# PROBABILISTIC MECHANICS MODELING OF FATIGUE CRACK GROWTH AND FRACTURE

A.B.O. Soboyejo
Department of Aerospace Engineering, Applied Mechanics and Aviation
The Ohio State University, Columbus, OH 43210

## 1. Introduction

In several important engineering applications, the need to develop simplified mechanistically-based methodologies has become a significant factor in the development of appropriate techniques in the choice of suitable engineering materials for the design of important mechanical components, structures and infrastructures. Several structural materials which have been used in such systems have suffered severe damages due to the initiation and propagation of fatigue cracks, and in several instances, these have resulted in catastrophic structural and mechanical failures. In addition to the above, the importance of environmental and climatological factors, including the effects of other essential variables need to be addressed and incorporated into the formulation of any new methodologies which may be proposed to solve the practical problems in this field. This research had been prompted by the success of the simple phenomenological and deterministic fracture mechanics approaches, during the last three decades, during which the single parameter Paris law had been used. The need for the use of simplified multiparameter equations to assess the combined effects of several essential random variables into one simple equation, and which can affect fatigue crack initiation, propagation and fracture, in structural materials, including the need for probabilistic fatigue life prediction, have become important in many engineering applications.

In view of the important needs in several practical applications, attempts have been made here to address the major issues, with a view to providing practically acceptable and simple engineering methodologies which can provide the much needed solution to problems in several structural and mechanical engineering design applications. The contributions from the results of research in this field are presented in this technical paper.

## 2. Need for multi-parameter fracture mechanics models

The formulation of an appropriate fracture mechanics failure model is critical in the development of the solutions which are needed in this field. Such failure models can be developed using either theoretical or experimental engineering mechanics procedures. As much as possible all the essential random variables which can contribute to the process must be included in the formulation of the model. One important observation is the fact that the rate of crack propagation with time , becomes greater and greater until final fracture occurs in the tertiary zone of fatigue life.

Fatigue crack initiation, propagation and fracture are indeed statistical or stochastic phenomena, which are controlled essentially by multi-parameter random variables. Therefore, any deterministic single variable approaches can only provide limited descriptions of such statistical ranges. Multiparameter approaches which are essentially mechanistically-based, are therefore needed, in order to provide the reliability-based solutions.

## 3. Formulation of fracture mechanics models

In earlier work [1,2] it has been successfully demonstrated that the following empirical formulation can be used to model crack growth rate, Y:

$$Y = \frac{da}{dN} = n_0 \prod_{i=1}^{k} X_i^{n_i} \qquad (1)$$

where a is the crack size, N is the number of cycles at a given stress level for the crack to develop to the size a, $X_i$'s are the essential random variables which can contribute to the phenomena in such as, $\Delta K$, stress intensity factor range, environmental temperature, physical and chemical nature of the structural material, and any other random variables which need to be considered.

Since the rate of the process as indicated in equation (1) increases with any increase in the number of cycles until failure, the choice of a reasonable assumption is finally made, after several other assumptions had been tried. The final choice is as follows:

$$\frac{da}{dN} = \frac{[\alpha(a - a_0) + \beta]^2}{\beta} \qquad (2)$$

where $\alpha$ and $\beta$ are functions of other essential variables ($X_1$, $X_2$, .... $X_k$) which can also affect crack growth and fracture and which can be expressed by the following mathematical functions:

$$\alpha = g_1(X_1, X_2, \ldots X_k)$$
$$\beta = g_2(X_1, X_2, \ldots X_k) \qquad (3)$$

Equation (2) is important because it gives good physical representation to the phenomenon of crack growth rate, which increases non-linearly from an initial value of $\beta$, to the critical or accelerating growth rate when final fracture occurs, particularly during the tertiary stage of fatigue fracture. For a given set of the random variables ($X_1$, $X_2$, .... $X_k$), the values of $\alpha$ and $\beta$ are constants. However, whenever any changes occur in these random variables, such as environmental considerations, or fatigue stresses in the structural material, the values of $\alpha$ and $\beta$ can also change.

