A NONSTATIONARY THREE-DIMENSIONAL PROBLEM OF CONCENTRATION STRESS IN ELASTIC-PLASTIC BODIES

P.A. STEBLJANKO

Dneprodzerzhinsk Industrial Institute, Dneprostroevskaja St., 2, Dneprodzerzhinsk, Ukraine, 322618

ABSTRACT

In the present paper we propose a new technique for the numerical solution of some nonstationary problems for elastic-plastic bodies with concentrators. To define a three-dimensional stress-strained state the fractional step method is used with the unknown values given at each step in the form of two-dimensional time - and coordinate - dependent splines. Such an approach gives a better (than in case of the finite-difference method) opportunity to numerically solve the nonstationary three-dimensional problems for elastic-plastic bodies of an arbitrary shape. A FORTRAN program based on this method is also developed.

KEYWORDS

Nonstationary, three-dimensional, concentrator, elastic - plastic, fractional steps, spline.

NUMERICAL SOLUTION

In case of ortogonal coordinate system α_i (i =1, 2, 3), the complete system comrising three equations of motion, six constitutive equations and six Cauchy equations given in terms of velocities, has the following form :

$$\overline{W}_{,t} = \sum_{i=1}^{3} A_i \cdot \overline{W}_{,i} + \overline{B} . \qquad (1)$$

Here three constituents of the displacement velocity vector ($W_1 = v_1$, $W_2 = v_2$, $W_3 = v_3$), six components of stress tensor ($W_4 = \sigma_{11}$, $W_5 = \sigma_{22}$, $W_6 = \sigma_{33}$,

$W_7 = \sigma_{12}$, $W_8 = \sigma_{13}$, $W_9 = \sigma_{23}$) and six constituents of deformation tenzor ($W_{10} = \varepsilon_{11}$, $W_{11} = \varepsilon_{22}$, $W_{12} = \varepsilon_{33}$, $W_{13} = \varepsilon_{12}$, $W_{14} = \varepsilon_{13}$, $W_{15} = \varepsilon_{23}$) are the components of vector $\bar{W}$. Vector $\bar{B}$ and A_1 , A_2 , A_3 matrices 15 x 15 are known. In general case they can be dependent on some constituents of vector $\bar{W}$ (Stebljanko, 1991). Introducing time grid

$$\omega_\tau = \{ t_p ; t_{p+1/3} = t_p + \tau_1 , t_{p+2/3} = t_{p+1/3} + \tau_2 ,$$
$$t_{p+1} = t_{p+2/3} + \tau_3 , t_0 , \tau = \tau_1 + \tau_2 + \tau_3 , p = 0, 1, 2, \dots \} , \qquad (2)$$

we can write a system equivalent to (1) using the fractional step method (Janenko, 1971)

$$(\bar{W}^{p+n/3} - \bar{W}^{p+(n-1)/3}) / \tau = \alpha \cdot \Lambda_n (\bar{W}^{p+n/3}) +$$
$$+ \beta \cdot \Lambda_n (\bar{W}^{p+(n-1)/3}) + \gamma_n \bar{B}^{p+(n-1)/3} , \qquad (3)$$

where $\Lambda_n(\dots) = A_n \cdot (\dots)_n$, $\gamma_1 + \gamma_2 + \gamma_3 = 1$, $\alpha + \beta = 1$.

The elements of matrices A_n are calculated for $t = t_{p+(n-1)/3}$, $\bar{W}^{p+n/3}$ is calculated for $t = t_{p+n/3}$ (n = 1, 2, 3) . The components of this vector are given at each fractional step in terms of two-dimensional splines based on the product of B-splines of the first-order on time and of the third - order on α_n (Zavjalov et al, 1980)

$$W_m(\xi, h) = [(a_m)_q \cdot (1-h) + (a_m)_r \cdot h] \cdot [(b_m)_{i+2} \cdot \xi^3/6 +$$
$$+ (b_m)_{i+1} \cdot (-\xi^3 + \xi^2 + \xi + 1/3)/2 + (b_m)_i \cdot (\xi^3/2 - \xi^2 + 2/3) + (b_m)_{i-1} \cdot (1-\xi)^3/6] , \qquad (4)$$

where m = 1,2,..., 15; $q = p + (n-1)/3$; $r = p + n/3$; $\xi = [\alpha_n - (\alpha_n)_{i-1}]/h_n$; $h = (t - t_q)/\tau_n$; $\xi, h \in [0,1]$.

