RESEARCH PAPER

CREEP FAILURE IN LINEAR AND NONLINEAR VISCOELASTIC MATERIALS

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Objectives

In all structural components the pre-existing cracks of unknown magnitude decrease the confidence level associated with the classic machine design methodology based on stress analysis of undamaged component as exemplified by the standard procedures of Strength of Materials and Theory of Elasticity. The degree of uncertainty is usually reflected by the variety of safety factors resulting from an empirical database and accepted almost universally in various design projects.

The primary objective of the paper is to provide a more rational basis for evaluation of the residual strength of damaged material, which may contain pre-existing cracks or micro-defects. The fail-safe design concept will be adopted, basing on the history of crack growth. The paper will focus on the phenomenon of time-dependent fracture propagation by creep mechanism.

Nomenclature

E_1, E_2, η	elastic moduli and dashpots coefficients characterizing standard linear solids
t	time
τ	The characteristic relaxation time for the standard linear solid
θ	nondimensional time, $\theta = t/\tau$
E_0, E_∞	The glassy (short time $t = 0$) and rubbery (long time $t = \infty$) elastic moduli
β	ratio E_1/E_2
K_G	the instantaneous critical K-factor
K	the applied K-factor
σ_0	applied constant stress
σ_G	Griffith critical stress evaluated at glassy state, i.e., $t = 0$
σ_∞	long-time tensile stress
n	ratio $[\sigma_G/\sigma_0]^2$
t_I	time elapsed from the instant of load application to the point of termination of the dormant stage of the crack development.
ρ	the structural constant related to the size of the process zone
a_0	initial crack length
a	current crack length
X	ratio $= a/a_0$
Δ	process zone size
$\dot{a}$	creep growth rate
θ_{cr}	nondimensional time which elapses from the instant of crack growth to the onset of catastrophic failure
Γ	reciproical of the nondimensional rate of crack growth, $\Gamma = \frac{a_0}{\tau}\frac{1}{da/dt}$

GROWTH OF VISCOELASTIC CRACK

The origin of the term VISCOELASTICITY lies in the simple models, such as those of Maxwell, Voigt, or the standard linear solid, which consist of springs and dashpots. The springs are elastic and represent the behavior of a Hookean solid, the dashpots are viscous and model the response of a Newtonian liquid, hence the Theory of Viscoelasticity describes the behavior of continua which exhibit a mixture of solid and liquid response.

While all solids under appropriate conditions will exhibit to various degrees viscoelastic behavior or sensitivity to the rate of loading in their response to mechanical loads, polymers appear particularly susceptible in even the most benign environment. Furthermore, polymers are finding more applications in which their load carrying capacity becomes an integral part of the performance of the structure.

Consequently, time-dependent fracture plays an important role in determining the service life of polymeric materials as well as viscoelastic components. Today, most of the fracture mechanics approaches to crack propagation in polymeric structures have modeled the material as being either linear or nonlinear viscoelastic.

While studying response of linear viscoelastic solids to existing "dormant" cracks contained within the solid Wnuk and Knauss (1970)(9) have found that the process of failure development can be divided into two stages:

1) Latent stage where $\dot{a} = 0$, i.e., the crack does not propagate;

2) Crack propagation stage for which $\dot{a} \neq 0$.

In this section we shall review the essential results of the theories based on the continuum mechanics.

1. <u>The stationary crack and the first critical time in linear viscoelastic solid.</u>

Consider a linear viscoelastic material which could be modeled by a standard linear solid, i.e. the three parameter model as shown below.

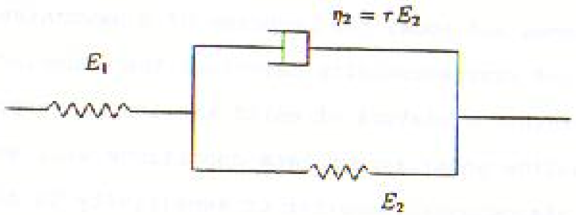


Fig. 1.1

E_1, E_2, η denote the elastic moduli and dashpot viscosity coefficients respectively, characterizing the mechanical response of standard linear solid. The plane strain creep compliance for this model D(t) is defined as the strain per unit stress induced by a step-load of magnitude σ_o.

$$D(t) = \frac{\epsilon(t)}{\sigma_0}$$

$$D(t) = \frac{1}{E_1}\{(1 + \frac{E_1}{E_2}) - \frac{E_1}{E_2}e^{-t/\tau}\} \qquad (1.1)$$

In which t denotes time, τ is the characteristic relaxation time, i.e. time required for stress relaxation of the element by the factor of "e", in which "e" denotes the Euler number. The relaxation of the element shown in Fig. 1.1 is governed by the formula.

$$E_{rel}(t) = \frac{E_1 E_2}{E_1+E_2}\left\{1 + \frac{E_1}{E_2}\exp\left[-\left(\frac{E_1+E_2}{E_2}\right)\left(\frac{t}{\tau}\right)\right]\right\} \qquad (1.1a)$$

in which E_1 denotes the instantaneous (or glassy) elastic modulus in contrast to the "rubbery" or infinite elastic modulus, $E_\infty = E_1E_2/(E_1+E_2)$ At the instant of load application, t = 0, one obtains

$$D(0) = \frac{1}{E_1} = \frac{1}{E_0} \qquad (1.1b)$$

For a very long time compared with the characteristic time, τ one has

$$D(\infty) = \frac{1}{E_1}\left[\frac{E_1+E_2}{E_2}\right] = \frac{E_1+E_2}{E_1 E_2} = \frac{1}{E_\infty} \qquad (1.1c)$$