Integrating mathematically equations (2) and from the initial conditions ($N_0$, $a_0$), up to (N,a); and with the appropriate changes in the mathematical variables, we have:

$$N = N_0 + (a - a_0)[\alpha(a - a_0) + \beta]^{-1} \qquad (4)$$

Equation (4) gives the prediction of life as a function of all the essential variables. It can be shown from equation 4 that the crack size, a, is given by:

$$a = a_0 + (N - N_0)\beta[1 - (N - N_0)\alpha]^{-1} \qquad (5)$$

From equation 5, the fracture life, $N_f$, can be shown to be:

$$N_f = N_0 + (\alpha^{-1}) \qquad (6)$$

From Equation 2 at the initial point when a=$a_0$, and N=$N_0$; we have the important result:

$$\left.\frac{da}{dN}\right|_{\substack{a=a_0\\N=N_0}} = \beta \qquad (7)$$

In other words, the variables $\alpha$ and $\beta$, which are expressed in equations 6 and 7; and as already shown in equation 3 are related directly not only to the engineering mechanics of this problem, but also to the physical, chemical, environmental and other essential random variables associated with the fracture growth and fracture.



In the tertiary fatigue zone, and for the special case which is widely encountered in practice, when the logarithm of the crack growth rate, In(da/dN) is plotted against $|\overline{\Delta K}|$, as defined in Fig. 2 and where $\Delta K$ is the stress intensity factor range; the following similar mathematical relationship has been successfully developed and can be applied to such problems (See Fig. 2):

$$|\overline{\Delta K}| = \frac{|\overline{Y}|}{\overline{\alpha}|\overline{Y}| + \overline{\beta}} \qquad (8a)$$

$$\frac{|\overline{Y}|}{|\overline{\Delta K}|} = \overline{\alpha}|\overline{Y}| + \overline{\beta} \qquad (8b)$$

where $\overline{\Delta K}$ = In($\Delta K/\Delta K_0$); $\overline{Y}$ = In($Y/Y_0$); $\overline{\alpha}$ and $\overline{\beta}$ are new constants, similar to the constants $\alpha$ and $\beta$. From Figure 2, it is quite evident that special care should be taken practically to estimate the location $Y_0$, which is the point where the tangency of the fatigue curve takes off into the tertiary zone. $|\overline{\Delta K}|$ is a measure of the increase in the stress intensity factor range, commencing from the beginning of the tertiary zone of fatigue crack growth and fracture. It has been shown that when experimental mechanics data of $|\overline{Y}||\overline{\Delta K}|^{-1}$ are plotted against $|\overline{Y}|$ as indicated in equation 8b, good correlations are obtained with the use of this model; and from the regression analysis, the values of $\overline{\alpha}$ and $\overline{\beta}$ can be obtained.

4. Probabilistic fracture mechanics considerations

In order to account for the possible variabilities in the random variables and also in order to have better understanding of the mechanisms, which are chiefly responsible for the phenomena of fatigue crack growth and fracture, inputs of mathematical statistics and probability theory must be linked together with the deterministic knowledge of fracture mechanics.

It therefore becomes necessary to develop probabilistic fracture mechanics relationships for any state of the process from (N, a) up to fracture, which will occur at ($N_f$, $a_f$). In this development the following are in fact the possible random variables of interest: $a_0$, a, $N_0$, N, $\alpha$, $\beta$, and $X_1$, $X_2$, $X_3$, ... $X_k$. $\alpha$ and $\beta$ are functions of the random variables $X_1$, $X_2$, $X_3$, ... $X_k$. Since for any given set of conditions $X_1$, $X_2$, $X_3$, ... $X_k$; $\alpha$ and $\beta$ are constants; equation 2 can be used simply to estimate the possible variation, $\Delta a$, in the crack size, as related to the possible corresponding variation $\Delta N$, in life N, thus:

$$(\Delta a) = \beta[1 - \beta(N - N_0)\alpha]^{-1}(\Delta N) \qquad (9)$$

Using the above results, therefore, at any point in time during the propagation of fatigue cracks and up to fracture; the probability of failure of the structural material F(a) and the reliability, R(a), which is the probability of successful operation of the structural materials, up to the level when the cracks have developed to the magnitude, a, can be shown, respectively to be:

$$F(a) = \int_0^a f(a^1)da^1 \text{ for } a \geq 0 \qquad (10)$$

where f(a) is the probability density function for the crack size, of magnitude a, and at life N. If this is assumed to be a normal or Gaussian distribution, then, we have:

$$f(a^1) = \frac{1}{(\Delta a)\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{a^1 - a}{\Delta a}\right)^2\right\} \quad \text{for } a^1 \geq 0 \qquad (12)$$

The mean value of the crack size a, is given by equation 5.

