ANALYTIC SOLUTION FOR FATIGUE LIFE PREDICTION UNDER BROAD BAND RANDOM LOADING

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ABSTRACT

The analytic solution of equivalent stress range based on the power spectral density of the stress in the critical position of structures or components and the statistical theory of the stress peak distribution of a stationary Gaussian random process is deduced for a quick fatigue life assessment under broad band random loading. The equivalent stress range in the analytic solution can be calculated directly by frequency domain parameters, obviating the need for cycle-by-cycle counting and damage summation. The fatigue life for structures or components under broad band random loading can be predicted quickly by means of this solution with constant-amplitude $S-N$ curve or $da/dN - \Delta K$ curve. The model offers a great advantage when compared with any other existing approximate models which are based on the stress peak distribution.

KEY WORDS

Analytic solution, equivalent stress range, fatigue life prediction, broad band random loading

INTRODUCTION

Fatigue is one of the most common causes of the in-service failure for components and structures. Most of structures are subjected to the Broad Band Random Loading(BBRL). For fatigue life assessment under BBRL, the most convenient method is the statistical analytic approach. For a stationary random loading with Gaussian amplitude distribution, the Probability Density Function(PDF) of stress peaks can be obtained by the Power Spectral Density(PSD) of the stress in the critical position of structures. From the PDF, the equivalent stress range S_h under BBRL is calculated. By application of experimental date(such as $S-N$ curve, $da/dN - \Delta K$ curve, etc.), the fatigue life can be predicted. Consequently, the S_h calculation is very important.

Through the PSD in the critical position of structures, some statistical quantities for fatigue assessment are obtained, which are showed in table 1 (Wirsching,1980; Chow and Li,1991)

Table 1 Definition and expression of statistical quantities

Definition	expression
Kth moment of spectral density fuction G(f), M_k	$\int_{-\infty}^{\infty} f^k G(f)df$
Root-mean-square stress, σ	$\sqrt{M_0}$
Expected rate of zero crossing with (+) slop, f_0	$\sqrt{M_2/M_0}$
Expected rate of peaks, n_0	$\sqrt{M_4/M_2}$
Irregularity factor, α	f_0/n_0
spectral width parameter, ϵ	$\sqrt{1-\alpha^2}$

The PDF of stress peaks under BBRL can be expressed as(Rice,1955):

$$p(s) = \frac{1}{(2\pi)^{\frac{1}{2}}\sigma}(1-\alpha^2)^{\frac{1}{2}} exp\left[-s^2(2\sigma^2(1-\alpha^2))^{-1}\right] + \frac{s}{2\sigma^2}\alpha\left[1 + erf(\frac{s}{\sigma(2\alpha^{-2}-2)^{\frac{1}{2}}})\right] exp(\frac{-s^2}{2\sigma^2}) \quad (1)$$

where s is stress peak and $erf(x)$ is error function. $erf(x)$ is defined as $erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x exp(-t^2)dt$. By putting $x = \alpha s/(\sqrt{2}\sigma\epsilon)$, Equation(1) can be simplified to

$$p(x) = \frac{\epsilon}{\sqrt{2\pi}\sigma} exp\left[-(\frac{x}{\alpha})^2\right] + \frac{\epsilon x}{\sqrt{2}\sigma}\left[1 + erf(x)\right] exp\left[-(\frac{\epsilon x}{\alpha})^2\right] \quad (2)$$

According to the $S-N$ curves or the Paris formula with the Palmgren-Miner linear accumulation damage rule, the equivalent stress range under random loading can be expressed as(Chow and Li,1991):

$$S_h = \left[\frac{\sum(\Delta S_i)^m n_i}{\sum n_i}\right]^{\frac{1}{m}} = \left[\int_0^{\infty}(2s)^m p(s)ds\right]^{\frac{1}{m}} \quad (3)$$

where ΔS_i is stress range for an individual cycle, n_i is numbers of cycles corresponding to ΔS_i and m is material constant. Substituting Equation(2) into Equation(3) yields