It is suggested to calculate the products of the splines coefficients instead of the coefficients themselves

$$(C_m)_k = (a_m)_q \cdot (b_m)_{i-k+3} ; \quad (C_m)_{k+4} = (a_m)_r \cdot (b_m)_{i-k+4} ,$$
$$k = 1, 2, 3, 4 \qquad (5)$$

Thus, the spline (4) can be reduced to

$$W_m(\xi, h) = \sum_{i=1}^{8} (C_m)_i \cdot \varphi_i(\xi, h) \qquad (6)$$

The functions $\varphi_i(\xi, h)$ can be easily obtained from (4). Approximation(6) gives values of W_m not only for the grid knots

$$\omega_h = \{ [(\alpha_1)_i, (\alpha_2)_j, (\alpha_3)_k] ; (\alpha_1)_i = (\alpha_1)_{i-1} + h_1 ; i = \overline{1, N_1} ,$$
$$(\alpha_2)_j = (\alpha_2)_{j-1} + h_2 ; j = \overline{1, N_2} ; (\alpha_3)_k = (\alpha_3)_{k-1} + h_3 , k = \overline{1, N_3} \} , \qquad (7)$$
$$h = \max(h_1, h_2, h_3) ,$$

but also between them, which is significant when treating the nonstationary problems for the bodies of combined shape. It must be noted that finite-difference approximation of the derivatives (3) allows only the first order, while (6) ensures the third order of approximation with respect on h (Zavjalov et al, 1980).Therefore, larger grids (than in case of difference approximation) can be chosen with the same accuracy of computation which leads to the greater stability of calculations.

The algorithm for the numerical solution of the system (3) consists of three steps. In the first step (n = 1) the auxiliary vector $\bar{W}^{p+1/3}$ is calculated. Vector $\bar{W}_p$ is initially known or calculated in the previous step. In the next steps (n = 2 ; 3) vectors $\bar{W}^{p+2/3}$ and $\bar{W}^{p+1}$ are calculated. The procedure is repeated further on. All the quantities entering the equation (1), (3) are reduced to the nondimensional form (Stebljanko, 1991).

Half of the coefficients (5) are determined by means of $\bar{W}^{p+(n-1)/3}$ using the formula

$$(C_m)_k = \sum_{\ell=1}^{4} d_{\ell k} W_m[(\ell-1)/3 ; 0] , \quad k = 1, 2, 3, 4 , \qquad (8)$$

where

$$d_{11} = d_{44} = -4 ; \quad d_{21} = d_{34} = 16,5 ; \quad d_{31} = d_{24} = -24 ;$$
$$d_{41} = d_{14} = 12,5 ; \quad d_{12} = d_{43} = 1,5 ; \quad d_{22} = d_{33} = -6 ;$$
$$d_{32} = d_{23} = 7,5 ; \quad d_{42} = d_{13} = -2 .$$

The rest of the coefficients $(C_m)_{k+4}$ are calculated at each fractional step, using the collocation conditions and the known conditions at the mesh boundaries of the ω_h grid. The latter may be treated as boundary conditions. In the case of several known values of W_m (three values) for boundary $(\alpha_n)_0$ (ξ =0), the following equation can be developed

$$[(C_m)_6 + 4(C_m)_7 + (C_m)_8]/6 = W_m(0,1) , \qquad (9)$$

and for boundary $(\alpha_n)_N$ (ξ = 1)

$$[(C_m)_5 + 4(C_m)_6 + (C_m)_7]/6 = W_m(1,1). \qquad (10)$$

Analogous conditions are being formulated for the interior meshes of the grid ω_h for all values of m. The lacking equations are derived from the collocation condition

$$(C_m)_5 \cdot \xi^3/6 + (C_m)_6 \cdot (-\xi^3 + \xi^2 + \xi + 1/3)/2 + (C_m)_7 (\xi^3/2 - \xi^2 + 2/3) +$$

$$+ (C_m)_8 \cdot (1-\xi)^3/6 - \alpha\tau/h_n \cdot \sum_{s=1}^{15} (A_n)_{ms} \left[(C_s)_5 \, \xi^2/2 + \right. \qquad (11)$$

$$\left. + (C_s)_6 (-3\xi^2/2 + \xi + 1/2) + (C_s)_7 \cdot (3\xi^2/2 - 2\xi) - (C_s)_8 \cdot (1-\xi)^2/2 \right] = Q_m(\xi),$$

which is obtained by substituting (6) into (3). Here $(A_n)_{ms}$ is the element of the matrix A_n, and $Q_m(\xi)$ is determined by the known values. To define $(C_m)_{k+4}$ we use the collocation condition (11), written for $\xi = n/3$ (n = 0, 1, 2, 3) and supplemented with the conditions of (9), (10) type. In general case a system consisting of 15 x 4 equations is to be numerically solved. If we confine ourselves to the explicit scheme only, when $\alpha = 0$ and $\beta = 1$, the system falls apart into 15 uniform subsystems, which are solved as follows

$$(C_m)_{k+4} = \sum_{\ell=1}^{4} d_{\ell k} \cdot Q_m[(\ell-1)/3]. \qquad (12)$$

In the case of some values of W_m being known at the mesh boundary (ξ = 0 ; 1 and η = 1), Q_m can be substituted for W_m in (12).