It is convenient to introduce the ratio

$$\frac{D(t)}{D(0)} = \left[1 + \frac{E_1}{E_2} - \frac{E_1}{E_2}\exp(-t/\tau)\right] \qquad (1.2)$$

and another ratio

$$\frac{D(\infty)}{D(0)} = \frac{E_0}{E_\infty} \qquad (1.2a)$$

Here, E_0 denotes the relaxation modulus at the glassy state, i.e., when time t = 0, while E_∞ is the relaxation modulus at the rubbery state, i.e., when time t = ∞

$$E_\infty = \frac{E_1 E_2}{E_1+E_2} \qquad (1.2b)$$

By normalizing the creep compliance function, by the compliance at the instant of load application D(0), we obtain a nondimensional function

$$\phi(t) = \frac{D(t)}{D(0)} = 1 + \beta - \beta e^{-\theta} \qquad (1.3)$$

For simplicity of the notation let

$$\theta = \frac{t}{\tau}, \quad \beta = \frac{E_1}{E_2},$$

The end of the first stage of fracture development is denoted by the symbol t_I and referred to here as the "first critical time." Wnuk [7] and Knauss [8] (1968) have derived the following formula which relates t_I to the applied load, as given by the K-factor, and the material characteristics provided by the creep compliance function D. They suggested

$$\frac{D(t_I)}{D(0)} = \left(\frac{K_G}{K}\right)^2 \tag{1.4}$$

Here K_G denotes the critical K-factor which corresponds to the instantaneous (Griffith) fracture immediately following the instant of load application. K is the applied K-factor, while t_I is the time which elapsed from the instant of load application to the point of termination of the dormant stage of the crack development.

For a step-load of magnitude σ_o ($\sigma_o < \sigma_{Griffith}$) and substituting for the values of K_G and K for a Griffith crack in eq. (1.4), one obtains:

$$\frac{D(t_I)}{D(0)} = \left(\frac{K_G}{K}\right)^2 = \left(\frac{\sigma_G\sqrt{\pi a_o}}{\sigma\sqrt{\pi a}}\right)^2 = \left(\frac{\sigma_G}{\sigma}\right)^2\left(\frac{a_o}{a}\right)$$

Since at the onset of crack propagation $a = a_o$, one finally obtains

$$\frac{D(t_I)}{D(0)} = \left(\frac{\sigma_G}{\sigma_o}\right)^2 \tag{1.5}$$

which is a well-known Wnuk-Knauss equation used as a predictor of the first critical time. Here σ denotes constant applied stress, σ_G is the Griffith critical stress evaluated for the polymer in its "glassy" state, i.e. when $t = 0$.

Considering the ratios

$$\frac{D(\infty)}{D(0)} = \frac{E_o}{E_\infty}$$

$$\frac{D(t_I)}{D(0)} = \left(\frac{\sigma_G}{\sigma}\right)^2 \tag{1.6}$$

hence

$$\frac{D(\infty)}{D(0)} = \left(\frac{\sigma_G}{\sigma_\infty}\right)^2 = \frac{E_o}{E_\infty}$$

and thus, one may derive the following relation:

$$\frac{D(\infty)}{D(o)} = \left(\frac{\sigma_G}{\sigma_o}\right)^2 = \frac{E_o}{E_\infty} \tag{1.7}$$

From this equation the lower bound of the strength observed in delayed failure experiment or the so-called long-time tensile strength can be predicted

$$\sigma_\infty = \sigma_G\left(\frac{E_\infty}{E_o}\right)^{1/2}$$

$$\sigma_G^2 = \sigma_\infty^{\ 2}\left[\frac{E_1(E_1+E_2)}{E_1\ E_2}\right] \tag{1.8}$$

$$= \sigma_\infty^{\ 2}(1+\beta) \tag{1.9}$$

Therefore

$$\left(\frac{\sigma_G}{\sigma_o}\right)^2 = \frac{\sigma_\infty^{\ 2}(1+\beta)}{\sigma_o^2} \tag{1.10}$$

Combining eqs. (1.3) and (1.5) gives:

$$\frac{D(t_I)}{D(0)} = 1 + \beta - \beta\exp(-\theta_I) \tag{1.11}$$

One can further reduce expression (1.11) as follows:

$$\frac{\sigma_\infty^2(1+\beta)}{\sigma_o^2} = 1+\beta-\beta\exp(-\theta_I)$$

Finally, this equation can be solved for the first critical time:

$$\theta_I = \ln\left\{\frac{\beta}{(1+\beta)\left[1-\left(\frac{\sigma_\infty}{\sigma_o}\right)^2\right]}\right\} \qquad (1.12)$$

Figure 1.2 illustrates the effects of load ratio, σ_∞/σ_G on the first critical time. Note that a double logarithmic scale has been used to plot the graphs shown in Fig. 1.2.

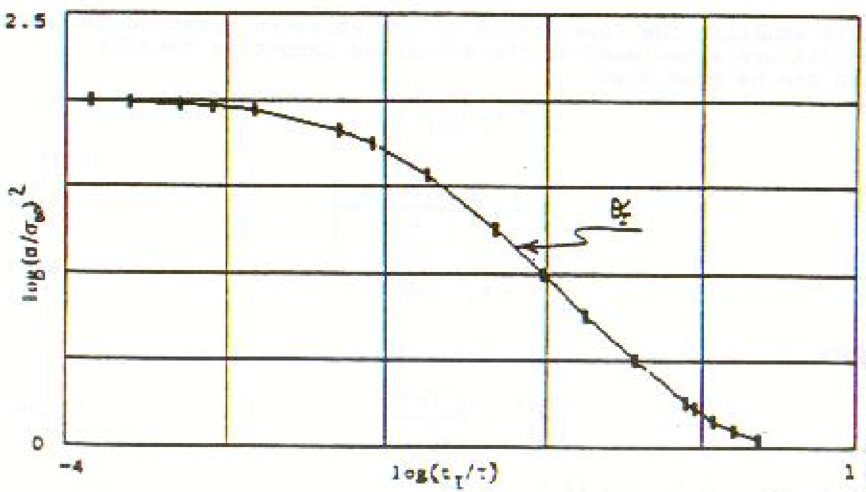


Fig. (1.2)

First critical time in standard linear viscoelastic solid vs. the loading ratio.

2. <u>Crack propagation stage and second critical time in linear viscoelastic solid.</u>

Next phase of time-dependent failure begins with an onset of crack propagation and the ensuing relations can be deduced by consideration of the governing equation of motion.

