THE DYNAMIC THERMOELASTICITY PROBLEM FOR A PLATE WITH A MOVING SEMI-INFINITE CUT

A.I. ZHORNIK

Taganrog Pedagogical Institute,
Department of Theoretical Physics
48 Initsiativnaya Street, Taganrog, 347900 Russia

E.M. KARTASHOV

Lomonosov Moscow Institute of Fine Chemical Technology,
Department of Higher and Applied Mathematics,
Malaya Pirogovskaya Street, Moscow, 119831 Russia

ABSTRACT

The heat-shock, i.e. action of stresses created by a sharp change in temperature of a thin plate with a semi-infinite cut, whose tip is moving at a constant velocity since the initial time is considered. The lateral surfaces of the plate are subjected to a linear heat transfer by radiation to the surroundings. It is assumed that the plate has initially a constant temperature that is equal to the surrounding temperature and in the cut suddenly appears some constant temperature that is not equal to the initial temperature and is invariable along the moving cut. The mathematical description of the heat-shock is obtained by the solution of the thermoelasticity equation with consideration of inertial members using the Wiener-Hopf technique. The main result is an expression obtained for the stress intensity factor in case of large and little intervals.

KEYWORDS

Propagation of a brittle crack, the stress intensity factor, fracture, dislocation.

STATEMENT OF THE PROBLEM

In a thin isotropic infinite plate of 2δ thickness along the ray of $y'=0$, $x'<0$ there is a semi-infinite cut that starts at the initial time to move to the region of $y'=0$, $x'>0$ at a constant velocity. The plate heats symmetrically with respect to the middle plane by heat exchange with the medium for temperature T_c which surrounds its lateral surfaces. The initial temperature of the plate is equal to the surrounding temperature. In no time the temperature T_1 arises on the edges of the cut. It is considered that the origin of the fixed (x', y')-system is at the point where the end of the cut is at the initial time.

The origin of the moving x,y - coordinate system is chosen to remain at the cut tip, i.e.,

$$x = x' - V_T t, \quad y = y', \quad t = t'$$

This problem represents a case of plane-stress state, a case of plane strain which is realized by zero heat exchange with the medium and by replacing of elastic constants combinations. In consequence of the symmetry with respect to the x-axis, the problem is considered for a half-plane $y \geq 0$ with the boundary and initial conditions given in the moving system of co-ordinates,

$$T(x,y,t) = T_1, \qquad x<0, \ y=0 \qquad (1)$$

$$\frac{\partial T(x,y,t)}{\partial y} = 0, \qquad x>0, \ y=0 \qquad (2)$$

$$T(x,y,t) = T_c, \qquad t=0 \qquad (3)$$

$$\sigma_{yy}(x,y,t) = 0, \qquad x<0, \ y=0, \ t>0 \qquad (4)$$

$$\sigma_{xy}(x,y,t) = 0, \qquad -\infty < x < \infty, \ y=0, \ t>0 \qquad (5)$$

$$U_y(x,y,t) = 0, \qquad x>0, \ y=0, \ t>0 \qquad (6)$$

$$U_y(x,y,t) = 0, \qquad t \leq 0 \qquad (7)$$

$$\frac{\partial U_y(x,y,t)}{\partial t} = 0, \qquad t \leq 0 \qquad (8)$$

The problem formulated by conditions of (1) - (3) was solved by Zhornik and Kartashov (1988) considering an analogous thermoelasticity problem in the quasi-static statement. For the same reason the solution for a temperature field will be the same and ha. the form:

$$T(x,y,t) - T_c = \Theta(x,y,t)\, e^{-\gamma x} \qquad (9)$$

$\Theta(x,y,t)$ is a function which has the following form according to the Laplace-Fourier transformations:

$$\tilde{\tilde{\Theta}}(\xi,y,t) = \int_{-\infty}^{\infty} e^{-i\xi x} dx \int_0^{\infty} e^{-st}\, \Theta(x,y,t)\, dt / \sqrt{2\pi} =$$

