3D SINGULAR STRESSES IN A CRACKED PLATE

X. M. SU and C. T. SUN
School of Aeronautics and Astronautics
Purdue University
West Lafayette, IN 47907, USA

ABSTRACT

Three dimensional solutions of singular stresses at the crack tip of a cracked plate are obtained by superimposing three stress expansions. The solutions, satisfying not only equilibrium and compatibility equations but also all boundary conditions, are of square root singularity throughout the plate thickness. Deformations related to the singular stresses are shown to be plane strain. All components of the singular stresses vanish on plate surfaces, and the profile of the stress intensity factor in the thickness direction cannot be determined by the asymptotic analysis alone.

KEYWORDS

Cracked Plate, 3D, Singular Stresses, Plane Strain

INTRODUCTION

The 3D singular stresses at the crack tip in a flat plate were attempted by many researchers. Recent publications include those of Benthem (1977), and Kawai, Fujitani and Kumagai (1977). While all the authors agreed upon a square root singularity for stresses away from free plate surfaces, conflicting results were reported on the singularities close to plate surfaces: the solution of Benthem (1977) was characterized by a stress singularity less than square root, while that by Kawai et al(1977) by a stress singularity higher than square root. The difference in singularities for the two above mentioned studies results from different approximations used, as the solution by Benthem (1977) satisfied boundary conditions on crack surface exactly but on plate surfaces only approximately and the solution by Kawai et al(1977) satisfied boundary condition on crack surfaces approximately but on plate surfaces exactly.

In the present paper, recent developments in the 3D plate analysis are exploited to obtain an asymptotic expansion of singular stresses at the crack tip that satisfy not only equilibrium and compatibility equations but also all boundary conditions exactly.

BASIC EQUATIONS

Consider a plate with a through-the-thickness cut. The plate thickness is 2h, and both upper and lower plate surfaces and the crack surfaces are traction free. Remote tensile loading is applied on the edges and is symmetric about the middle plane of the plate. A cylindrical coordinate system is established with the (r, θ) plane in the middle plane of the plate. The origin of the coordinate system is at the crack tip with the plane $\theta = 0$ parallel to the crack surface.

The problem described above is among the thick plate problems which have been studied by Gregory (1992). Gregory has rigorously proved that the most general state of stress σ_{ij} (i,j=1,2,3) for the thick plate problem can be uniquely decomposed into three parts

$$\sigma_{ij} = \sigma_{ij}^{PS} + \sigma_{ij}^{S} + \sigma_{ij}^{PF} \qquad (i,j = 1,2,3) \tag{1}$$

where σ_{ij}^{PS} represents the (exact) plane stress state; σ_{ij}^{S} the shear state stress; and σ_{ij}^{PF} the so called Papkovich-Fadle state stress. The three stress fields are generated by three different stress functions.

Plane Stress State

This is the well known (exact) plane stress (Timoshenko and Goodier, 1970). With respect to Cartesian coordinates x_1, x_2 and x_3, the stresses are obtained from stress function ψ as

$$\sigma_{11}^{PS} = \psi,_{22} + k(z)\overline{\nabla}^2\psi,_{22} \qquad \sigma_{22}^{PS} = \psi,_{11} + k(z)\overline{\nabla}^2\psi,_{11} \tag{2}$$

$$\sigma_{12}^{PS} = -\psi,_{12} - k(z)\overline{\nabla}^2\psi,_{12} \qquad \sigma_{13}^{PS} = \sigma_{23}^{PS} = \sigma_{33}^{PS} = 0 \tag{3}$$

where

$$k(z) = \frac{\nu(1-3(\frac{z}{h})^2)h^2}{6(1+\nu)}, \tag{4}$$

a comma indicates partial differentiation, ν is Poisson's ratio, and ψ is a two dimensional function satisfying the biharmonic equation

$$\overline{\nabla}^2\overline{\nabla}^2\psi = (\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2})(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2})\psi = 0 \tag{5}$$