$$S_h = 2\sqrt{2}\sigma\left[\frac{\epsilon^{m+2}}{2\sqrt{\pi}}\Gamma(\frac{m+1}{2}) + \frac{\alpha}{2}\Gamma(\frac{m+2}{2}) + \alpha Z\right]^{\frac{1}{m}} \quad (4)$$

where

$$Z = \int_0^{\infty} erf(x)(\frac{\epsilon x}{\alpha})^{m+1} exp\left[-(\frac{\epsilon x}{\alpha})^2\right] d(\frac{\epsilon x}{\alpha}) \quad (5)$$

and the Γ-function is defined as

$$\Gamma(x) = 2\int_0^{\infty} y^{2x-1} exp(-y^2)dy$$

for $x > 0$.

Because Equation(5) contains the transcendental function $erf(x)$, an exact solution is difficult to obtain. Many approximate models are suggested, such as Chaudhury and Dover model(Chaudhury and Dover, 1985), Kam and Dover model(Kam and Dover,1988), Chow and Li model(Chow and Li, 1991),etc., but every model has its shortcomings and suitable range. In this paper, the analytical solution of Equation (5) is deduced, which completely solves the problem. This model has a greater advantage than that of any other model based on the stress peak distribution.

REVIEW OF APPROXIMATE MODELS

The model of Chaudhury and Dover(CDM)

As the value of $erf(x)$ falls between zero and one, $erf(x)$ is assumed to be 0.5 in the Chaudhury and Dover model, and Equation(5) was changed as (Chaudhury and Dover,1985):

$$Z = \frac{1}{2}\int_0^{\infty}(\frac{\epsilon x}{\alpha})^{m+1} exp\left[-(\frac{\epsilon x}{\alpha})^2\right] d(\frac{\epsilon x}{\alpha}) = \frac{1}{4}\Gamma(\frac{m+2}{2}) \quad (6)$$

It can be seen that the model is simplified by the use of 0.5 to replace $erf(x)$ in the integration range$(0,\infty)$ and inevitably causes certain error.

The model of Kam and Dover(KDM)

Kam and Dover modified the CDM. Equation(5) in the Kam and Dover model is approximately expressed as(Kam and Dover,1988):

$$Z = erf(x)\int_0^{\infty}(\frac{\epsilon x}{\alpha})^{m+1} exp\left[-(\frac{\epsilon x}{\alpha})^2\right] d(\frac{\epsilon x}{\alpha}) = \frac{1}{2} erf(x)\Gamma(\frac{m+2}{2}) \quad (7)$$

From iterative calculation and fitting analysis, the error function $erf(x)$ of Equation(7) can be approximately expressed as(Kam,1990):

$$erf(x) = 0.3012\alpha + 0.4916\alpha^3 + 0.9181\alpha^8 - 2.3534\alpha^4 - 3.3307\alpha^5 + 15.6524\alpha^6 - 10.7846\alpha^7 \quad (8)$$

for $\alpha < 0.96$ and $erf(x) = 1.0$ for $\alpha \geq 0.96$.

According to integration theory, the $erf(x)$ can not be excluded from the integration. Besides, as $x = \alpha s/(\sqrt{2}\sigma\epsilon)$, $erf(x)$ is not only the function of α, but it also depends on s and σ. Calculations show that the error increases as ϵ increases.

The model of Chow and Li(CLM)

C.L.Chow and D.L.Li obtained a series solution of equivalent stress range S_h by using the series expression of $erf(x)$. The series solution is as follows(Chow and Li,1991):

A series expression by using to replace $erf(x)$ can be written as

$$I_1(x) = \frac{2}{\sqrt{\pi}} exp(-x^2)\sum_{k=0}^{\infty}\frac{2^k x^{2k+1}}{(2k+1)!!} \quad (9)$$

Substituting Equation(9) into Equation(5) yields

$$Z = \frac{\epsilon^{m+2}}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{2^k \alpha^{2k+1}}{(2k+1)!!} \Gamma(\frac{2k+m+3}{2}) \tag{10}$$

