A LAGRANGIAN DESCRIPTION OF THE MOVING CRACK PROBLEM

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ABSTRACT.

The aim of this paper is to present a general Lagrangian description of a bidimensional medium with a moving crack, using a mapping of the moving domain to a fixed one, and then, in the framework of linear elasticity, to compute the Lagrangian expression of the energy release rate and of the tearing modulus for each current crack growth.
This kind of method has already been successfully used for a stationary crack, or more precisely for an infinitesimal virtual crack growth, to compute the energy release rate for the initial crack length [Destuynder-Djaoua. 1981]. We suggest to extend this description to finite crack growth to analyze the moving crack problem.
The advantages of this method are firstly to avoid remeshing or introducing special numerical parameters such as relaxation forces to simulate the crack growth, secondly to obtain exact derivatives of the potential energy.

KEYWORDS.

Moving crack, arbitrary mapping, Lagrangian representation, energy release rate, tearing modulus.

INTRODUCTION.

The moving crack problem is generally treated with specific numerical techniques, such as remeshing or nodal forces relaxation, directly in the discretized model.
Nevertheless, in some particular cases, the moving crack problem has already been treated in the framework of a continuum model. For

instance, in the case of the steady-state crack growth in an infinite tensile strip, the complete analysis of the continuum problem has been achieved for an elastoplastic material [Nguyen, Rahimian 1981]. For particular domains such as circular disks, using a conformal mapping of the moving domain to a fixed one, the problem has also been treated for an elastic material [Fedelich 1990]. For the elastodynamic case, a moving finite element method based on a mixed Eulerian-Lagrangian description has been performed [Koh et al. 1988]. This method operates on the discretized model and uses a mapping defined locally for each finite element, and leads to a non-symmetric form of the variational equations, due to the convective term.

To avoid the difficulties of the numerical techniques used to describe the crack kinematics (in particular remeshing), we propose an arbitrary Lagrangian description of the continuum problem, for any kind of two dimension cracked domain and for a large class of boundary conditions, which leads to a symmetric variational problem on the initial cracked domain. For the quasi-static elastic case, we can find an exact Lagrangian expression of the derivative of the energy as a function of the current crack length, represented by an integral defined on the initial cracked domain.

STATEMENT OF THE MOVING CRACK PROBLEM.

Definition of the problem on the current moving cracked domain.

We consider an homogeneous 2D continuum medium, with a straight moving crack , represented by a family of domains Ω_α of $\mathbb{R}^2$, with a moving line $\overline{AP}$, as presented in figure 1.

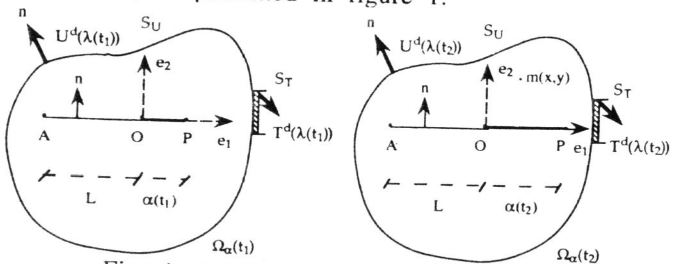

Fig. 1. Family of domains with a moving crack.

The evolution parameter of the medium is the crack extension $\alpha(t)$, and we note $\lambda(t)$ the loading parameter. The stationary part of the boundary of Ω_α is subjected to mixed conditions $T^d(\lambda)$ and $U^d(\lambda)$, and the moving crack is free of stress. The current crack length is $L+\alpha(t)$, with $\alpha(0)=0$.