THREE-DIMENSIONAL PLATE WITH CONCENTRATORS

The author employs the above-mentioned technique to find numerical solutions to some nonstationary problems for the elastic-plastic plate which is modelled by a three-dimensional body. A notch and holes of three types were chosen as concentrators. If the body is undergoing plastic deformations at the near-concentrator area, we use the theory proposed by Shevchenko (1987).

In tests 1 - 4 a nondimentional velocity of displacement (Stebljanko, 1991) is locally predetermined. The direction is indicated by an arrow. The rest of the surface area is free from load.

The results of the numerical solution (τ = 0,01 ; h_1 = 0,1 ; h_2 = 0,1 ; h_3 = 0,1) are presented in the form of the stress intensity fields σ_u / σ_s (σ_s is the elastisity limit) at different instants of time. Here

$$\sigma_u = \left\{ \left[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 \right]/2 + 3(\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2) \right\}^{1/2}.$$

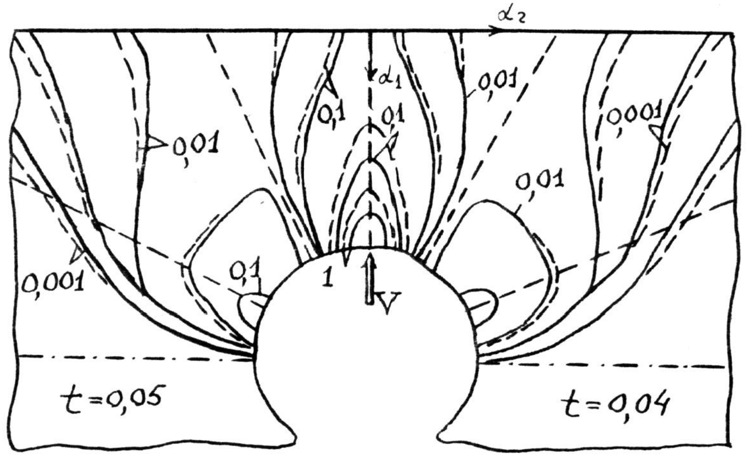

Fig. 1. Test. 1. Field of σ_u / σ_s -const.

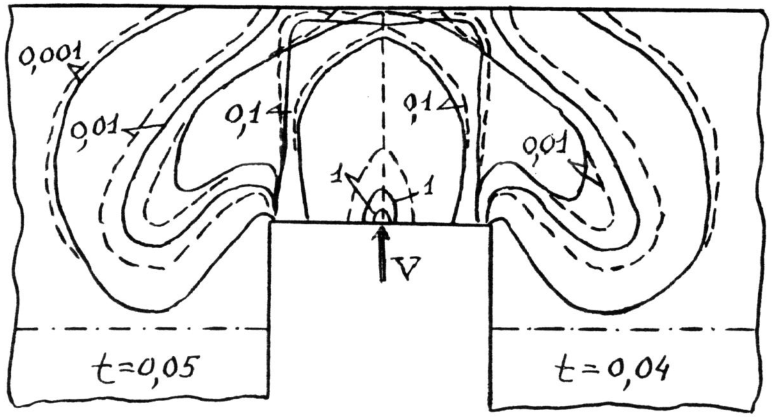

Fig. 2. Test 2. Field of σ_u / σ_s -const.

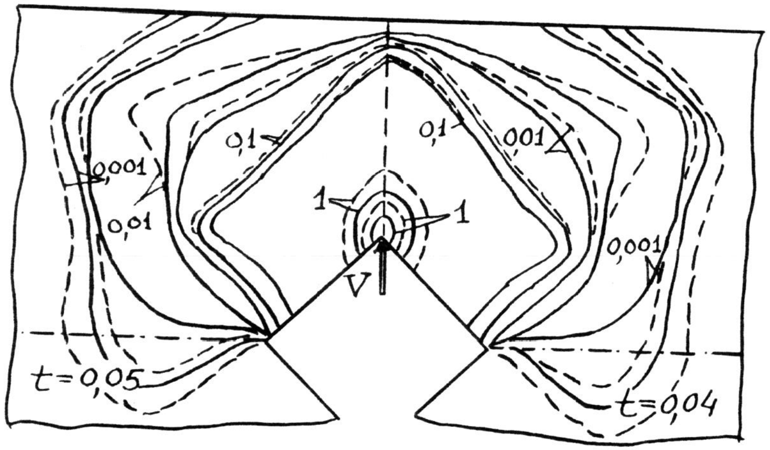

Fig. 3. Test 3. Field of σ_u / σ_s -const.