$$= (T_1 - T_c)\, e^{-\frac{i\pi}{4}} \sqrt{\gamma+\beta}\; e^{-\sqrt{\xi^2+\beta^2}|y|} / \sqrt{2\pi} \times$$

$$\times s(-\xi + i\gamma)\sqrt{-\xi + i\beta} \qquad (10)$$

where $\gamma = V_T/2k$, $k = \lambda/\rho c$ = thermal diffusivity of the plate; λ = its heat conduction; ρ = density; c = specific heat;

$$\beta = \sqrt{x^2 + s/(k)}, \quad x^2 = \gamma^2 + \varkappa^2, \quad \varkappa^2 = \alpha/\lambda\delta$$

α = coefficient of linear heat transfer to the medium em' racing the lateral surfaces of the plate.
The case of the stationary problem of heat conduction for $t \to \infty$ $(s \to 0)$ was investigated by Salganik and Chertkov (1969) and the problem of heat conduction for a fixed cut was considered by Poberezhny and Gaivas (1982), Kozlov et al.(1985).

SOLUTION OF THE THERMOELASTICITY PROBLEM

The solution of the dynamic thermoelasticity problem will have the form:

$$\sigma_{ij} = \sigma_{ij}^T + \sigma_{ij}^P \qquad (11)$$

$$U_i = U_i^T + U_i^P \qquad (12)$$

σ_{ij}^T and U_i^T = the thermoelasticity problem for a plate wi out cut satisfies the boundary conditions of (5), (7), (8) and its solution is discovered by means of the thermoelastic potential of displacement $F(x,y,t)$ in the x,y -co-ordinates from the equation below:

$$\left(1 - \frac{a^2}{d^2}\right)\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} - (1+\nu)\alpha_T\, \Theta(x,y,t)\, e^{-\gamma x} =$$

$$= a^2\left[-\frac{2}{d}\frac{\partial^2 F}{\partial x \partial t} + \frac{\partial^2 F}{\partial t^2}\right] \qquad (13)$$

where $a = 1/c_{\text{н}} = \sqrt{\rho(1-\nu)/2G}$ = longitudinal wave slowness; $d = 1/V_T$ = slowness of the cut end; G = the shear modulu. , ν = Poisson's ratio; α_T = the coefficient of linear thermal expansion.
The normal stress which is necessary for the subsequent solution formulated according to Laplace-Fourier has the form:

$$\tilde{\tilde{\sigma}}^T(\xi,y,s) = \sqrt{\frac{2}{\pi}}\, G(1+\nu)\alpha_T \frac{T_1 - T_c}{s} \times$$

$$\times \frac{\sqrt{-i}\sqrt{\gamma+\beta}\left[\xi^2 - b^2(-\xi - ids)^2/2d^2\right]}{(-\xi - i0)\sqrt{-\xi + i(\gamma+\beta)}\left[-2i\gamma\xi - \gamma^2 + \beta^2 + \frac{a^2}{d^2}(-\xi - id)^2 - \eta^2\right]} \times$$

$$\times \left[e^{-\gamma_1|y|} - \frac{\gamma_1 e^{-\alpha|y|}}{\sqrt{1 - \frac{a^2}{d^2}(-\xi + isa_2)^{1/2}(-\xi - isa_1)^{1/2}}}\right] \qquad (14)$$

where $\eta \to 0$, $\gamma_1 = \sqrt{(-\xi + i\gamma)^2 + \beta^2}$

$$\mathcal{L} = \sqrt{1-a/d}\,\sqrt{1+a/d}\,(-\xi+isa_2)^{1/2}(-\xi-isa_1)^{1/2},$$

$$a_1 = a/(1+a/d), \quad a_2 = a/(1-a/d),$$

$b = 1/c_\perp = \sqrt{\rho/G}$ = shear wave slowness.

