DYNAMIC FRACTURE OF ELASTIC-PLASTIC BEAM WITH AN EDGE CRACK

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ABSTRACT

The elastic–plastic line–spring model is proposed to analyse the dynamic fracture of a cracked beam of ductile material in present paper. It can be expected from the numerical results that this method for the analysis of dynamic fracture under elastic–plastic conditions will be valuable due to its computational simplicity.

KEYWORDS

Dynamic fractrue, cracked beam, elastic–plastic model.

COMPUTATIONAL MODEL

Supposing that an edge crack is located at the point of maximum bending moment in an elastic–plastic beam, the plastic defomation are limited to the vicinity of the cracked cross section. Therfore, the beam element containing the cracked cross section is replaced by the elastic–plastic line–spring and the elastic–plastic line–spring model by Parks (1981) is proposed to analyse the dynamic fracture of a cracked beam of ductile material in present paper. By the reason that the inertia forces only appear in the equations of motion of a beam and not emerge in the constitutive relations of the Line–spring beam element, the resultant equations are simplified greatly.

NONLINEAR CONSTITUTIVE RELATIONS OF ELASTIC–PLASTIC LINE–SPRING

The constitutive relations of the cracked beam element, i.e., the Line–spring, are derived from the edge–cracked strip subjected to axial force N and bending moment M, as shown in Fig.1(b). The stress intensity factor of the edge crack is

$$K_I = \sqrt{h}\,[N / h g_N(\zeta) + M / h^2 g_B(\zeta)] \tag{1}$$

Where $\zeta = a / h$, $g_M(\zeta)$ *and* $g_B(\zeta)$ are taken from the handbook by Tada et al.(1973). The dimensionless stress intensity factor K_I is then

$$\overline{K}_I = Q_i g_i(\zeta) \tag{2}$$

where $\overline{K}_I = K_I / \sigma_0 \sqrt{h}$, $Q_1 = N / \sigma_0 h$, $Q_2 = M / \sigma_0 h^2$, $g_1(\zeta) = g_M(\zeta)$, $g_2(\zeta) = g_B(\zeta)$ and σ_0 = yield stress.

The strain energy induced by the crack is

$$U_c = \int_0^a dU_c / da\, da = \int_0^a G da = (1-\nu^2)/E\int_0^a K_I^2 da \qquad (3)$$

From eq.(1) and eq.(2) and using $\triangle = \partial U_c/\partial N$ and $\theta = \partial U_c/\partial M$, the linear constitutive relations in elastic phase are obtained as follows

$$q_i = \alpha_{ij} Q_j \qquad (i,j=1,2) \qquad (4)$$

where $q_1 = \triangle/h$, $q_2 = \theta$, $\triangle$ and θ are relative axial displacement and rotation of two ends of the edge cracked strip respectively, and

$$\alpha_{ij} = 2(1-\nu^2)\sigma_0/E\int_0^\zeta g_i(\zeta)g_j(\zeta)d\zeta \qquad (5)$$

In elastic–plastic phase, q_i can be expressed as the sum of elastic and plastic defomations

$$q_i = q_i^e + q_i^p \qquad (i=1,2) \qquad (6)$$

The eqs.(4)can be rewritten as

$$Q_i = S_{ij} q_j^e \qquad (i,j=1,2) \qquad (7)$$

The yield condition of the cracked beam element is derived from the plastic limit analysis of the edge cracked strip subjected to axial force N and bending moment M and can be written as

$$\varphi(Q_i,\zeta) = 0 \qquad (8)$$

Thus the following equations are given

$$\dot q_i^p = \lambda\partial\varphi/\partial Q_i \qquad (i=1,2) \qquad (9)$$

Differentiating eq.(8), we have

$$\partial\varphi/\partial Q_i\dot Q_i + \partial\varphi/\partial\zeta\dot\zeta = 0 \qquad (10)$$

From eqs.(7), we have

$$\dot Q_i = S_{ij}\dot q_j^e + dS_{ij}/d\zeta q_j^e\dot\zeta \qquad (i,j=1,2) \qquad (11)_a$$

Use eq.(6) and eqs.(9), the eq.$(11)_a$ become

$$\dot Q_i = S_{ij}(\dot q_j - \lambda\partial\varphi/\partial Q_j) + dS_{ij}/d\zeta q_j^e\dot\zeta \qquad (i,j=1,2) \qquad (11)_b$$