Knauss (1970) [9] and Wnuk (1971) [10] have suggested an equation of motion very similar in its structure to equation (1.4). The only difference is that now t_I is replaced by time $\delta t = \Delta/\dot a$. As can be seen, δt denotes the time which is needed for the crack trip to traverse its own process zone (Δ). Applying equation (1.4) and replacing t_I with $\delta t = \Delta/\dot a$ one obtains

$$1+\beta-\beta\exp\left[-\frac{\Delta}{\dot a}\right] = \left(\frac{K_c}{K}\right)^2 \qquad (2.1)$$

Introducing the nondimensional crack length

$$X = \frac{a}{a_o}, \quad \dot X = \frac{\dot a}{a_o} \qquad (2.2)$$

and the stress intensity ratio

$$\left(\frac{K_c}{K}\right)^2 = n\left(\frac{a_o}{a}\right) = \frac{n}{X} \qquad (2.3)$$

and substituting these quantities into equation (2.1) yields

$$1+\beta-\beta e^{-\rho\Gamma} = n\left(\frac{a_o}{a}\right) \qquad (2.4)$$

where

$$\Gamma = \frac{1}{\tau\dot X} = \frac{a_o}{\tau}\frac{1}{da/dt} \qquad (2.5)$$

$$\rho = \Delta/a_o$$

The scaling constant, ρ, as can be seen by setting the quantity $\Delta/\dot{a}\tau$ equal to $\rho\Gamma$ represents the size of the process zone Δ adjacent to the crack tip, namely, $\rho = \Delta/a_o$. Solving for the reciprocal of the nondimensional rate of crack propagation, Γ, one obtains:

$$\exp(-\rho\Gamma) = \frac{1 + \beta - n\left(\frac{a_o}{a}\right)}{\beta}$$

$$\rho\Gamma = \ln\left[\frac{\beta}{1 + \beta - n\left(\frac{a_o}{a}\right)}\right] \qquad (2.6)$$

Hence

$$\frac{\Delta\rho}{\dot{a}\tau} = \ln\left[\frac{\beta}{1 + \beta - n\left[\frac{a_o}{a}\right]}\right] \qquad (2.7)$$

$$\frac{\tau}{\Delta}\frac{da}{dt} = \frac{1}{\ln\left[\frac{\beta}{1 + \beta - n\left[\frac{a_o}{a}\right]}\right]}$$

$$\frac{dX}{d\theta} = \frac{\rho}{\ln\left[\frac{\beta}{1 + \beta - n\left(\frac{a_o}{a}\right)}\right]} \qquad (2.8)$$

This is the nonlinear first order differential equation which governs motion of the crack during the second phase of fracture development.

Integrating Eq. (2.8) one obtains:

$$\int_1^{\theta} \rho\, d\theta = \int_1^{X} \ln\left[\frac{\beta}{1 + \beta - \frac{n}{X}}\right] dX \qquad (2.9)$$

From that it follows

$$\theta(X) = \frac{1}{\rho}\int_1^{X} \ln\left[\frac{\beta}{1 + \beta - \frac{n}{X}}\right] dX \qquad (2.10)$$

Here X denotes the current crack length normalized by its initial value, a_o.

Equation (2.10) indicates the relation between the size of a propagating crack (X) and time (θ) needed for fracture development at various loading ratios (n). The results obtained numerically from Equation (2.10) are shown in Fig. 2.1.

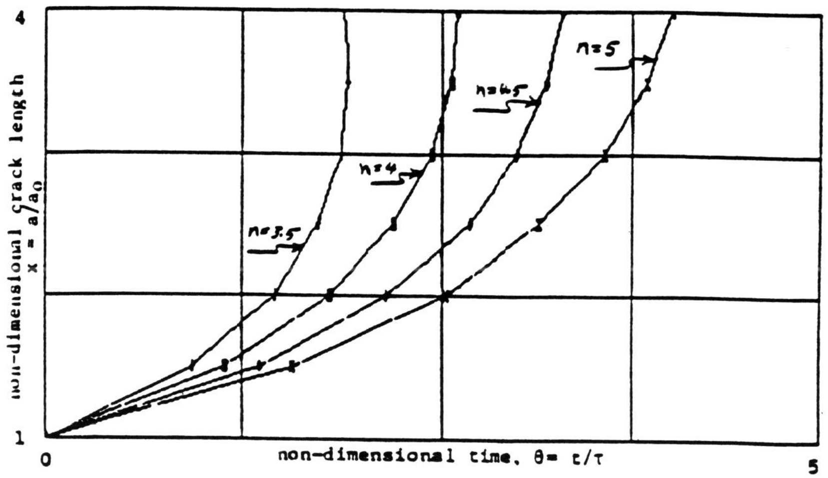


Fig. (2.1)

Histories of crack extension in a standard linear visco-elastic solid computed for 4 different loading ratios n.

As can be seen from Fig. 2.1, the function $X = X(\theta)$ passes through a maximum at a certain value of time θ, which is identified with the second critical time θ_{II}. To find this point of maximum ($X_{max.}$, θ_{II}) one needs to seek a solution to the equation $dX/d\theta = \infty$. Using Eq. (2.8), it is noted that the rate $d\theta/dX$ approaches zero when the expression

$$\ln\left[\frac{\beta}{1 + \beta - n(a_o/a)}\right] = 0 \qquad (2.11)$$

This occurs when the argument of the logarithmic function approaches one, i.e.,

$$1 + \beta - \frac{n}{X} = \beta \qquad (2.12)$$

Hence, one obtain the predicted point of maximum on the curves shown in Fig. 2.1, namely

$$X_{max} = n \qquad (2.13)$$

and

$$\theta_{II} = \frac{1}{\rho}\int_1^n \ln\left[\frac{\beta}{1 + \beta - \frac{n}{X}}\right] dX \qquad (2.14)$$

These equations were used to generate curves shown in Fig. 2.2.