An alternative solution can be obtained by using the asymptotic expansion

$$I_2(x) = 1 - exp(-x^2)\frac{1}{\sqrt{\pi}x}\left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k(2k-1)!!}{(2x^2)^k}\right] \tag{11}$$

This expansion leads to the following form for Z:

$$Z = \frac{1}{2}\Gamma(\frac{m+2}{2}) - \frac{\epsilon^{m+2}}{2\sqrt{\pi}\alpha}\Gamma(\frac{m+1}{2}) - \frac{\epsilon^{m+2}}{2\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{(-1)^k(2k-1)!!}{2^k \alpha^{2k+1}} \Gamma(\frac{m-2k+1}{2}) \tag{12}$$

By numerical analysis, Chow and Li turned the above two series into a single solution, that is a united expression

$$Z = \frac{\epsilon^{m+2}}{\sqrt{\pi}} \sum_{k=0}^{k_1} \frac{2^k \alpha^{2k+1}}{(2k+1)!!} \Gamma(\frac{2k+m+3}{2}) \tag{13}$$

for $\epsilon \geq \epsilon_1$, and

$$Z = \frac{1}{2}\Gamma(\frac{m+2}{2}) - \frac{\epsilon^{m+2}}{2\sqrt{\pi}\alpha}\Gamma(\frac{m+1}{2}) + \frac{\epsilon^{m+2}}{4\sqrt{\pi}\alpha^3}\Gamma(\frac{m-1}{2}) \tag{14}$$

for $\epsilon < \epsilon_1$, in which ϵ_1 and k_1 determined by a computer program.

In fact, $I_1(x)$ and $I_2(x)$ can be used to express $erf(x)$ only x in a certain range. Therefore, it is not very satisfactory to simply replace $erf(x)$ in Eqution(5), as in the integration range of $(0, \infty)$. Although the united formula can improve the accuracy of the calculation, it still has some limitations.

Thus it is evident that all the approximate models suffer inaccuracy and limitation.

DERIVATION OF THE ANALYTICAL SOLUTION

Equation(5) can be transformed as follows

$$\begin{aligned}
Z &= \int_0^{\infty} erf(x)(\frac{\epsilon x}{\alpha})^{m+1} exp\left[-(\frac{\epsilon x}{\alpha})^2\right] d(\frac{\epsilon x}{\alpha}) \\
&= \int_0^{\infty} \left[\frac{2}{\sqrt{\pi}}\int_0^x exp(-t^2)dt\right] (\frac{\epsilon x}{\alpha})^{m+1} exp\left[-(\frac{\epsilon x}{\alpha})^2\right] d(\frac{\epsilon x}{\alpha}) \\
&= -\frac{1}{2}\int_0^{\infty} \left[\frac{2}{\sqrt{\pi}}\int_0^x exp(-t^2)dt\right] (\frac{\epsilon x}{\alpha})^m d\left(exp\left[-(\frac{\epsilon x}{\alpha})^2\right]\right) \\
&= -\frac{1}{\sqrt{\pi}}\left[(\frac{\epsilon x}{\alpha})^m \left[\int_0^x exp(-t^2)dt\right] exp\left[-(\frac{\epsilon x}{\alpha})^2\right]\right]_0^{\infty} \\
&+ \frac{1}{\sqrt{\pi}}\int_0^{\infty} m(\frac{\epsilon x}{\alpha})^{m-1}\frac{\epsilon}{\alpha}\left[\int_0^x exp(-t^2)dt\right] exp\left[-(\frac{\epsilon x}{\alpha})^2\right] dx
\end{aligned}$$