The boundary conditions for σ_{ij}^{P}, u_i^{P} -solutions of isothermic theory of elasticity have the form of (5) - (8), and also:

$$\sigma_{yy}^{P}(x,y,t) = -\sigma_{yy}^{T}(x,y,t), \quad x<0, \quad y=0. \qquad (15)$$

Fundamental Solution of the Elastic Problem. To find solutions for σ_{ij}^{T} and u_i^{T} it is considered a fundamental solution σ_{ij}^{*}, u_i^{*} of elasticity theory about a stressed state of the plate with a semi-infinite cut moving at a constant velocity v_T when normal concentrated forces f, moving subsequently at a constant distance l from the cut end, instantly apply on the edges of the cut at the distance l from its end. The problem as in the preceding item is considered in the moving x, y coordinates of (5) - (8) and also:

$$\sigma_{yy}^{*}(x,y,t) = -f\,\delta(x+l)H(t), \quad x<0, \quad y=0, \qquad (16)$$

where $\delta(x)$ = delta Dirac function; $H(t)$ = Heaviside function. For solution of this problem it is used an original method proposed by Freund (1974) for a fixed cut.

The stress intensity factor $K_I(t)$ $\sigma_{yy}^{*}(x,0,t)$ which is necessary for further consideration, has the form:

$$K_I(t) = \sqrt{2/\pi l}\,\frac{f}{\pi}(1-a/d)^{\frac{1}{2}}\int_{a_2}^{t/l}\left[(h-a_2)^{\frac{1}{2}}\Big/\left(\frac{t}{l}-h\right)^{\frac{1}{2}} \times (c_2-h)\,S_+(-h)\right]dh\,H(t-a_2 l), \qquad (17)$$

where

$$S_\pm(\lambda) = \exp\left\{-\frac{1}{\pi}\int_{a_2,a_1}^{b_2,b_1} \arctan\left[4\eta^2|\mathcal{L}(\mp\eta)| \times |\beta(\mp\eta)|\Big/\left(2\eta^2-b^2-\frac{b^2}{d^2}\eta^2 \mp 2b^2\frac{\eta}{d}\right)^2\right]\frac{d\eta}{\eta\pm\lambda}\right\} \qquad (18)$$

$$\mathcal{L}(\lambda) = (1+a/d)^{\frac{1}{2}}(1-a/d)^{\frac{1}{2}}(a_1-\lambda)^{\frac{1}{2}}(a_2+\lambda)^{\frac{1}{2}},$$

$$\beta(\lambda) = (1+b/d)^{\frac{1}{2}}(1-b/d)^{\frac{1}{2}}(b_1-\lambda)^{\frac{1}{2}}(b_2+\lambda)^{\frac{1}{2}},$$

$c_2 = c/(1-c/d)$, $c = 1/c_R$, c_R = Rayleigh wave velocity.

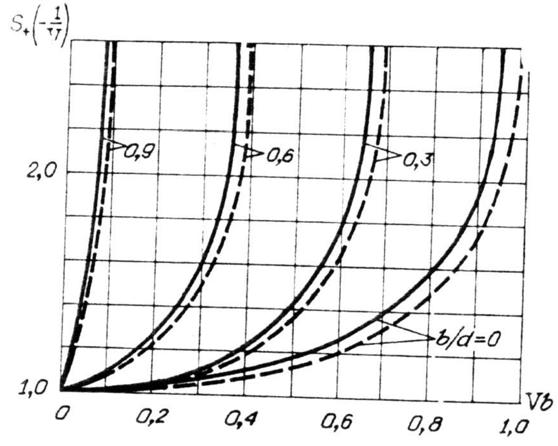

Fig. 1. Relation between $S_+(\frac{1}{V})$ and Vb for various b/d.
——— ν =0,4; – – – ν =0,25