We assume small displacements and no volume force. For each couple (α,λ), the equations of the local problem are:

$$\begin{cases} \varepsilon = (\nabla \mathbf{u})_S = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T) \text{ in } \Omega_\alpha, \\ \mathbf{u} = U^d(\lambda) \text{ on } S_U, \\ \sigma = \frac{\partial \Phi}{\partial \varepsilon}(\nabla \mathbf{u})_S, \\ \operatorname{div} \sigma = 0 \text{ in } \Omega_\alpha, \\ \sigma.\mathbf{n} = T^d(\lambda) \text{ on } S_T, \end{cases} \qquad (1)$$

where Φ is the elastic potential.

The moving boundary conditions on the crack are:

$$\sigma.\mathbf{n} = 0,\ \forall m(x,y),\ m \in \overline{AP}\ (y=0,\ -L<x<\alpha(t)). \qquad (2)$$

Remark: we assume that $\alpha(t)$ is known all along the evolution process, the determination of $\alpha(t)$ is a problem we will not discuss in this paper. The following method is applicable even when only the evolution law of $\alpha(t)$ is given.

To produce the variational form of the problem, we look for the displacement field **u**, among the admissible fields **v**, which minimizes the potential energy

$$W(\mathbf{v},\alpha,\lambda) = \int_{\Omega_\alpha} \Phi(\nabla \mathbf{v})_S \, dV - \int_{S_T} T^d(\lambda).\mathbf{v} \, dS . \qquad (3)$$

Mapping of the moving domain Ω_α to the initial domain

We introduce a one-to-one C^1 geometrical mapping f_α, which connects the point M(X,Y) of Ω_0 to the current material point m(x,y) of Ω_α, such that f_α^{-1} maps the family of cracks $\overline{AP}$ (length $L+\alpha$) into the initial crack $\overline{AO}$ (length L), as presented in the figure 2.

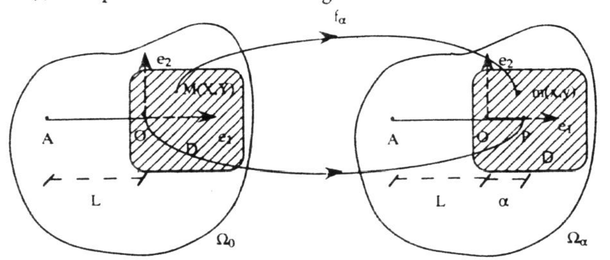

Fig. 2. Mapping defined on the subdomain D

Formally, this mapping , which becomes identity function out of the sub-domain D of Ω_0, has the following properties:

$$\begin{cases} f_\alpha(D) = D, \\ f_\alpha(M) = M, \text{in } \Omega_0\text{-}D, \\ f_\alpha(\overline{AO}) = \overline{AP}, \\ f_\alpha(O) = P, \\ f_0 = I^d. \end{cases} \quad (4)$$

With such a transformation, starting from any current domain Ω_α with the moving crack, we can restore the initial crack on the initial domain Ω_0.

Definition of the problem on the initial cracked domain.

With the help of the previous mapping, we can change the spatial variable in the displacement field:

$$\forall M,\ M \in \Omega_0,\ \mathbf{v}_\alpha(M) \equiv \mathbf{v}(f_\alpha(M)). \quad (5)$$

The new displacement field $\mathbf{v}_\alpha$ associated to the point M does not describe a specific material particle as in classical Lagrangian representation but a family of material particles $f_\alpha(M)$, determined by the choice of the mapping. At each step of the evolution, the set of points M of Ω_0 represents an arbitrary Lagrangian configuration. If we note $\mathbf{F}_\alpha$ the gradient of f_α, the moving boundary condition (2) now becomes stationary:

$$(\sigma.\mathbf{F}_\alpha^{T}).\mathbf{n} = 0,\ \forall M(X,Y),\ M \in \overline{AO}\ (Y=0,\ -L<X<0). \quad (6)$$

With the variable switching (5), the new problem, defined on the fixed domain Ω_0, lies in searching, among the admissible fields $\mathbf{v}$, the solution $\mathbf{u}_\alpha$ which minimizes the new expression of the potential energy:

$$\widehat{W}(\mathbf{v},\alpha,\lambda) = \int_{\Omega_0} \Phi(\nabla\mathbf{v}.\mathbf{F}_\alpha^{-1})_S \det\mathbf{F}_\alpha \, dV - \int_{S_T} T^d.\mathbf{v}\, dS. \quad (7)$$

Remark: this formulation may be extended to non-elastic materials with internal variables "transported" on the fixed domain. These internal variables and their evolution law are described on the arbitrary Lagrangian configuration.

