\boldsymbol{\sigma}] = [\boldsymbol{\sigma}]_-. \tag{4.10}$$

The identity (4.6), with $d = 0$, now takes the form

$$[\mathbf{U}]_+^{\mathrm{T}} * \langle\boldsymbol{\sigma}\rangle_+ = -[\mathbf{U}]_+^{\mathrm{T}} * \langle\boldsymbol{\sigma}\rangle_- + \langle\mathbf{U}\rangle^{\mathrm{T}} * [\boldsymbol{\sigma}]_- + \langle\boldsymbol{\Sigma}\rangle_-^{\mathrm{T}} * [\mathbf{u}]_-. \tag{4.11}$$

The functions $\langle\boldsymbol{\sigma}\rangle_-$, $[\boldsymbol{\sigma}]_-$ are known, from the given boundary conditions[1]. The function $\langle\boldsymbol{\sigma}\rangle_+$ is not known, but it has the asymptotic form

$$\langle\boldsymbol{\sigma}\rangle_+ \sim \mathbf{K}(x_2, t)H(X)/\sqrt{2\pi X} \quad \text{as } X \to 0, \tag{4.12}$$

where the stress intensity vector $\mathbf{K}$ has components (K_{II}, K_{III}, K_I). It is unknown, but it can be found by applying (4.11) as $X \to +0$. For any $X > 0$, the convolution of the two '$-$' functions is zero, identically. Also, as $X \to 0$, the convolution of the two '$+$' functions can be evaluated from their asymptotic forms (4.8), (4.12), giving

$$[\mathbf{U}]_+^{\mathrm{T}} * \langle\boldsymbol{\sigma}\rangle_+ \sim \mathbf{K}. \tag{4.13}$$

Thus, (4.11) gives

$$\mathbf{K} = -[\mathbf{U}]_+^{\mathrm{T}} * \langle\boldsymbol{\sigma}\rangle_- + \langle\mathbf{U}\rangle^{\mathrm{T}} * [\boldsymbol{\sigma}]_-, \tag{4.14}$$

in which the convolutions are evaluated for $X = 0$.

The perturbations of the stress intensity factors in the case of the perturbed crack also follow, by exploiting the fact that (4.6) is an identity in ϵ, d and X. Their evaluation requires expansion of the identity to lowest order in d and to first order in ϵ; it is also necessary to expand the unperturbed field, and the weight function, to two non-trivial terms in X.

The techniques required to complete the solution include Fourier transforms, complex variable theory, matched expansions and generalized functions. The final result of the perturbation theory is to represent the stress intensity factor in the form $\mathbf{K} = \mathbf{K}^{(0)} + \Delta\mathbf{K}$, with the perturbation given by

$$\begin{aligned}\Delta\mathbf{K} = \epsilon\Big\{&\mathbf{Q}^{\mathrm{T}} * \big((\phi\mathbf{I} - \psi^*\boldsymbol{\Theta})\mathbf{K}^{(0)}\big) - (\mathbf{E}\phi_{,2} + \boldsymbol{\Omega})\mathbf{K}^{(0)} \\ &+ \left(\frac{\pi}{2}\right)^{1/2}(\phi\mathbf{A}^{(0)} + \psi^*\mathbf{L})\Big\} + \Big\{[\mathbf{U}]^{\mathrm{T}} * \langle\mathbf{P}^{(1)}\rangle - \langle\mathbf{U}\rangle^{\mathrm{T}} * [\mathbf{P}^{(1)}]\Big\}_{(X=0)}.\end{aligned} \tag{4.15}$$

[1] In fact, for the conditions given, $[\boldsymbol{\sigma}]_- = 0$, but the formula also applies more generally, when the upper and lower crack faces are loaded differently.

The terms in (4.15) are defined in full in Willis and Movchan (1996). The function $\mathbf{Q}$ is obtained from the asymptotic expansion of $[\mathbf{U}]$ as $X \to 0$. The unperturbed stress field ahead of the crack has the form

$$\boldsymbol{\sigma} \sim \mathbf{K}^{(0)}/\sqrt{2\pi X} - \mathbf{P}^{(0)*} + \mathbf{A}^{(0)}\sqrt{X}, \tag{4.16}$$

$\psi^*(x_2, t) = \psi(0, x_2, t)$ and $\mathbf{P}^{(1)}$ defines the perturbation of the boundary condition, obtained by formal expansion of (4.2). The matrix $\mathbf{L}$ depends on $\mathbf{A}^{(0)}$, $\dot{\mathbf{K}}^{(0)}$ and $\mathbf{K}^{(0)}_{,2}$, and $\boldsymbol{\Omega}$ depends on $\psi_{,1}$ and $\psi_{,2}$; the other matrices are constants.

The solution has recently been exploited, in the special case of Mode I loading with the crack confined to the plane $x_3 = 0$, to explain the presence of Wallner lines on a crack surface[2]. The full solution will facilitate a variety of studies of crack perturbation, and crack stability; these will require additional information in the form of fracture criteria, and perhaps also further asymptotic analysis.

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