Shear Stress State

Shear stress state is derived from a three dimensional potential function ϕ as

$$\begin{aligned} \sigma_{11}^{S} &= 2\phi,_{12} & \sigma_{22}^{S} &= -2\phi,_{12} \\ \sigma_{12}^{S} &= \phi,_{22} - \phi,_{11} & \sigma_{13}^{S} &= \phi,_{23} \\ \sigma_{23}^{S} &= -\phi,_{13} & \sigma_{33}^{S} &= 0 \end{aligned} \tag{6}$$

where

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} + \frac{\partial^2\phi}{\partial x_3^2} = 0 \tag{7}$$

and

$$\phi,_3|_{x_3=\pm h} = 0 \qquad \int_{-h}^{h} \phi dx_3 = 0 \tag{8}$$

Papkovich-Fadle State

Papkovich-Fadle state stress is generated by a three dimensional bi-harmonic function χ as

$$\begin{aligned} \sigma_{11}^{PF} &= \chi,_{1133} + \nu\nabla^2\chi,_{22} & \sigma_{22}^{PF} &= \chi,_{2233} + \nu\nabla^2\chi,_{11} \\ \sigma_{12}^{PF} &= \chi,_{1233} - \nu\nabla^2\chi,_{12} & \sigma_{13}^{PF} &= -\overline{\nabla}^2\chi,_{13} \\ \sigma_{23}^{PF} &= -\overline{\nabla}^2\chi, & \sigma_{33}^{PF} &= \overline{\nabla}^2\overline{\nabla}^2\chi \end{aligned} \tag{9}$$

where

$$\nabla^2\nabla^2\chi = 0 \tag{10}$$

and

$$\chi|_{x_3=\pm h} = \chi,_3|_{x_3=\pm h} \tag{11}$$

It can easily be verified that any combinations of the above three types of stresses satisfy not only the equilibrium equations and compatibility equations but also the boundary conditions on free plate surfaces. The decomposition of the three dimensional stress field in a thick plate into PS, S and PF stresses was shown by Gregory(1992) to be unique.

ASYMPTOTIC SOLUTIONS

Expansion of PS State Stresses

PS state stress consists of two parts: conventional plane stress and the modification which is added to satisfy compatibility equations of elasticity. The stress function of PS state stress is determined by solving a conventional plane stress problem with the boundary conditions obtained from averaging traction on the edges through-the-thickness, i.e.,

$$\frac{1}{h}\int_{-h}^{h} \sigma_\theta|_{\theta=\pm\pi} dz = 0, \qquad \frac{1}{h}\int_{-h}^{h} \sigma_{r\theta}|_{\theta=\pm\pi} dz = 0 \tag{12}$$

Following Williams (1952), stress function in Eqn (5) is expanded in terms of r as

$$\psi(r,\theta) = r^{1+\rho}\tilde{\psi}(\theta) \tag{13}$$

where ρ is an arbitrary positive number, and $\tilde{\psi}(\theta)$ is determined by

$$(1+\rho)^2(\rho-1)^2\tilde{\psi} + [(\rho-1)^2 + (1+\rho)^2]\tilde{\psi}'' + \tilde{\psi}'''' = 0 \tag{14}$$

Stresses obtained with above stress functions satisfy boundary conditions in Eqn (12) only when

$$sin 2\rho\pi = 0 \tag{15}$$

which leads to eigenvalues

$$\rho = \frac{m}{2}, \qquad m=1, 2, 3, ... \tag{16}$$

Only the odd integers of m in the above equation give singular stresses and are thus taken for the asymptotic solutions. To retain all the possible singular stress terms, we take m=1, 3, 5.

For Mode-I (symmetric mode), PS stresses are

$$\begin{aligned}
\sigma_{rr}^{PS} &= -\frac{3}{2}B_1k(z)cos(\frac{\theta}{2})r^{-\frac{5}{2}} + \frac{3}{2}B_2k(z)cos(\frac{\theta}{2})r^{-\frac{3}{2}} \\
&+[\frac{5}{4}B_1cos\frac{\theta}{2} - \frac{3}{4}D_1cos\frac{3}{2}\theta - \frac{15}{2}B_3k(z)cos\frac{3}{2}\theta]r^{-\frac{1}{2}} && (17)\\
\sigma_{\theta\theta}^{PS} &= \frac{3}{2}B_1k(z)cos(\frac{\theta}{2})r^{-\frac{5}{2}} - \frac{3}{2}B_2k(z)cos(\frac{\theta}{2})r^{-\frac{3}{2}} \\
&+[\frac{3}{4}B_1cos\frac{\theta}{2} + \frac{3}{4}D_1cos\frac{3}{2}\theta + \frac{15}{2}B_3k(z)cos\frac{3}{2}\theta]r^{-\frac{1}{2}} && (18)\\
\sigma_{r\theta}^{PS} &= -\frac{3}{2}B_1k(z)sin(\frac{\theta}{2})r^{-\frac{5}{2}} - \frac{3}{2}B_2k(z)sin(\frac{\theta}{2})r^{-\frac{3}{2}} \\
&+[\frac{1}{4}B_1sin\frac{\theta}{2} + \frac{3}{4}D_1sin\frac{3}{2}\theta + \frac{15}{2}B_3k(z)sin\frac{3}{2}\theta]r^{-\frac{1}{2}} && (19)\\
\sigma_{zz}^{PS} &= \sigma_{rz}^{PS} = \sigma_{\theta z}^{PS} = 0 && (20)
\end{aligned}$$

where B_n and D_n are constants.