$$\begin{aligned}
&+ \frac{1}{\sqrt{\pi}}\int_0^{\infty} (\frac{\epsilon x}{\alpha})^m exp(-x^2) exp\left[-(\frac{\epsilon x}{\alpha})^2\right] dx \\
&= \frac{1}{\sqrt{\pi}}\int_0^{\infty} m(\frac{\epsilon x}{\alpha})^{m-1}\left[\int_0^x exp(-t^2)dt\right] exp\left[-(\frac{\epsilon x}{\alpha})^2\right] d\left(\frac{\epsilon x}{\alpha}\right) \\
&+ \frac{1}{\sqrt{\pi}}\int_0^{\infty} (\frac{\epsilon x}{\alpha})^m exp\left[-(\frac{x}{\alpha})^2\right] dx \\
&= \frac{m}{2}\int_0^{\infty} erf(x)(\frac{\epsilon x}{\alpha})^{m-1} exp\left[-(\frac{\epsilon x}{\alpha})^2\right] d(\frac{\epsilon x}{\alpha}) + \frac{\alpha\epsilon^m}{2\sqrt{\pi}}\Gamma(\frac{m+1}{2})
\end{aligned} \tag{15}$$

By defining

$$I(m) = \int_0^{\infty} erf(x)(\frac{\epsilon x}{\alpha})^{m+1} exp\left[-(\frac{\epsilon x}{\alpha})^2\right] d(\frac{\epsilon x}{\alpha}) \tag{16}$$

$$I(m-2) = \int_0^{\infty} erf(x)(\frac{\epsilon x}{\alpha})^{m-1} exp\left[-(\frac{\epsilon x}{\alpha})^2\right] d(\frac{\epsilon x}{\alpha}) \tag{17}$$

the recurrence formula can be yielded, that is

$$Z = I(m) = \frac{m}{2}I(m-2) + \frac{\alpha\epsilon^m}{2\sqrt{\pi}}\Gamma(\frac{m+1}{2}) \tag{18}$$

When m is an even number, Equation(18) becomes

$$\begin{aligned}
Z &= \frac{m}{2}\frac{m-2}{2}\cdots\frac{2}{2}I(0) + \frac{\alpha\epsilon^m}{2\sqrt{\pi}}\Gamma(\frac{m+1}{2}) + \frac{m}{2}\frac{\alpha\epsilon^{m-2}}{2\sqrt{\pi}}\Gamma(\frac{m-1}{2}) \\
&+ \cdots + \frac{m}{2}\frac{m-2}{2}\cdots\frac{4}{2}\frac{\alpha\epsilon^2}{2\sqrt{\pi}}\Gamma(\frac{2+1}{2}) \\
&= (\frac{m}{2})!I(0) + \frac{\alpha}{2\sqrt{\pi}}\sum_{k=1}^{m/2}\frac{(\frac{m}{2})!}{k!}\epsilon^{2k}\Gamma(\frac{2k+1}{2})
\end{aligned} \tag{19}$$

Using Equation(16) leads to

$$\begin{aligned}
I(0) &= \int_0^{\infty} erf(x)(\frac{\epsilon x}{\alpha}) exp\left[-(\frac{\epsilon x}{\alpha})^2\right] d(\frac{\epsilon x}{\alpha}) \\
&= -\frac{1}{2}\int_0^{\infty}\left[\frac{2}{\sqrt{\pi}}\int_0^x exp(-t^2)dt\right] d\left(exp\left[-(\frac{\epsilon x}{\alpha})^2\right]\right) \\
&= -\frac{1}{\sqrt{\pi}}\left[\left[\int_0^x exp(-t^2)dt\right] exp\left[-(\frac{\epsilon x}{\alpha})^2\right]\right]_0^{\infty} \\
&+ \frac{1}{\sqrt{\pi}}\int_0^{\infty} exp(-x^2) exp\left[-(\frac{\epsilon x}{\alpha})^2\right] dx \\
&= \frac{1}{\sqrt{\pi}}\int_0^{\infty} exp\left[-(\frac{x}{\alpha})^2\right] dx \\
&= \frac{\alpha}{2}
\end{aligned} \tag{20}$$

Substituting Equation(20) into Equation(19) yields

$$Z = \frac{\alpha}{2}(\frac{m}{2})! + \frac{\alpha}{2\sqrt{\pi}}\sum_{k=1}^{m/2}\frac{(\frac{m}{2})!}{k!}\epsilon^{2k}\Gamma(\frac{2k+1}{2}) \tag{21}$$