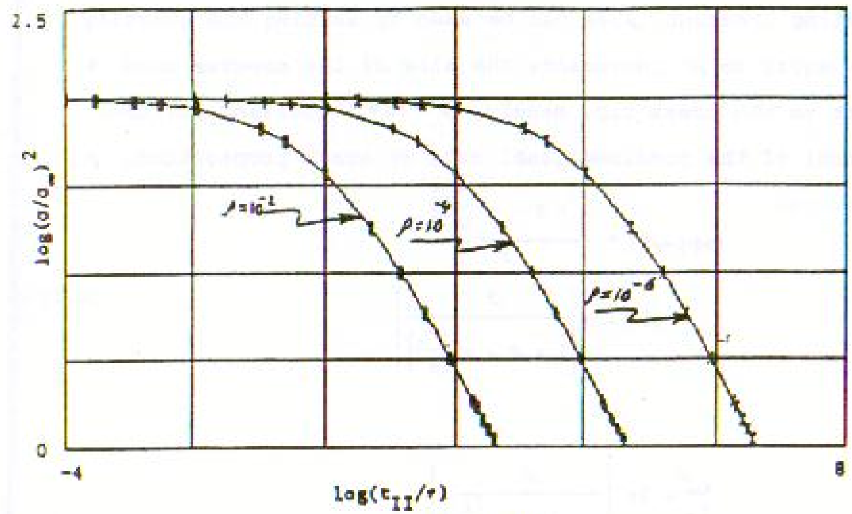


Fig. (2.2)

Second critical time in standard linear viscoelastic solid

Finally, superimposing Figures (1.2) and (2.2) one may construct the so called "map" of time-dependent fracture which shows both the first (θ_I) and the second (θ_{II}) critical times as functions of the load ratio σ_o/σ_∞; this is shown in Fig. (2.3). Note that the ratio σ_o/σ_∞ used on the vertical axis in Fig. (2.3) is related to the previously introduced quantity $n = (\sigma_G/\sigma_o)^2$ as follows:

$$\frac{\sigma_\infty}{\sigma_o} = \sqrt{\frac{n}{1+\beta}} \qquad (2.15)$$

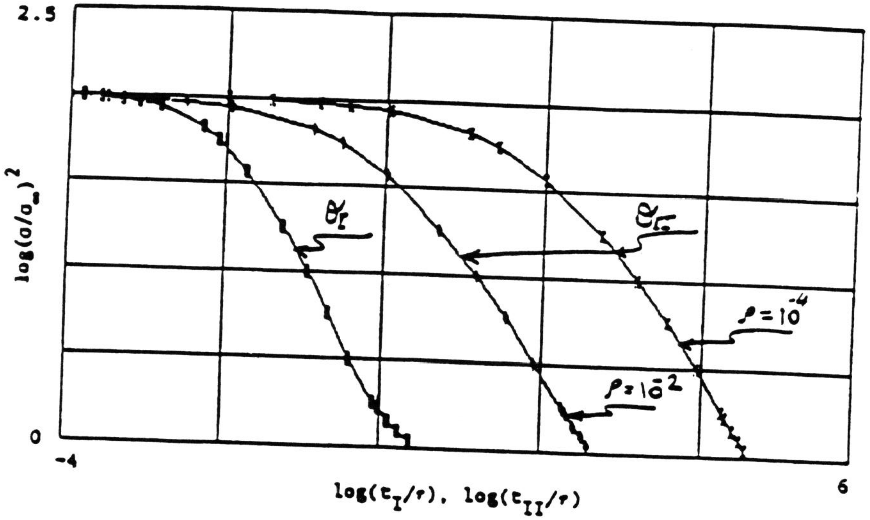


Fig. (2.3)

First and second critical time in viscoelastic solid

3. Governing equation of moving crack

Knauss (1968) (8), Wnuk (1969) (4) and Schapery (1975) (18) suggested that the motion of the crack is controlled by the mechanical properties of the viscoelastic medium through the relation

$$\psi(\delta t) = \frac{D(\delta t)}{D(0)} = \left(\frac{K_G}{K}\right)^2 \qquad (3.1)$$

The time increment δt equals the time used by the crack front to traverse the process zone of length Δ, thus

$$\delta t = \frac{\Delta}{\dot{a}} \qquad (3.2)$$

Symbols K_G and K denote the stress intensity factors corresponding to the critical (Griffith) load and the actual load, respectively. The K-factor incorporates variables such as the applied stress, crack size and geometry, namely

$$K = \sigma\sqrt{\pi a}\,\Phi\left(\frac{a}{w}\right) = \sigma\sqrt{\pi a_o}\sqrt{\frac{1}{X}}\,\Phi\left(\frac{a}{w}\right) \qquad (3.3)$$

The function Φ is geometry dependent, see Appendix; for instance for the center crack panel (CCP) specimen, shown in Fig. 3.1, one has

$$\Phi\left(\frac{a}{w}\right) = \frac{1}{\sqrt{\cos(\pi a/w)}} \qquad (3.4)$$

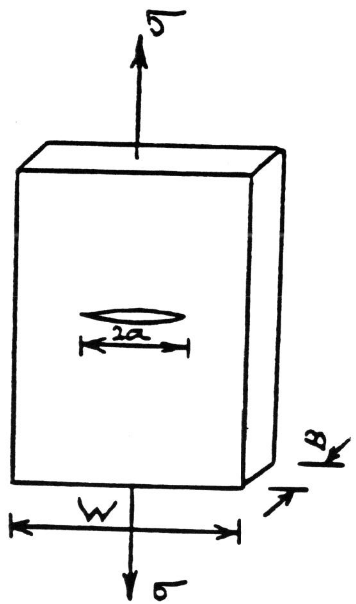


Fig. 3.1

When the instantaneous critical load σ_C (Griffith load) is introduced, then K-factor requires the critical value

$$K_C = \sigma_C\sqrt{\pi a_o}\;\phi(\frac{a_o}{w}) \qquad (3.5)$$

Hence, dividing expressions (3.5) and (3.3), one may define the ratio

$$\left(\frac{K_C}{K}\right)^2 = \left(\frac{\sigma_C}{\sigma}\right)^2\left[\frac{\Phi(\frac{a}{w}) \;/\; \Phi(\frac{a_o}{w})}{X}\right]$$

$$= \frac{n}{X}\left[\frac{\Phi(\frac{a}{w})}{\Phi(\frac{a_o}{w})}\right] = \frac{n}{X}\left[\phi(a_o, a/w)\right] \qquad (3.6)$$