Substituting eqs.$(11)_b$ into eqs.(10),we have

$$\partial\varphi/\partial Q_i[S_{ij}(\dot q_j - \lambda\partial\varphi/\partial Q_j) + dS_{ij}/d\zeta q_j^e\dot\zeta] + \partial\varphi/\partial\zeta\dot\zeta = 0 \qquad (12)$$

It follows from eq.(12)that

$$\lambda = [\partial\varphi/\partial Q_i S_{ij}\dot q_j + (\partial\varphi/\partial Q_i dS_{ik}/d\zeta\alpha_{kj}Q_j + \partial\varphi/\partial\zeta)\dot\zeta]/A \qquad (13)$$

where $A = \partial\varphi/\partial Q_i S_{ij}\partial\varphi/\partial Q_j$.

Putting eq.(13) into eqs.$(11)_b$, then the nonlinear constitutive relations in elastic–plastic phase are derived

$$\dot Q_i = A_{ij}\dot q_j + B_i\dot\zeta \qquad (i,j=1,2) \qquad (14)$$

where

$$A_{ij} = S_{ij} - 1/A S_{il}\partial\varphi/\partial Q_l\partial\varphi/\partial Q_m S_{mj},$$

$$B_i = dS_{ik}/d\zeta\alpha_{kj}Q_j - 1/A(S_{il}\partial\varphi/\partial Q_l\partial\varphi/\partial Q_m dS_{mk}/d\zeta\alpha_{kj}Q_j + S_{ij}\partial\varphi/\partial Q_j\partial\varphi/\partial\zeta) = 0$$

When the crack is static, $\dot\zeta = 0$ and eqs.(14) become

$$\dot q_i = C_{ij}\dot Q_j \quad or \quad \dot Q_i = A_{ij}\dot q_j \qquad (15)_{a,b}$$

For a growing crack, eqs.(14) can be rewritten as

$$\dot q_i = C_{ij}(\dot Q_j - B_j\dot\zeta) \qquad (i,j=1,2) \qquad (16)$$

Substituting eqs.(16) into eq.(13), we obtain

$$\lambda = [\partial\varphi/\partial Q_i S_{ik}C_{kj}\dot Q_j + (\partial\varphi/\partial Q_i dS_{ik}/d\zeta\alpha_{kj}Q_j - \partial\varphi/\partial Q_i S_{ik}C_{kj}B_j + \partial\varphi/\partial\zeta)\dot\zeta]/A \qquad (17)$$

Substituting eq.(17) into eqs.(9), we obtain

$$\dot q_i^p = \partial\varphi/\partial Q_i[(\partial\varphi/\partial Q_l dS_{lk}/d\zeta\alpha_{kj}Q_j - \partial\varphi/\partial Q_i S_{ik}C_{kj}B_j + \partial\varphi/\partial\zeta)\dot\zeta + \partial\varphi/\partial Q_l S_{lk}C_{kj}\dot Q_j]/A \qquad (18)$$

CRACK TIP OPENING DISPLACEMENT AND CRACK GROWTH RATE

The crack tip opening displacement rate $\dot\delta_t$ and the relative plastic displacement rates $\dot\triangle^p$ and $\dot\theta^p$ have the following relation, see Parks(1981)

$$\dot\delta_t = \dot\triangle^p + (h/2-a)\dot\theta^p \qquad (19)$$

Using the dimensionless crack tip opening displacement $\delta_t^* = \delta_t/h$, eq.(19) becomes

$$\dot\delta_t^* = \dot q_1^p + (1/2-\zeta)\dot q_2^p \qquad (20)$$

Putting eqs.(18) into eq.(20), we have

$$\dot\delta_t^* = \mu_i\partial\varphi/\partial Q_i[\partial\varphi/\partial Q_l S_{lk}C_{kj}\dot Q_j + (\partial\varphi/\partial Q_l dS_{lk}/d\zeta\alpha_{kj}Q_j - \partial\varphi/\partial Q_l S_{lk}C_{kj}B_j + \partial\varphi/\partial\zeta)\dot\zeta]/A \qquad (21)$$

where $\mu_1 = 1$ and $\mu_2 = 1/2-\zeta$.