CASE OF LINEAR ELASTICITY.

Lagrangian expression of the energy release rate.

In the case of linear elasticity, we can develop the elastic potential as following:

$$\Phi(\varepsilon) = \frac{1}{2}\mathbf{A}:\varepsilon:\varepsilon,$$

where $\mathbf{A}$ is a fourth-order positive symetric tensor. The expression (7) of the energy becomes:

$$\widehat{W}(\mathbf{v},\alpha,\lambda) = \frac{1}{2}\int_{\Omega_0} \mathbf{A}:(\nabla\mathbf{v}.\mathbf{F}_\alpha^{-1}):(\nabla\mathbf{v}.\mathbf{F}_\alpha^{-1}) \det\mathbf{F}_\alpha \, dV - \int_{S_T} T^d.\mathbf{v}\, dS . \quad (8)$$

To pull off, in the mapping f_α, the crack growth parameter α from the specific geometrical function, we choose the mapping as following:

$$\forall M,\ M \in \Omega_0,\ f_\alpha(M) = M + \alpha\overrightarrow{\xi(M)} ,$$

where $\vec{\xi}$ is a vector field, which becomes zero outside of the domain D. Then, we can develop the inverse of the transformation gradient $\mathbf{F}_\alpha^{-1}$:

$$\mathbf{F}_\alpha^{-1} = \frac{1}{\det\mathbf{F}_\alpha}(\mathbf{I^d} + \alpha\mathbf{C}),$$

$$\text{with} \quad \mathbf{C} = -\nabla\vec{\xi} + \mathrm{div}\vec{\xi}\ \mathbf{I}^d.\ \det\mathbf{F}_\alpha = 1 + \alpha\,\mathrm{div}\vec{\xi} + \alpha^2 \det\nabla\vec{\xi}.$$

At equilibrium, the expression of the potential energy becomes:

$$\begin{cases} \widetilde{W}(\alpha,\lambda) \equiv \widehat{W}(\mathbf{u}_\alpha,\alpha,\lambda) = \frac{1}{2}\int_{\Omega_0} \frac{1}{\det\mathbf{F}_\alpha}\mathbf{A}:(\nabla\mathbf{u}_\alpha):\nabla\mathbf{u}_\alpha \, dV \\ + \alpha\int_D \frac{1}{\det\mathbf{F}_\alpha}\mathbf{A}:(\nabla\mathbf{u}_\alpha):(\nabla\mathbf{u}_\alpha.\mathbf{C})\, dV + \frac{\alpha^2}{2}\int_D \frac{1}{\det\mathbf{F}_\alpha}\mathbf{A}:(\nabla\mathbf{u}_\alpha.\mathbf{C}):(\nabla\mathbf{u}_\alpha.\mathbf{C})\, dV \\ - \int_{S_T} T^d.\mathbf{v}\, dV , \end{cases} \quad (9)$$

where $\mathbf{u}_\alpha = \mathbf{u} \circ f_\alpha$ is the solution of the minimization problem of the functional $\widehat{W}$. Starting from the definition of the energy release rate:

$$G(\alpha,\lambda) = -\frac{\partial \widetilde{W}(\alpha,\lambda)}{\partial \alpha}.$$