It is noted that terms of singularity $r^{-\frac{5}{2}}$ and $r^{-\frac{3}{2}}$ appear in Eqns (17-19). These terms satisfy the two dimensional conditions in Eqns (12) but violate exact three dimensional boundary conditions on crack surfaces as $\sigma_{r\theta}^{PS}$ is not zero when $\theta = \pm\pi$. Thus, the S and PS state stresses are needed to make the crack surfaces traction free.

Expansion of S State Stresses

Stress function ϕ is expanded as

$$\phi(r,\theta,z) = \sum_{n=1}^{\infty} \phi_n(r,\theta)cos(\frac{n\pi z}{h}) \tag{21}$$

where ϕ_n is determined by

$$\nabla^2\phi_n - (\frac{n^2\pi^2}{h^2})\phi_n = 0 \tag{22}$$

which is obtained by substituting Eqn (21) into Eqn (7).

Each ϕ_n is again expanded as

$$\phi_n(r,\theta) = \sum_{k=1}^{\infty} \phi_{nk}(\theta)r^{k-\frac{3}{2}} \tag{23}$$

or,

$$\phi_n(r,\theta) = (a_{n1}sin\frac{\theta}{2})r^{-\frac{1}{2}} + (a_{n2}sin\frac{\theta}{2})r^{\frac{1}{2}} + [a_{n3}sin\frac{3\theta}{2} + \frac{1}{2}(\frac{n\pi}{h})^2(a_{n1}sin\frac{\theta}{2})]r^{\frac{3}{2}} + ... \tag{24}$$

where a_{ni} (i=1,...) are constants. The corresponding stress components can be obtained according to Eqn(6).

Expansion of PF State Stresses

The stress function is assumed to be (see Gregory(1992))

$$\chi(r,\theta,z) = \sum_{\lambda} \chi_\lambda(r,\theta)E_\lambda(z) \tag{25}$$

where $E_\lambda(z)$ is defined by

$$E_\lambda(z) = (\frac{z}{h} - 1)sin[\lambda(\frac{z}{h} + 1)] + (\frac{z}{h} + 1)sin[\lambda(\frac{z}{h} - 1)] \tag{26}$$

and λ is the root of the equation

$$sin2\lambda + 2\lambda = 0 \tag{27}$$

Without loss of generality, the summation in Eqn (25) will run only through roots whose real parts are positive (see Gregory (1992)). The roots are thus in conjugate pairs.

Substituting Eqn (25) into Eqn (10) leads to

$$\nabla^2\chi_\lambda - (\frac{\lambda}{h})^2\chi_\lambda = 0 \tag{28}$$

Consistent with the expansions of PS stress, χ_λ is assumed to be

$$\chi_\lambda = \sum_{k=1}^{\infty} \chi_{\lambda k}(\theta)r^{k-\frac{3}{2}} \tag{29}$$

or

$$\begin{aligned}
\chi &= [cos\frac{\theta}{2}(\sum d_{\lambda 1}E_\lambda(z))]r^{-\frac{1}{2}} + [cos\frac{\theta}{2}(\sum d_{\lambda 2}E_\lambda(z))]r^{\frac{1}{2}} + \\
&+[cos\frac{3\theta}{2}(\sum d_{\lambda 3}E_\lambda(z)) + \frac{1}{2}cos\frac{\theta}{2}(\sum(\frac{\lambda}{h})^2 d_{\lambda 1}E_\lambda(z))]r^{\frac{3}{2}} + ... && (30)
\end{aligned}$$

where $d_{\lambda i}$ are constants. PF stresses can be easily worked out with above stress function and formulas in Eqn(9)

PS, S and PF stresses individually satisfy not only all equilibrium equations and compatibility equations but also the boundary conditions at the two plate surfaces. However, the boundary conditions at the crack surfaces are satisfied only when the three stresses are combined. These crack surface boundary conditions are

$$\sigma_\theta|_{\theta=\pm\pi} = \sigma_{r\theta}|_{\theta=\pm\pi} = \sigma_{z\theta}|_{\theta=\pm\pi} = 0 \tag{31}$$