Fig. 1. represents a dependence $S_+(\frac{1}{V})$ on Vb for various cut tip velocities b/d. Here is V, velocity of edge dislocation which is moving in the positive direction of the x-axis and starts the motion from the cut tip. The dislocation velocity is measered within

$$0 \leqslant V < c_{\parallel} - v_T$$

The curves for a fixed cut $b/d=0$ are given by Parton and Boriskovsky (1985). In general case the alteration of t is an integral of (17) that requires calculation. But in case of relatively large t, when $t/l > b_2$ all the peculiarities of the integral, namely the branch points of $h = a_2$, t/l, b_2 excepting the simple pole of c_2 we can eliminate, replacing the real integral (17) by a contour integral along the contour which embraces the cut (a_2, t/l) with moving clockwise. The final form for $K_I(t)$ when $t/l > b_2$ is:

$$\sqrt{\pi l}\,K_I(t)/\sqrt{2}f(1-a/d)^{1/2} =$$

$$= 1-\gamma_2 H(c_2 l - t)\Big/\left[\gamma_2\left(\frac{c_2}{a_2}-\frac{t}{a_2 l}\right)\right]^{1/2} S_+(-c_2),$$

$$\gamma_2 = c_2/a_2 - 1.$$

In case of a fixed cut $d \to \infty$ the latter formula turns into the solution obtained by Freund (1974).

General Loading. In case when the load $-\sigma_{yy}^{T}(x,0,t)$ is applied to the cut edges, the stress intensity factor is represented as:

$$K_I^{d}(t) = \int_{-\infty}^{0} dx \int_{0}^{t} \frac{\partial \sigma_{yy}^{T}(x,0,\tau)}{\partial \tau} K_I(x, t-\tau)\,d\tau \qquad (19)$$

where $K_I(x,t)$ results from (17) by replacing l with $-x>0$, and we also take $f=1$. Then to (19) the Laplace transform is applied:

$$\bar{K}_I^d(s) = \int_{-\infty}^{0} s\,\tilde{\sigma}_{yy}^{T}(x,0,s)\,\bar{K}_I(x,s)\,dx. \tag{20}$$

And then using the Parceval ratio (Sneddon, 1951) $K_I^d(s)$ has the form:

$$\bar{K}_I^d(s) = \int_{-\infty}^{\infty} s\,\tilde{\tilde{\sigma}}_{yy}^{T}(\xi,0,s)\,\tilde{\tilde{K}}_I(-\xi,s)\,d\xi, \tag{21}$$

where $\tilde{\tilde{\sigma}}_{yy}^{T}(\xi,0,s)$ is given in (14), $\tilde{\tilde{K}}_I(-\xi,s)$ is obtained from (17) using the above-mentioned replacements and applying to (17) the Laplace-Fourier transform with parameters of s and $-\xi$ and it will have the form:

$$\tilde{\tilde{K}}_I(-\xi,s) = -\frac{i}{\sqrt{\pi}}\left(1-\frac{a}{d}\right)^{\frac{1}{2}} \frac{(-i\xi - a_2 s)^{\frac{1}{2}}}{s(c_2 s + i\xi)\,S_+(i\xi/s)}. \tag{22}$$

The substitution of (14) and (22) into (21) comes $\bar{K}_I^d(s)$ to the form:

$$\bar{K}_I^d(s) = -2\pi A a^{\frac{3}{2}} b^2 \sqrt{1+a/d}\Big/\left(\sqrt{\beta^2+\gamma^2}+as\right)\sqrt{s}\,S_+(0) +$$

$$+\frac{Aa^2\sqrt{\gamma+\beta}}{s}\int_{-\infty}^{\infty}\Big\{(\xi^2+\eta_2^2)/i(\xi-i0)\sqrt{\gamma+\beta+i\xi}\;\times$$

$$\times\left[\sqrt{(-\xi+i\gamma)^2+\beta^2}+\eta_1\right]\sqrt{a_1 s - i\xi}\left(s+\frac{i\xi}{c_2}\right)S_+\!\left(\frac{i\xi}{s}\right)\Big\}\,d\xi, \tag{23}$$