Here, the nondimensional load, crack size and geometry parameters are defined as follows:

- load parameter, $n = (\sigma_C/\sigma)^2$
- crack site parameter, $X = a/a_o$
- geometry parameter, $\phi = \Phi(a/w) \;/\; \Phi(a_o/w)$

For example, for the CCP geometry

$$\phi = \sqrt{\frac{\cos(\pi a_o/w)}{\cos(\pi a/w)}} \qquad (3.7)$$

Replacing the LHS of Eq. (3.1) by the linear form in terms of logarithm of δt, i.e.,

$$A + B\log(\frac{\delta t}{\tau}) \;, \qquad \delta t = \Delta/\dot a \qquad (3.5)$$

and using expression (3.6) for the RHS of Eq. (3.1) one obtains

$$A + B\log(\frac{\Delta}{\tau\dot a}) = \log(\frac{n}{X}\phi) \qquad (3.6)$$

$$A + B\log(\frac{\rho}{\dot X}) = \log(\frac{n}{X}\phi) \qquad (3.7)$$

Here, the "constants" A and B depend on the stress level σ (or n), and they are determined experimentally. For a given geometry and at constant

applied load σ = const., equation (3.7) can be solved for the normalized velocity $\dot X = dX/d(t)$ and then integrated. The solution for X results readily from Eq. (3.7)

$$\dot X = \frac{\rho}{\exp\left\{\frac{1}{B}\left[\log(\frac{n\phi}{X}) - A\right]\right\}} \qquad (3.8)$$

Integration yields the relation between current crack length X and time ϑ ($\vartheta = t/\tau$), namely (see Appendix for details)

$$\vartheta = \frac{1}{\rho}\int_1^X \exp\left[\frac{\log(\frac{n\phi(x)}{X})}{B}\right]\cdot\exp(-\frac{A}{B})dX \qquad (3.9)$$

For any given function $\phi = \phi(X)$ this integral can be evaluated numerically. When $\phi = 1$, the Griffith configuration is considered, and then Eq. (3.8) can be integrated in a closed form

$$\vartheta = \frac{1}{\rho}\,\frac{n^{1/B}}{\exp(A/B)}\cdot\frac{B}{B-1}\cdot\left[X^{\frac{B-1}{B}} - 1\right] \qquad (3.10)$$

This provides the relation between nondimensional crack length X and the nondimensional time ϑ. Examples of curves that result from Eqs. (3.8) and (3.10) are given in Figs. 3.3, 3.4, and 3.5.

It is seen that at a certain crack size, say $X = X_{cr}$, velocity becomes either unbounded or very large. This instant $\vartheta = \vartheta_{II}$ corresponds to the transition of slowly moving crack to catastrophic fracture. Thus, at $X = X_{cr}$ normalized velocity $\dot\vartheta$ becomes V_{cr} and time ϑ_{II} is referred to as the <u>second critical time</u>. Therefore the total life-time of the specimen consists of the incubation time θ_I, discussed previously, and the second critical time θ_{II} (see Fig. 3.2).

$$\text{life-time} = \theta_* = \theta_I + \theta_{II} \qquad (3.11)$$

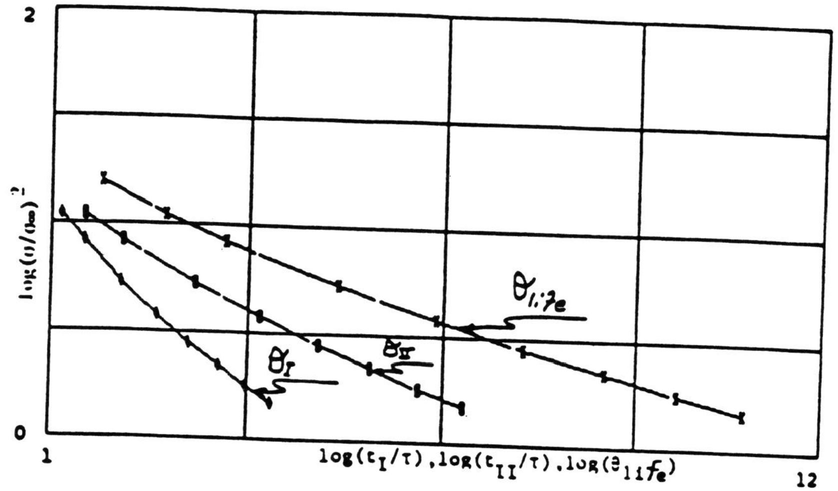


Fig. (3.2)

The life expectation of the specimen in viscoelastic medium.

Formula for θ_I was given in Section 2. Now, an estimate of the critical X is provided by the solution to X in terms of velocity V, obtained from Eq. (3.8) as follows (ϕ = 1)

$$X_{cr} = \frac{n}{\exp\left[B \ln\left(\frac{\phi}{V_{cr}}\right) + A\right]} \tag{3.12}$$

With this value substituted for X in Eq. (3.9) the second critical time may be evaluated:

$$\theta_{II} = \frac{1}{\phi} \frac{n^{1/B}}{\exp(A/B)} \frac{B}{B-1} \left[X_{cr}^{\frac{B-1}{B}} - 1\right] \tag{3.13}$$

The only problem is how to estimate the quantity V_{cr} which appears in Eq. (3.12). The estimation is done by inspection of curves shown in Fig. 3.2.