As far as life prediction is concerned, at any point N, the corresponding failure probability of having the value at any life N, F(N); and the corresponding value of the reliability R(N) are as follows:

$$F(N) = \int_0^N f(N^1)dN^1 \quad \text{for } N \geq 0 \qquad (13)$$

$$R(N) = \int_N^\infty f(N^1)dN^1 \quad \text{for } N \geq 0 \qquad (14)$$

where f(N) is the probability density function for life N at the crack size, a. If this is assumed to be a normal or Gaussian distribution, then we have:

$$f(N^1) = \frac{1}{(\Delta N)\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{N^1 - N}{\Delta N}\right)^2\right\} \quad \text{for } N^1 \geq 0 \qquad (15)$$

The mean value of life N is given by equation 4.

The relationship between crack size, a, and life N, shows that a is increasing, as N is also increasing up to fracture which occurs at the point $N_f$, $a_f$. Therefore, the following results can be established a the points [a, N] and [ $N \pm \Delta N$, $a \pm \Delta a$].

$$R(a) = R(N) \qquad (16)$$

$$F(a) = F(N) \qquad (17)$$

$$f(a)\bullet(\Delta a) = f(N) \bullet (\Delta N) \qquad (18)$$

The results shown above in equations 16, 17, and 18 are important conclusions which can be used in several important engineering applications [3].

It is necessary to mention here that a model which has the same mathematical form as the one which has been developed here in this paper, had been successfully used to model the stress transfer from concrete to steel, due to creep phenomenon in reinforced concrete structures (7). Other studies in related fields of non-linear mechanics have also been successfully carried out by others [5,6], using the same mathematical model as indicated essentially in equations 4 and 5.

5. Practical example (See Fig 3)

From the experimental mechanics data gathered for the fatigue crack growth of aluminum alloys, 2219-T851 at R=0.1 in reference [7], the model proposed here has been used to predict the fatigue crack growth and fracture. Good correlations are obtained when the data are fitted against the model, using equations 8a and 8b. For the tertiary fatigue zone which is mainly of interest here, the predictions given by the new model proposed here are compared with the actual fatigue data obtained from the experimental mechanics procedures in Reference 7. Table I gives the details of some of the results of such comparisons. It is quite clear from these results shown in Table I that the agreements between the predictions from this model, and the actual experimental mechanics fatigue data, are quite encouraging and should be acceptable for use in several practical engineering applications.



6. Conclusions and Summary

(i) A simplified methodology is hereby presented for the characterization of fatigue crack growth and fracture. The method should have significant impact on engineering design applications.

(ii) Even though the methodology proposed here had been developed for tertiary stage of fatigue crack growth and fracture, the result of the methodology proposed here can be mathematically extended with some modifications and applied to the prediction of fatigue crack growth of structural materials for either the primary stage or the secondary stage of fatigue crack growth.

(iii) The model proposed here demonstrates significant contributions which probabilistic fracture mechanics can make in the modeling of fatigue crack growth and fracture, in several important engineering applications.

7. References

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Table I

| (1) In $(\Delta K/\Delta K_0)$ From data | (2) In (da/dN) Prediction from model | (3) In (da/dN) From data | (4) Ratio of (2/3) |
|---|---|---|---|
| 0 | 6.00 | 6.00 | 1.0000 |
| 1 | 5.92 | 5.90 | 1.0034 |
| 3 | 5.86 | 5.88 | 0.9966 |
| 5 | 5.70 | 5.73 | 0.9948 |
| 8 | 5.40 | 5.42 | 0.9963 |
| 10 | 4.90 | 4.88 | 1.0004 |
| 13 | 4.18 | 4.20 | 0.9952 |