$$A = G(1+\nu)\alpha_T(T_1-T_c)(1-c/d)/ca^2\sqrt{1+a/d}\,\sqrt{2}\,\pi,$$

$$\eta_1 = \sqrt{1-a^2/d^2}\,\sqrt{a_2 s + i\xi}\,\sqrt{a_1 s - i\xi},$$

$$\eta_2 = \sqrt{1-b^2/d^2}\,\sqrt{b_2 s + i\xi}\,\sqrt{b_1 s - i\xi}.$$

For a fixed cut $d\to\infty$ (23) is obtained by general asymptotic method (Kozlov et al.,1988). In expression of (23) integration is conducted along the real axis embracing the origin of the coordinates from below. Calculation of $K_I^d(s)$ is connected with calculating of the contour integral in the lower half-plane of ξ, which embraces the cut along the imaginary axis from $-ia_1 s$ till $-i(\beta-\gamma)$.
Having made the integration and proceeding to the original of t, $K_I^d(t)$ for large and little intervals comes to the form:

$$K_I^* = 2\,\frac{b^2}{a^2}\left(1-\frac{c}{d}\right)\sqrt{\frac{\delta}{ak}}\,\sqrt{\tau}\Big/\sqrt{1-\frac{a}{d}}\;\frac{c}{a}\,\sqrt{\pi}\,S_+(0), \tag{24}$$

$$\tau \ll a^2k^2\left(1+\sqrt{1+4\text{æ}^2a^2k}\right)/2\delta^2.$$

$$K_I^* = \frac{1}{\sqrt{\pi}}\sum_{k=1}^{\infty}(2\gamma^*)^{k-1}\frac{\Gamma\left(k-\frac{1}{2}\right)}{\Gamma(k+1)}\left(k-\frac{1}{2}\right)\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}(\text{æ}^*)^{2m}\;\times$$

$$\times\Bigg\{\frac{\tau^{\frac{1}{2}\left(k-\frac{1}{2}\right)+m}}{2\Gamma\left(\frac{1}{2}k+\frac{3}{4}\right)\left(\frac{k}{2}-\frac{1}{4}+m\right)}\;\times$$

$$\times\,{}_2F_2\left[-\frac{1}{2}k+\frac{1}{4},\;\frac{1}{2}\left(k-\frac{1}{2}\right)+m;\;\frac{1}{2},\;\frac{1}{2}\left(k-\frac{1}{2}\right)+m+1;\;-(\gamma^*)^2\tau\right]-$$

$$-\frac{\tau^{\frac{1}{2}\left(k+\frac{1}{2}\right)+m}}{\Gamma\left[\frac{1}{2}\left(k+\frac{1}{2}\right)\right]\left(\frac{k}{2}+\frac{1}{4}+m\right)}\,\gamma^*\;\times$$

$$\times\,{}_2F_2\left[-\frac{1}{2}k+\frac{3}{4},\;\frac{1}{2}\left(k+\frac{1}{2}\right)+m;\;\frac{3}{2},\;\frac{1}{2}\left(k+\frac{1}{2}\right)+m+1;\;-(\gamma^*)\tau\right]\Bigg\} \tag{25}$$

$$\tau \to a^2k^2\left(1+\sqrt{1+4\text{æ}^2a^2k}\right)/2\delta^2$$

$$K_I^* = -K_I^d/\sqrt{2}\,G(1+\nu)\alpha_T(T_1-T_c)\sqrt{\delta}\,\sqrt{1-a/d}$$

$\tau = kt/\delta^2$ = Fourier criterion; $\gamma^* = \gamma\delta$, $\text{æ}^* = \text{æ}\delta$,
${}_2F_2\,(a,b;c,d;e)$ = generalized hypergeometrical function; $\Gamma(a)$ = gamma function.
The dependence of K_I^* on τ for great intervals under different intensities of heat exchange æ^* and moving cut velocities γ^* is given on Fig. 2 - 3:

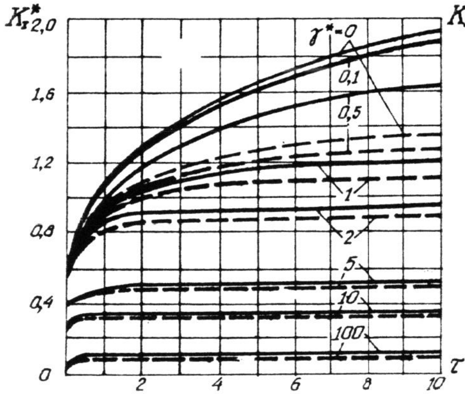

Fig.2. Relation between K_I^* and τ for various γ^*.
———— æ^* = 0;
– – – – æ^* = 0,5