When m is odd number, from the recurrence formula(18), Z can be expressed as

$$Z = \frac{m}{2}\frac{m-2}{2}\cdots\frac{3}{2}I(1) + \frac{\alpha\epsilon^m}{2\sqrt{\pi}}\Gamma(\frac{m+1}{2}) + \frac{m}{2}\frac{\alpha\epsilon^{m-2}}{2\sqrt{\pi}}\Gamma(\frac{m-1}{2})$$

$$+\cdots+\frac{m}{2}\frac{m-2}{2}\cdots\frac{5}{2}\frac{\alpha\epsilon^3}{2\sqrt{\pi}}\Gamma(\frac{3+1}{2})$$

$$= \frac{m!!}{2^{\frac{m-1}{2}}}I(1) + \frac{\alpha}{2\sqrt{\pi}}\sum_{k=2}^{(m+1)/2}\frac{2^{k-1}m!!}{2^{\frac{m-1}{2}}(2k-1)!!}\epsilon^{2k-1}\Gamma(k) \qquad (22)$$

From Equation(16), I(1) can be derived as

$$I(1) = \frac{1}{\sqrt{\pi}}\int_0^\infty\int_0^x exp(-t^2)exp\left[-(\frac{\epsilon x}{\alpha})^2\right]dtd(\frac{\epsilon x}{\alpha}) + \frac{\alpha\epsilon}{2\sqrt{\pi}}\Gamma(1) \qquad (23)$$

By putting $y = \epsilon x/\alpha$, Equation(23) can be simlified to

$$I(1) = \frac{1}{\sqrt{\pi}}\int_0^\infty\int_0^{\frac{\alpha}{\epsilon}y} exp(-t^2)exp(-y^2)dtdy + \frac{\alpha\epsilon}{2\sqrt{\pi}}\Gamma(1) \qquad (24)$$

From integral transformation(see Fig.1), I(1) becomes

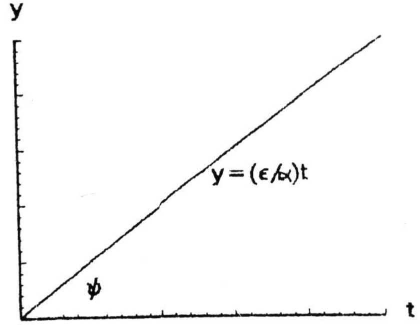

Figure 1: $y - t$ curve

$$I(1) = \frac{1}{\sqrt{\pi}}\int_\psi^{\frac{\pi}{2}}d\theta\int_0^\infty exp(-r^2)rdr + \frac{\alpha\epsilon}{2\sqrt{\pi}}\Gamma(1)$$

$$= \frac{1}{2\sqrt{\pi}}\left[\frac{\pi}{2} - tg^{-1}(\frac{\epsilon}{\alpha})\right] + \frac{\alpha\epsilon}{2\sqrt{\pi}}\Gamma(1) \qquad (25)$$

Substituting Equation(25) into Equation(22) yields

$$Z = \frac{m!!}{2^{\frac{m+1}{2}}\sqrt{\pi}}\left[\frac{\pi}{2} - tg^{-1}(\frac{\epsilon}{\alpha})\right] + \frac{\alpha m!!}{2^{\frac{m+1}{2}}\sqrt{\pi}}\sum_{k=1}^{(m+1)/2}\frac{2^{k-1}}{(2k-1)!!}\epsilon^{2k-1}\Gamma(k) \qquad (26)$$