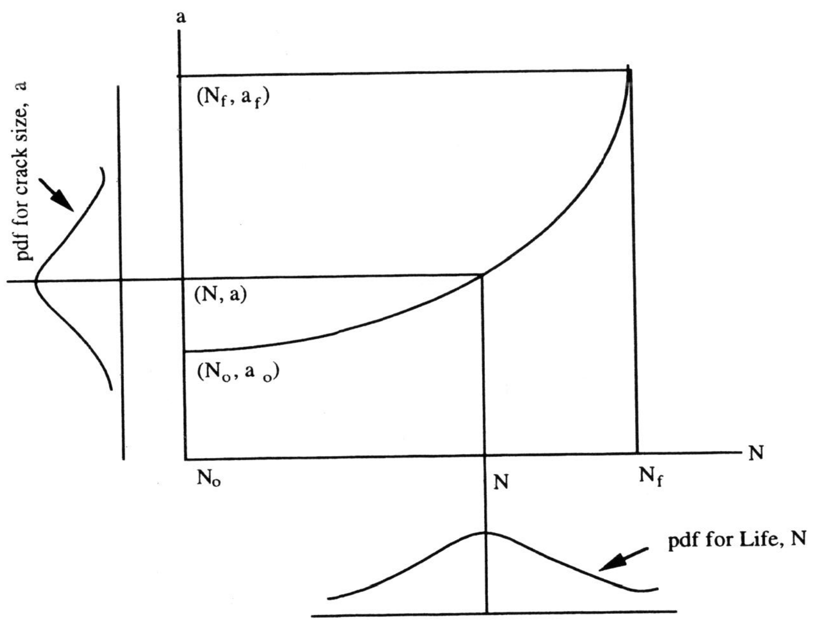


Fig. 1 Crack Size (a) against Life (N)



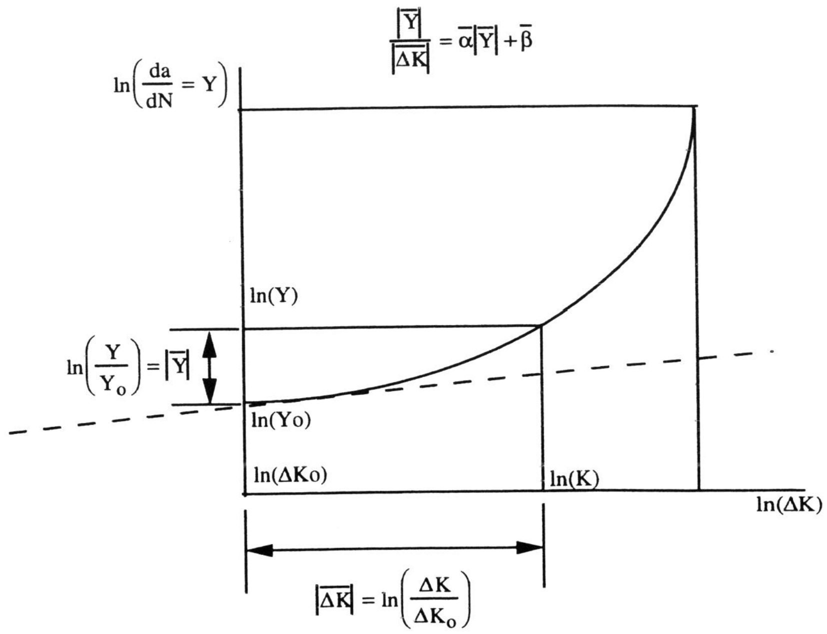


Fig. 2 Plot of ln (da/dN) against ln(ΔK)

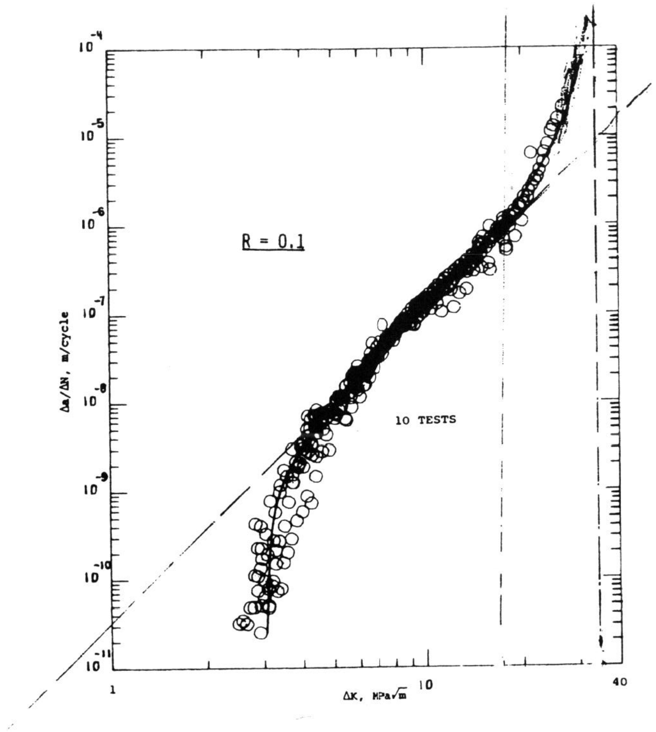


Fig. 3 Crack Growth Data of Aluminum Alloy 2219-T851 at R = 0.1 (Ref. 7)