Thus, for a given load level (fixed n), the critical velocity V_{cr} is read from Fig. 3.2, then this value is plugged into Eq. (3.13) resulting in the prediction of the second critical time, θ_{II}. This process may be shown schematically as follows

$$n \xrightarrow{(\dot{X},X \text{ curve})} V_{cr} \longrightarrow X_{cr} \xrightarrow{\text{Eq.(3.13)}} \theta_{II}$$

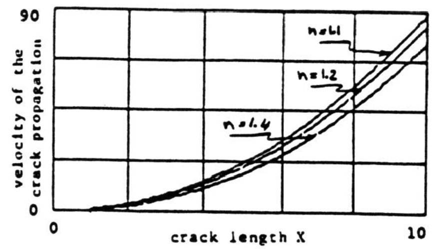


Fig. (3.3)

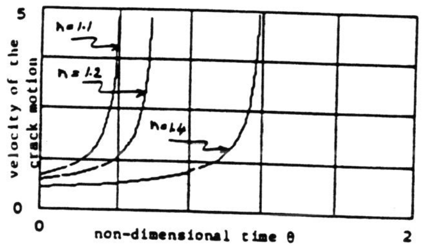


Fig. (3.4)

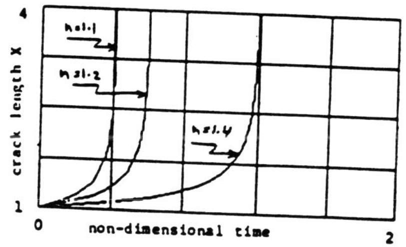


Fig. (3.5)

Appendix

•The derivation of *equation of motion* for the crack propagation in viscoelastic media. Wnuk[7] and Knauss[8] (1968), have derived the following formula which relates "t_I", i.e., the time elapsed from the instant of load application to the point of termination of the dormant stage of crack devolpment, where $\dot{a} = 0$, that is when $0 \leq t < t_I$. As follows

$$\frac{D(t_I)}{D(0)} = (\frac{K_G}{K})^2 \quad \text{(A.1)}$$

Substituting for the values of $K_G = \sigma_G\sqrt{\pi a_0}$ and $K = \sigma\sqrt{\pi a}$ in the above expression, one can obtain

$$\frac{D(t_I)}{D(0)} = \{\frac{\sigma_G\sqrt{\pi a_0}\Phi(a_0/W)}{\sigma\sqrt{\pi a}\Phi(a/W)}\}^2 \quad \text{(A.2)}$$

or

$$\frac{D(t_I)}{D(0)} = (\frac{\sigma_G}{\sigma})^2\frac{a_0}{a}(\frac{\Phi(\omega)}{\Phi(\omega X)})^2 \quad \text{(A.3)}$$

We have defined $n = (\sigma_G/\sigma)^2$, $X = a/a_0$
and $\phi(X) = \{\Phi(\omega)/\Phi(\omega X)\}^2$, one can rewrite equation (A.3) as follow

$$\frac{D(t_I)}{D(0)} = (\frac{K_G}{K})^2 = \frac{n}{X}\,\phi(X) \quad \text{(A.4)}$$

We show in section 1 for the termination of the dormant stage when $\dot{a} = 0$, that.

$$\frac{D(t_I)}{D(0)} = 1 + \beta - \beta\exp(-\theta_I) \quad \text{(A.5)}$$

where; $\theta_I = t_I/\tau$. By equating equation (A.4) and equation (A.5) one can obtain

$$\frac{D(t_I)}{D(0)} = \frac{n}{X}\phi(X) = 1 + \beta - \beta\,exp(-\theta_I) \quad \text{(A.6)}$$

For chosen time interval equation (A.6) can be represented by *straight line* equation if it is plotted in Logarithmic scale, this straight line equation can be written as

$$\ln(n) = A + B\,ln(\theta_I) \quad \text{(A.7)}$$

or

$$\ln(\theta) = \frac{1}{B}\{\ln(n) - A\} \quad \text{(A.8)}$$

Solving for the value of θ, one can obtain

$$\theta_I = exp\,\{\frac{1}{B}[\ln(n) - A]\} \quad \text{(A.9)}$$

or it can be shown in another form as

$$\theta_I = \frac{[e^{\ln(n)}]^{\frac{1}{B}}}{e^{\frac{A}{B}}} = \frac{n^{\frac{1}{B}}}{e^{\frac{A}{B}}} \quad \text{(A.10)}$$

Therefore the first critical time can be approximated by log-log approach in the following equation

$$\theta_I = \frac{n^{\frac{1}{B}}}{e^{\frac{A}{B}}} \quad \text{(A.11)}$$

Now, in crack propagation stages for which $\dot{a} \neq 0$.Knauss(1970)[9], and Wnuk (1971)[10], have suggested the same equations the only difference is that now t_f is replaced by time $\delta t = \Delta/\dot{a}$, where δt is the time which is needed for the crack tip to traverse it's own process zone (Δ). Applying these condition into equation (A.4) one can get

$$\frac{D(\delta t)}{D(0)} = (\frac{K_G}{K})^2 = \frac{n\ \phi(X)}{X} \tag{A.12}$$

For a chosen interval of time equation (A.12), can be replaced by a *straight line* when a double logarithmic scale is used such as double logarithmic linearization procedure leads to an equation such as

$$\ln\{\psi(t)\} = A + B\ \ln(t/\tau) \qquad t_1 \le t \le t_2 \tag{A.13}$$

or

$$\ln\{\psi(t)\} = A + B\ ln(\delta t/\tau), \qquad \frac{\delta t}{\tau} = \frac{\rho}{X} \tag{A.14}$$

Where; $\rho = \Delta/a_0$, therefore equation (A.12) can be written as

$$\ln\{\frac{n\ \phi(X)}{X}\} = A + B\ln(\frac{\rho}{X}) \tag{A.15}$$

or

$$\ln(\frac{\rho}{X}) = \frac{1}{B}\{ln[\frac{n\ \phi(X)}{X}] - A\} \tag{A.16}$$

$$\frac{\rho}{X} = \frac{\rho}{exp\ \{\frac{1}{B}[\ln(\frac{n\phi(X)}{X} - A]\}} \tag{A.17}$$

by integrating both side of equation (10.72) leads to

$$\int_1^X exp\{\frac{1}{B}[\ln(\frac{n\ \phi(X)}{X}) - A]\}dX = \rho \int_{\theta_I}^{\theta} d\theta \tag{A.18}$$