For a static crack, eq.(21) becomes

$$\dot\delta_t^* = \mu_i\partial\varphi/\partial Q_i\ (\partial\varphi/\partial Q_l S_{lk}C_{kj}\dot Q_j)\ /A \qquad (22)$$

For a growing crack, the crack growth rate is

$$\dot a = E\dot\delta_t/\ (\sigma_0 T_{\delta d}) = E/\ (\sigma_0 T_{\delta d})\ [\dot\triangle^p + (h/2-a)\dot\theta^p] \qquad (23)_a$$

and the dimensionless crack growth rate is

$$\dot\zeta = E/\ (\sigma_0 T_{\delta d})\ [\dot q_1^p + (0.5-\zeta)\dot q_2^p] \qquad (23)_b$$

where $T_{\delta d} = E/\sigma_0 d\delta_R/da$ (Tearing modulus)

Putting eqs.(18) into eq.$(23)_b$, we have

$$\dot\zeta = D_j\dot Q_j \qquad (24)$$

where

$$D_j = \mu_i\partial\varphi/\partial Q_i\partial\varphi/\partial Q_l S_{lk}C_{kj}/[\sigma_0 T_{\delta d}/\ (EA)\ + \mu_i\partial\varphi/\partial Q_i (\partial\varphi/\partial Q_l S_{lk}C_{kj}B_j - \partial\varphi/\partial Q_l dS_{lk}/d\zeta\alpha_{kj}Q_j - \partial\varphi/\partial\zeta)].$$

Substituting eq.(24) into eq.(21), we have

$$\dot\delta_t^* = \mu_i\partial\varphi/\partial Q_i[\partial\varphi/\partial Q_l S_{lk}C_{kj} + (\partial\varphi/\partial Q_l dS_{lk}/d\zeta\alpha_{km}Q_m - \partial\varphi/\partial Q_l S_{lk}C_{km}B_m + \partial\varphi/\partial\zeta)D_j]\dot Q_j/A \qquad (25)$$

For an unstable crack, its propagation rate can be given by

$$\dot\zeta = f\ (\delta_t^*) \qquad (26)$$

GOVERNING EQUATIONS AND DYNAMIC FRACTURE OF A SIMPLY SUPPORTED BEAM

A simply supported beam of retangular cross section subjected to uniformly distributed loading with a high speed $p = p_0 t / t_0$ is considered,as shown in Fig.(a). A cracked beam element is located at the middle of beam.

For the analysis of uncraked part in the beam, the elementary beam theory is adopted. From the equations of motion of the beam and its boundary conditions, it foloows that

$$N = -\rho h l \ddot{\Delta} / 2 \qquad (27)_a$$

$$M = -\rho h l^3 \ddot{\theta} / 6 + k l^2 f_0(\bar{t}) \qquad (27)_b$$

where

$$f_0(\bar{t}) = \bar{t}/2 - 16/\pi^3 \sum_{n=1,3,\ldots}^{\infty} (-1)^{\frac{(n-1)}{2}} 1/n^3 \, Sin\omega_n \bar{t} / \omega_n ,$$

$$\bar{t} = t / t^*, \; t^* = h / c_0, \; c_0 = \sqrt{E/\rho}, \; \rho = \text{mass density},$$

$$k = p_0 t^* / t_0 \text{ and } \omega_n = (n\pi h)^2 / (8\sqrt{3} l^2).$$

Adopting the dimensionless quantities Q_i and q_i, eqs.(27) can be rewritten as

$$Q_1 = -El/(2\sigma_0 h)\ddot{q}_1 \qquad (28)_a$$

$$Q_2 = -El^3/(6\sigma_0 h^3)\ddot{q}_2 + kl^2/(\sigma_0 h^2) f_0(\bar{t}) \qquad (28)_b$$

or

$$\ddot{q}_i = -D_{ij} Q_j + f_i(\bar{t}) \qquad (29)$$

where

$$\dot{q}_i = dq_i / d\bar{t}, \; \ddot{q}_i = d^2 q_i / d\bar{t}^2, \; D_{11} = 2\sigma_0 h / (El), \; D_{22} = 6\sigma_0 h^3 / (El^3),$$

$$D_{12} = D_{21} = 0, \quad f_1(\bar{t}) = 0, \quad f_2(\bar{t}) = 6kh / (El) \, f_0(\bar{t}).$$

In elastic phase,substituting eqs.(7) into eqs.(29), we have

$$\ddot{q}_i + D_{ij} S_{jl} q_l = f_i(\bar{t}) \qquad (30)$$

In elastic–plastic phase with a static crack, it follows from eqs.$(15)_b$ and eqs.(29) that

$$\dot{Q}_i = A_{ij} p_j \qquad (31)_a$$

$$\dot{p}_i = -D_{ij} Q_j + f_i(\bar{t}) \qquad (31)_b$$

where $p_i = \dot{q}_i$.