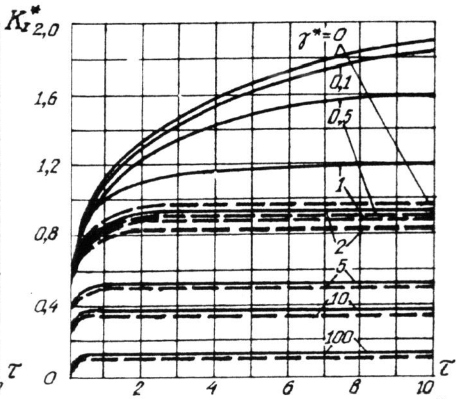

Fig.3. Relation between K_I^* and τ for various γ^*.
———— æ^* = 0,1;
– – – – æ^* = 1

In quasi-static case (25) was obtained by Zhornik and Kartashov (1988) and for $\gamma^*=0$ by Kozlov et al.(1988). The quasi-static case for $\gamma^*=0$ is realized by Poberezhny and Gaivas (1982), also by Kozlov et al.(1985). As one can see from Fig. 2-3, the stress intensity factor for $T_1-T_c>0$ is negative and that is why the obtained solution we can use specifically in a problem on closer definition of the dependence of crack velocity on the dynamic strain energy-release rate for PMMA with different average molecular weights (Döll, 1976).
There is a general close agreement between the calculated (Broberg, 1960) and experimentally measured variation of heat output with crack speed variation.
However the measured heat values are generally less than those predicted. The possible reason for existence of an energy gap may be that the stress intensity factor acting at the crack tip might be reduced due to thermal stresses. It can be taken into consideration using the results of the present work.
The consideration of thermal stresses in the quasi-static statement using Irwin models was made by Lucas (1969).

REFERENCES

1. Broberg,K.B.(1960) The propagation of a brittle crack. - Arkiv for fysik, 18, No.10, p.159-192
2. Döll,W.(1976) Application of an energy balance and an energy method to dynamic crack propagation. - Int.J.of Fracture, Vol.12, No.4, p.595-605
3. Freund,L.B.(1974) The stress intensity factor due to normal impact loading of the faces of a crack. - Int.J.Engng. Sci., Vol.12, p.179-189
4. Kozlov,V.A.,Mazya,V.G. and Parton,V.Z.(1985) Asymptotics of stress intensity factors in quasi-static temperature problems for the cut region.- Prikl.Mat.Mekh.,49,No.4, 627-636
5. Kozlov,V.A.,Mazya,V.G. and Parton,V.Z.(1988) On heat shock in the region of a crack.- Prikl.Mat.Mekh.,52,No.2;318-326
6. Lucas,R.N.(1969) A quasi-static thermoelastic analysis of a propagating crack. -Int.J.Solids & Structures,V.5;175-190
7. Parton,V.Z. and Boriskovsky,V.G.(1985) Dynamic Mechanics of Fracture. - Mashinostroyeniye, Moscow
8. Poberezhny,O.V. and Gaivas,I.V.(1982) Influence of non-stationary temperature field and of heat output of a plate upon stress intensity factors. -Prikl.Mekh.,18,No.6, 124-127
9. Salganik,R.L. and Chertkov,V.Y.(1969) On strength reduction under the action of shrinkage strain. - Mekh. tverdogo tela, No.3, p.126-133
10. Sneddon,I.N.(1951) Fourier Transforms. - McGraw-Hill Book Company,Inc., New York
11. Zhornik,A.I. and Kartashov,E.M.(1988) Temperature fields and temperature stresses arising in a plate with a moving semi-infinite crack. In: Metody i algoritmy parametricheskogo analiza lineinyh i nelineinyh modeley perenosa.(Kartashov,E.M.), No.6, p.27-40 - Moscow