Asymptotic Analysis of Order $r^{-\frac{5}{2}}$

Adding the three stresses obtained in the expansions above, the amplitudes of stresses of order $r^{-\frac{5}{2}}$ are

$$\begin{aligned}
\sigma_{rr}^{(-\frac{5}{2})} &= -\sigma_{\theta\theta} = F(z)cos(\frac{\theta}{2}) && (32)\\
\sigma_{r\theta}^{(-\frac{5}{2})} &= F(z)sin(\frac{\theta}{2}) && (33)\\
\sigma_{rz}^{(-\frac{5}{2})} &= \sigma_{\theta z}^{(-\frac{5}{2})} = \sigma_{zz}^{(-\frac{5}{2})} = 0 && (34)
\end{aligned}$$

where

$$F(z) = -\frac{3}{2}B_1 k(z) - \frac{3}{2}\sum a_{n1} cos\frac{n\pi}{h}z + \frac{3}{4}\sum[(1-\nu)E_\lambda(z)'' - \nu\frac{\lambda^2}{h^2}E_\lambda(z)]d_{\lambda 1} \tag{35}$$

and the superscript $(-\frac{5}{2})$ denotes the stress amplitude associated with the singular term $r^{-\frac{5}{2}}$. The boundary conditions in Eqns (31) lead to

$$F(z) = 0 \tag{36}$$

and consequently,

$$\sigma_{rr}^{(-\frac{5}{2})} = \sigma_{\theta\theta}^{(-\frac{5}{2})} = \sigma_{r\theta}^{(-\frac{5}{2})} = \sigma_{rz}^{(-\frac{5}{2})} = \sigma_{\theta z}^{(-\frac{5}{2})} = \sigma_{zz}^{(-\frac{5}{2})} = 0 \tag{37}$$

Asymptotic Analysis of Order $r^{-\frac{3}{2}}$

Similar to the analysis of order $r^{-\frac{5}{2}}$, the boundary conditions in Eqn(31) yield

$$-\frac{3}{2}B_2 k(z) + \frac{1}{2}\sum a_{n2} cos\frac{n\pi}{h}z + \frac{1}{4}\sum[(1-\nu)E(z)_\lambda'' - \nu\frac{\lambda^2}{h^2}E_\lambda(z)]d_{\lambda 2} = 0 \tag{38}$$

and

$$\sum_{n=1}^{\infty}\frac{n\pi}{h}a_{n1} sin\frac{n\pi}{h}z = \sum\frac{\lambda^2}{h^2}d_{\lambda 1}E_\lambda(z)' \tag{39}$$

Using the above relations, all stresses of this order can be shown to vanish identically.

Asymptotic Solution of Order $r^{-\frac{1}{2}}$

Boundary conditions in Eqns (31) require

$$-\{\frac{15}{2}B_3 k(z) - \frac{3}{2}\sum a_{n3} cos\frac{n\pi}{h}z - \frac{3}{4}\sum[(1-\nu)E_\lambda(z)'' - \nu\frac{\lambda^2}{h^2}E_\lambda(z)]d_{\lambda 3}\} =$$
$$-\{\frac{1}{4}\sum\frac{n^2\pi^2}{h^2}a_{n1} cos\frac{n\pi}{h}z - \frac{1}{8}\sum\frac{\lambda^2}{h^2}[(1-\nu)E_\lambda(z)'' - \nu\frac{\lambda^2}{h^2}E_\lambda(z)]d_{\lambda 1}\} \tag{40}$$

$$\sum_{n=1}^{\infty} a_{n2}\frac{n\pi}{h} sin\frac{n\pi}{h}z + \sum\frac{\lambda^2}{h^2}d_{\lambda 2}E_\lambda(z)' = 0 \tag{41}$$

After applying Eqns (40) and (41) in conjunction with Eqn (39), the amplitudes of stresses of order $r^{-\frac{1}{2}}$ are

$$\sigma_{rr}^{(-\frac{1}{2})} = K(z)(5cos\tfrac{\theta}{2} - cos\tfrac{3\theta}{2}) \qquad \sigma_{\theta\theta}^{(-\frac{1}{2})} = K(z)(3cos\frac{\theta}{2} + cos\frac{3\theta}{2}) \tag{42}$$

$$\sigma_{r\theta}^{(-\frac{1}{2})} = K(z)(sin\tfrac{\theta}{2} + sin\tfrac{3\theta}{2}) \qquad \sigma_{zz}^{(-\frac{1}{2})} = (\sum_\lambda\frac{\lambda^4}{h^4}E_\lambda(z)d_{\lambda 1})cos\frac{\theta}{2} \tag{43}$$

$$\sigma_{rz}^{(-\frac{1}{2})} = \sigma_{\theta z}^{(-\frac{1}{2})} = 0 \tag{44}$$

where

$$K(z) = \{1 + \frac{1}{2B_1}\sum_\lambda\frac{\lambda^2}{h^2}[\nu\frac{\lambda^2}{h^2}E_\lambda(z) + (1+\nu)E_\lambda(z)'']d_{\lambda 1}\}B_1 \tag{45}$$