In elastic–plastic phase with a growing stable crack, it follows from eqs.(14), (24) and eqs.(29) that

$$\dot{\zeta} = D_l A_{lj} p_j / (1 - D_k B_k) \qquad (32)_a$$

$$\dot{Q}_i = [A_{ij} + D_l A_{lj} / (1 - D_k B_k)] p_j \qquad (32)_b$$

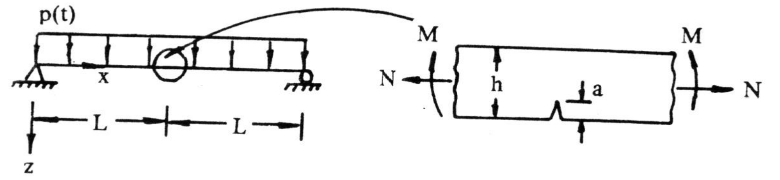
$$\dot{p}_i = -D_{ij} Q_j + f_i(\bar{t}) \qquad (32)_a$$

For a propagating unstable crack, eq.$(32)_a$ is replaced by eq.(26).

NUMERICAL RESULTS AND CONCLUSION

We had comprehensively analysed the dynamic fracture process of a simply supported beam with an edge crack that contains elastic,elastic–plastic and crack growth phase. The numerical results are given in Fig.2 to 5. Plots of the axial force Q_1 as a function of time $\bar{t}$ for various l / h are given in Fig.2, from which it can be seen that the axial force Q_1 is a pressure and its effect is important. Fig.3 and 4 give variation of relative rotation θ and crack tip opening displacement δ_t^* with time $\bar{t}$ for various ζ respectively. Fig.5 shows variation of crack growth rate ζ with time $\bar{t}$ for various l / h.

In present paper, we have advanced the elastic–plastic line–spring model so that it can be applied to the analysis of dynamic fracture in elastic–plastic range of structure. We can expect that this method for the analysis of dynamic fracture will be valuable due to its computational simplicity.



(a) (b)

Fig.1

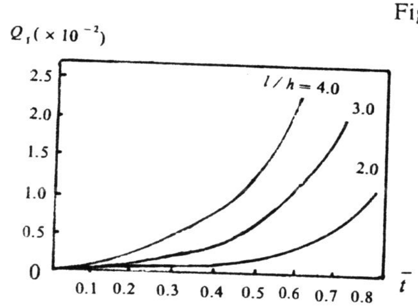


Fig.2

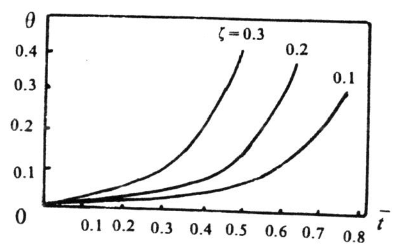


Fig.3

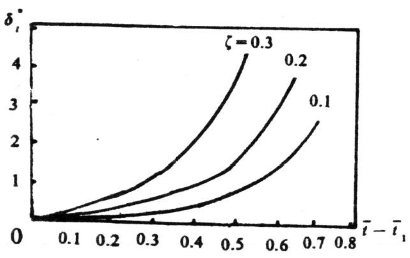


Fig.4

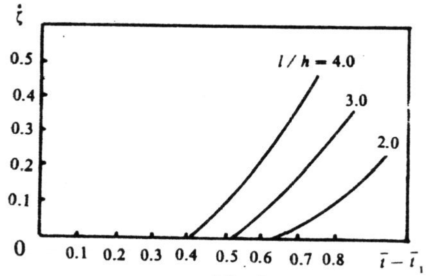


Fig.5

REFERENCES

[1]Parks,D.M. (1981): The inelastic Line–spring estimates of elastic–plastic fracture mechnics parameters for surface–cracked plates and shells. J. Press. Tech. 103, 246–254

[2]Tada, H., Paris, P. C. and Iwrin, G. R. (1973): The stress analysis of cracks Handbook. Del. Research Coporation.