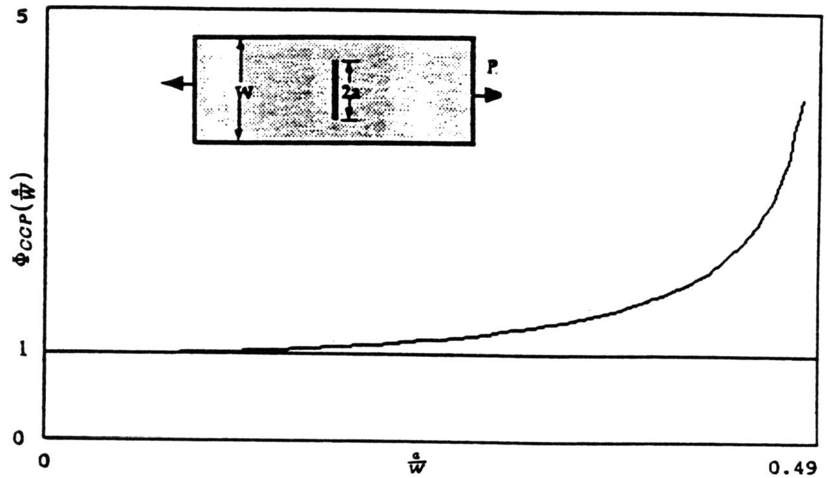
Thus, the approximation of the non-dimensional critical time can now be predicted from the above equation as

$$\theta_{II} = \frac{1}{\rho}\int_1^{X_{cr.}} exp\{\frac{1}{B}[\ln(\frac{n\phi(X)}{X} - A]\}dx \tag{A.19}$$

The auxiliary shape function for

Center Crack Panel, CCP

$$\Phi_{CCP}(\frac{a}{W}) = \sqrt{\frac{1}{\cos(\pi\frac{a}{W})}}[1 - 0.025(\frac{2a}{W})^2 + 0.06(\frac{2a}{W})^4]$$



$\frac{a}{W}$	$\Phi_{CCP}(\frac{a}{W})$
0	1
0.1	1.025
0.2	1.112
0.3	1.304
0.4	1.799
0.5	1.278×10^8

The auxiliary shape function for

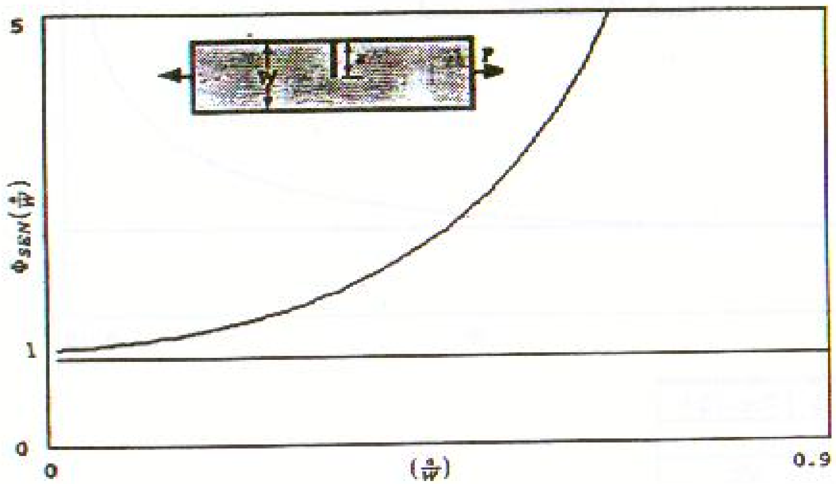
Single Edge Notch, SEN

$\Phi_{SEN}(\frac{a}{W}) = f_1(\frac{a}{W})\, f_2(\frac{a}{W})$

Where;

$f_1(\frac{a}{W}) = \{(\frac{2}{\pi(\frac{a}{W})})\tan(\frac{\pi\frac{a}{W}}{2})\}^{\frac{1}{2}}$

$f_2(\frac{a}{W}) = \{\frac{0.752+2.02(\frac{a}{W})+0.37[1-\sin(\frac{\pi(\frac{a}{W})}{2})]^3}{\cos[\frac{\pi(\frac{a}{W})}{2}]}\}$

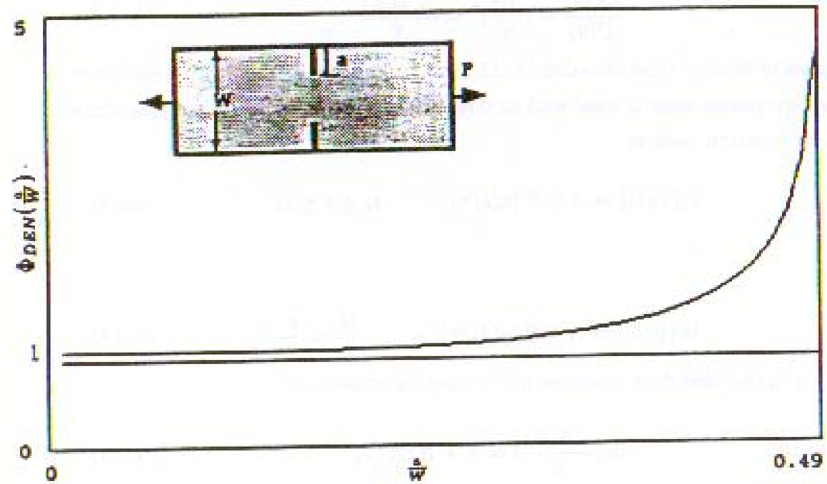


$\frac{a}{W}$	$\Phi_{SEN}(\frac{a}{W})$
0.0001	1.1222
0.1	1.196
0.2	1.367
0.3	1.655
0.4	2.108
0.5	2.827

The auxiliary shape function for

Double Edge Notch, DEN

$\Phi_{DEN}(\frac{a}{W}) = \{\frac{1.122-0.561(\frac{2a}{W})-0.205(\frac{2a}{W})^2+0.471(\frac{2a}{W})^3+0.190(\frac{2a}{W})^4}{\sqrt{1-(\frac{2a}{W})}}\}$