In the above formulas, $d_{\lambda 1}$ are restricted by Eqns (36) and (39). Integration of Eqn (39) gives

$$\sum_{n=1}^{\infty} a_{n1} cos\frac{n\pi}{h}z = -\sum\frac{\lambda^2}{h^2}d_{\lambda 1}E_\lambda(z) - \sum\frac{4\lambda}{h^2}d_{\lambda 1} \tag{46}$$

The last term in the equation above comes from the integration constant which is determined by using Eqn (8). Substituting Eqn (46) into Eqn (36) gives the sole condition for $d_{\lambda 1}$ as

$$2B_1 k(z) = \sum_\lambda[(2-\nu)\frac{\lambda^2}{h^2}E_\lambda(z) + (1-\nu)E_\lambda(z)'' + \frac{8\lambda}{h^2}]d_{\lambda 1} \tag{47}$$

DISCUSSIONS

From the asymptotic analysis above, it appears that the lowest order nonzero three-dimensional crack tip stresses of a plate are of inverse square root singularity with the amplitudes given by Eqns (42-44).

The in-plane components of the three-dimensional near tip singular stress field are of square root type singularity with the intensity factor K(z) varying in the thickness direction. The natural question is whether the corresponding near tip deformation is a state of plane strain, i.e., $\epsilon_{zz} = 0$.

As stated in Section 3, the constants $d_{\lambda 1}$ in Eqn (45) are restricted only by Eqn (47). Differentiating the two sides of Eqn (47) twice with respect to z gives

$$-\frac{2\nu}{1+\nu}B_1 = \sum_\lambda[(2-\nu)\frac{\lambda^2}{h^2}E_\lambda'' + (1-\nu)E_\lambda'''']d_{\lambda 1} \tag{48}$$

or, by using definition of E_λ in Eqn (26),

$$2\nu B_1 = (1-\nu^2)\sum_\lambda\frac{\lambda^4}{h^4}E_\lambda d_{\lambda 1} - \nu(1+\nu)\sum_\lambda\frac{\lambda^2}{h^2}E_\lambda'' d_{\lambda 1} \tag{49}$$

Substitution of results given by Eqns (42-44) into the following expression

$$\epsilon_{zz} = \frac{1+\nu}{E}\sigma_{zz} - \frac{\nu}{E}(\sigma_{rr} + \sigma_{\theta\theta} + \sigma_{zz}) \tag{50}$$

gives

$$\epsilon_{zz} = \frac{1}{E}[-2\nu B_1 + (1-\nu^2)\sum_\lambda\frac{\lambda^4}{h^4}E_\lambda d_{\lambda 1} - \nu(1+\nu)\sum\frac{\lambda^2}{h^2}E_\lambda'' d_{\lambda 1}]cos\frac{\theta}{2} \tag{51}$$

In view of the relation of Eqn (49), it is evident that ϵ_{zz}=0

The result that plane strain deformation prevails over the entire thickness of the plate near the crack tip is somewhat unexpected. One would anticipate a plane stress state close to the traction free plate surfaces, where $\sigma_{zz} = 0$ but not necessarily ϵ_{zz}=0. To further investigate this paradox, we note that Eqn (49) reduces to

$$2B_1 = -(1+\nu)\sum_\lambda\frac{\lambda^2}{h^2}E_\lambda''(\pm h)d_{\lambda 1} \tag{52}$$

at $z = \pm h$ since $E_\lambda(\pm h) = 0$ by definition. It is easy to see from Eqn (45) that

$$K(\pm h) = 0 \tag{53}$$

Consequently, all the components of the singular stress field vanish identically on the free plate surfaces. Thus, on plate surfaces, the distinction between the plane stress and plane strain states vanishes.

The implication of the above finding is that the stress intensity factor K(z) vanishes on the plate surfaces. These results are consistent with the finite element results obtained by Shivakumar and Raju (1990) which indicate that the strain energy release rate seems to drop to zero when approaching the plate surfaces.

At this point, one may surmise whether there is a unique functional form for K(z). Since K(z) depends on $d_{\lambda 1}$ (see Eqn (45)), to answer the above question, we must find out whether the coefficients $d_{\lambda 1}$ can be determined unambiguously. Unfortunately, except for the trivial case when $\nu = 0$ (and thus $d_{\lambda 1} = 0$), $d_{\lambda 1}$ cannot be determined uniquely by the crack tip asymptotic analysis alone (see Su and Sun, 1997).

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