The case of combined deformation of hole boundary, when $\vartheta_1 \neq 0$ and $\vartheta_2 \neq 0$, is considered in Test 4.

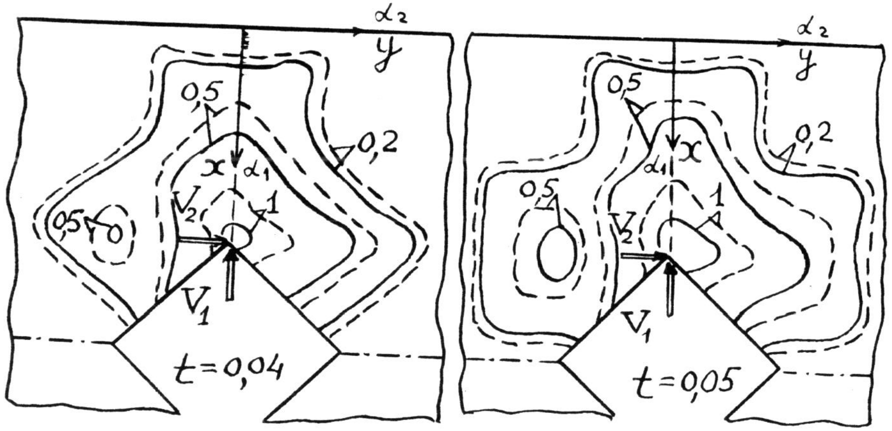

Fig. 4. Test 4. Field of σ_u / σ_s -const.

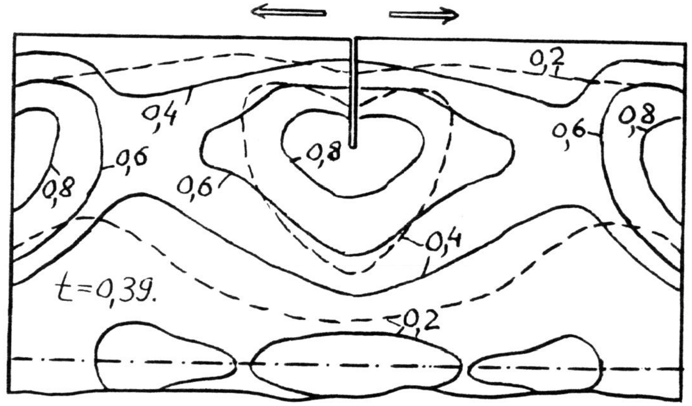

Fig. 5. Test 5. Field of σ_u / σ_s -const.

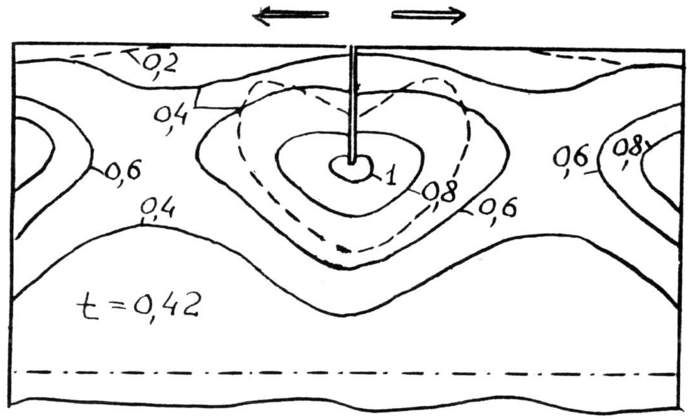

Fig. 6. Test 5. Field of σ_u / σ_s -const.

In test 5 a three-dimensional plate with a notch is investigated (τ = 0,03 ; h_1 = h_2 = 0,2 ; h_1 = 0,1). When $\alpha_1 = 0$, $\vartheta_2 = a \cdot sign(\alpha_2)$ ($a \neq 0$ if $t \leq 0,18$ and $a \equiv 0$ if $t > 0,18$).

In Fig. 1-5 the stress intensity field for mid-plane $\alpha_3 = 0$ is shown with a firm line; and for $\alpha_3 = \pm H/2$ with a dotted line. Here H is a plate thickness (H = 0,2).

CONCLUSION

It should be noted that the technique outlined above makes it possible to employ, with the same accuracy of numerical calculations, ω_h grid with the far more greater step h than the finite - difference method. (As an a priori estimate we can write $h^3 \approx h$, where h_* denotes the integration step of the finite - difference method.)

Thus, employing cubic B-splines for approximating the unknown values v_i , σ_{ij} , ε_{ij} and using the fractional step method we develop an ecónomical technique for solving nonstationary three-dimensional problems for the elastic - plastic bodies of an arbitrary shape.

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