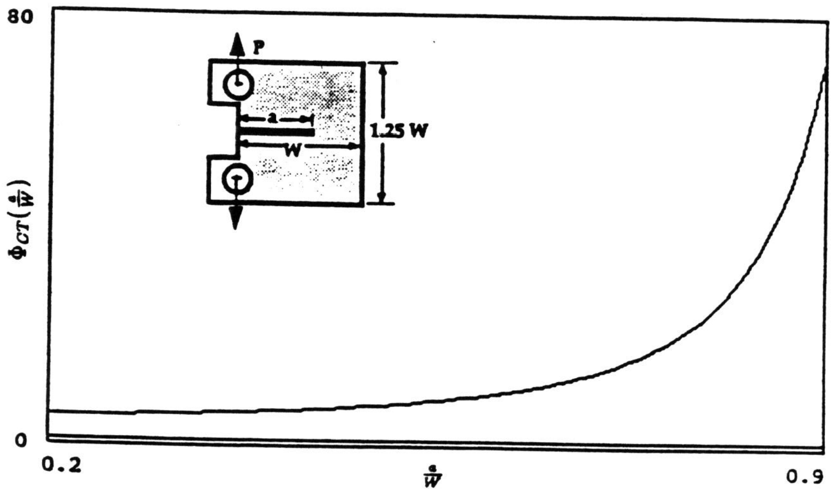


$\frac{a}{W}$	$\Phi_{DEN}(\frac{a}{W})$
0.001	1.122
0.1	1.118
0.2	1.132
0.3	1.226
0.4	1.567
0.5	1.019×10^8

The auxiliary shape function for

Compact Tension, CT

$$\Phi_{CT}(\tfrac{a}{W}) = \{\frac{1}{\sqrt{\pi \frac{a}{W}}}\}\{\frac{2+\frac{a}{W}}{(1-\frac{a}{W})^{\frac{3}{2}}}\}\{0.866 + 4.64(\tfrac{a}{W}) - 13.32(\tfrac{a}{W})^2 + 14.72(\tfrac{a}{W})^3 - 5.6(\tfrac{a}{W})^4\}$$

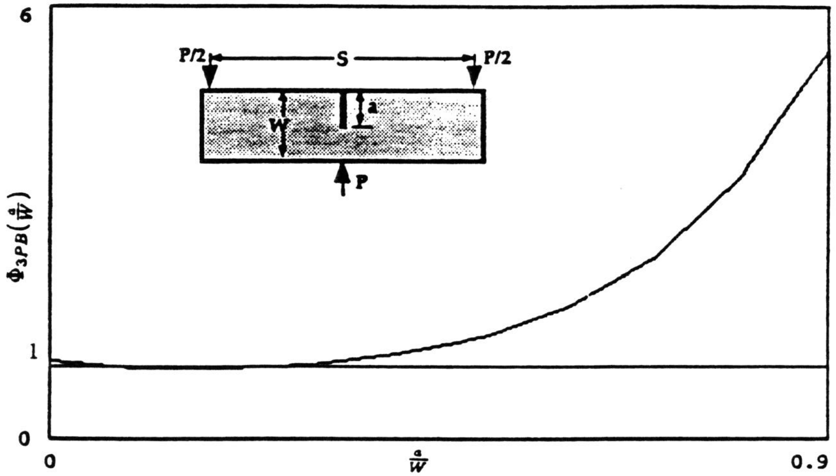


$\frac{a}{W}$	$\Phi_{CT}(\frac{a}{W})$
0.2	5.392
0.3	5.79
0.4	6.493
0.5	7.707

The auxiliary shape function for

Three Point Bending, 3PB

$$\Phi_{3PB}(\tfrac{a}{W}) = \frac{\frac{1}{\sqrt{\pi}}}{\{1+2(\frac{a}{W})\}(-\frac{a}{W})^{\frac{3}{2}}}\{1.99 - [\tfrac{a}{W} - (\tfrac{a}{W})^2][2.15 - 3.93(\tfrac{a}{W} + 2.7(\tfrac{a}{W})^2]\}$$



$\frac{a}{W}$	$\Phi_{3PB}(\frac{a}{W})$
0	1.09
0.1	0.986
0.2	0.981
0.3	1.043
0.4	1.173
0.5	1.826

The auxiliary shape function for

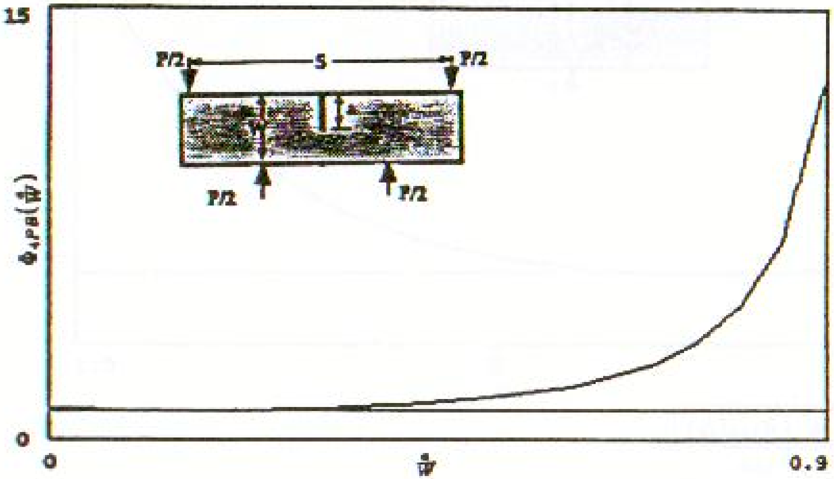
Four Point Bending, 4PB

$$\Phi_{4PB}(\frac{a}{W}) = g_1(\frac{a}{W})\{0.923 + 0.199\, g_2(\frac{a}{W})\}$$

Where;

$$g_1(\frac{a}{W}) = \frac{1}{\cos(\frac{\pi a}{2W})}\sqrt{\frac{\tan(\frac{\pi a}{2W})}{\frac{\pi a}{2W}}}$$

$$g_2(\frac{a}{W}) = \{1 - \sin(\frac{\pi a}{2W})\}^4$$



$\frac{a}{W}$	$\Phi_{4PB}(\frac{a}{W})$
0	1.122
0.1	1.041
0.2	1.035
0.3	1.098
0.4	1.234
0.5	1.475

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