Therefor the analytic solution of the equivalent stress range is

$$S_h = 2\sqrt{2}\sigma\left[\frac{\epsilon^{m+2}}{2\sqrt{\pi}}\Gamma(\frac{m+1}{2}) + \frac{\alpha}{2}\Gamma(\frac{m+2}{2}) + \alpha Z\right]^{\frac{1}{m}} \qquad (27)$$

where

$$Z = \frac{\alpha}{2}(\frac{m}{2})!\left[1 + \frac{1}{\sqrt{\pi}}\sum_{k=1}^{m/2}\frac{\epsilon^{2k}}{k!}\Gamma(\frac{2k+1}{2})\right] \qquad (28)$$

as m is an even number and

$$Z = \frac{m!!}{2^{\frac{m+1}{2}}\sqrt{\pi}}\left[\frac{\pi}{2} - tg^{-1}(\frac{\epsilon}{\alpha}) + \alpha\sum_{k=1}^{(m+1)/2}\frac{2^{k-1}}{(2k-1)!!}\epsilon^{2k-1}\Gamma(k)\right] \qquad (29)$$

as m is an odd number.

DISCUSSION

In one extreme case, at the Rayleigh peak distribution, which corresponds to $\epsilon = 0$(or $\alpha = 1$), the results can be obtained from Equation(27)~(29) as

$$S_h = 2\sqrt{2}\sigma\left[\Gamma(\frac{m+2}{2})\right]^{\frac{1}{m}} \qquad (30)$$

At the Gaussian peak ditribution, that is for $\epsilon = 1$(or $\alpha \to 0$), the solution can be obtained from Equation(27) ~(29), which is

$$S_h = 2\sqrt{2}\sigma\left[\frac{1}{2\sqrt{\pi}}\Gamma(\frac{m+1}{2})\right]^{\frac{1}{m}} \qquad (31)$$

According to the PDF of the Rayleigh ditribution and Gaussian distribution it has been shown that(Chow and Li,1991) Equation(30) and (31) are the correct solutions of the two extreme cases respectively. So, the analytical solution is suitable to all cases($\epsilon = 0 \sim 1$).

Digital results of S_h are showed in Fig.2, indicating that the S_h has a very good changing property. Therefore, when m is not an integer, a linear interpolation scheme can be used to evaluate the S_h from the two nearest m values with integer m.

CONCLUSION

The paper deals with analytical solution for calculating the equivalent stress range under wide band random loading and it is based on the stress peak distribution of a stationary Gaussian process . The formula of the equivalent stress range is

$$S_h = 2\sqrt{2}\sigma\left[\frac{\epsilon^{m+2}}{2\sqrt{\pi}}\Gamma(\frac{m+1}{2}) + \frac{\alpha}{2}\Gamma(\frac{m+2}{2}) + \alpha Z\right]^{\frac{1}{m}}$$

where

$$Z = \frac{\alpha}{2}(\frac{m}{2})!\left[1 + \frac{1}{\sqrt{\pi}}\sum_{k=1}^{m/2}\frac{\epsilon^{2k}}{k!}\Gamma(\frac{2k+1}{2})\right]$$

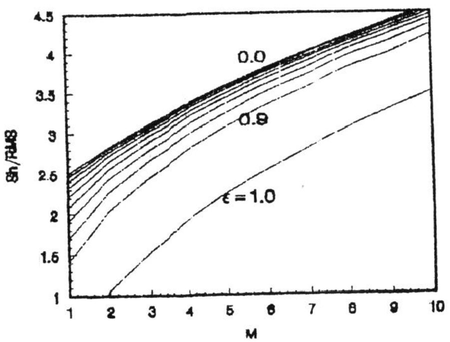

Figure 2: S_h/σ versus m

as m is an even number and

$$Z = \frac{m!!}{2^{\frac{m+1}{2}}\sqrt{\pi}}\left[\frac{\pi}{2} - tg^{-1}(\frac{\epsilon}{\alpha}) + \alpha \sum_{k=1}^{(m+1)/2} \frac{2^{k-1}}{(2k-1)!!}\epsilon^{2k-1}\Gamma(k)\right]$$

as m is an odd number.

The solution is valued for the cases($\epsilon = 0 \sim 1$). The equivalent stress range can be calculated by the frequent domain parameters (σ, α, ϵ) and the fatigue life can be predicted by means of this solution. The model has a great advantage over any other approximate model which is based on the stress peak distribution.

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