Noting that $\mathbf{u}_\alpha$ is optimal, and developing in the expression (9) the quantities $\mathbf{C}$ and $\det \mathbf{F}_\alpha$, the Lagrangian expression of G is:

$$(10)\left\{\begin{aligned} G(\alpha,\lambda) = \frac{1}{2}\int_D & \frac{\alpha^2(-\operatorname{div}\vec{\xi}^{\,3} + 2\operatorname{div}\vec{\xi}\det\nabla\vec{\xi}) + 2\alpha(-\operatorname{div}\vec{\xi}^{\,2} + \det\nabla\vec{\xi}) - \operatorname{div}\vec{\xi}}{(1+\alpha\operatorname{div}\vec{\xi} + \alpha^2\det\nabla\vec{\xi})^2}\,A:\nabla u_\alpha:\nabla u_\alpha\, dV \\ + \int_D & \frac{(1 + \alpha^2(-\det\nabla\vec{\xi} + \operatorname{div}\vec{\xi}^{\,2}) + 2\alpha\operatorname{div}\vec{\xi}\,)}{(1+\alpha\operatorname{div}\vec{\xi} + \alpha^2\det\nabla\vec{\xi})^2}\,A:\nabla u_\alpha:(\nabla u_\alpha.\nabla\vec{\xi})\, dV \\ -\frac{1}{2}\int_D & \frac{\alpha^2\operatorname{div}\vec{\xi} + 2\alpha}{(1+\alpha\operatorname{div}\vec{\xi} + \alpha^2\det\nabla\vec{\xi})^2}\,A:(\nabla u_\alpha.\nabla\vec{\xi}):(\nabla u_\alpha.\nabla\vec{\xi})\, dV. \end{aligned}\right.$$

Remark: for $\alpha=0$, the expression (10) becomes:

$$G(0,\lambda) = \int_D \mathbf{A}:\nabla\mathbf{u}:(\nabla\mathbf{u}.\nabla\vec{\xi} - \tfrac{1}{2}\operatorname{div}\vec{\xi}\nabla\mathbf{u})\, dV. \quad (11)$$

which is a result obtained by a domain perturbation method for a stationary crack [Destuynder-Djaoua 1981]. The expression (11) is in fact a Lagrangian form of the Rice integral . Our result (10) is an extension of (11) for finite crack growth.

A special selection of the mapping.

If we choose an incompressible geometrical transformation ($\det\mathbf{F}_\alpha = 1$), the elastic problem may be written in the simple form for each (α,λ):

$$\left\{\begin{aligned} & \int_{\Omega_0} \mathbf{A}:\nabla\mathbf{u}_\alpha:\nabla\mathbf{v}\, dV - \alpha\int_D \mathbf{A}:(\nabla\mathbf{u}_\alpha.\nabla\vec{\xi}):\nabla\mathbf{v}\, dV - \alpha\int_D \mathbf{A}:(\nabla\mathbf{u}_\alpha):(\nabla\mathbf{v}.\nabla\vec{\xi})\, dV \\ & +\alpha^2\int_D \mathbf{A}:(\nabla\mathbf{u}_\alpha.\nabla\vec{\xi}):(\nabla\mathbf{v}.\nabla\vec{\xi})\, dV = \int_{S_T} T^d.\mathbf{v}\, dS, \qquad (12) \\ & \forall\ \mathbf{v}, \text{ admissible displacement field.} \end{aligned}\right.$$

The linear system (12) has constant coefficients (that means that in the discretized problem, if we use the Finite Element Method, the "element stiffness matrix" corresponding to those coefficients are computed once only during the problem evolution).

For each (α,λ), the energy release rate takes the form:

$$G(\alpha,\lambda) = \int_D \mathbf{A}:\nabla\mathbf{u}_\alpha:(\nabla\mathbf{u}_\alpha.\nabla\vec{\xi})\, dV - \alpha\int_D \mathbf{A}:(\nabla\mathbf{u}_\alpha.\nabla\vec{\xi}):(\nabla\mathbf{u}_\alpha.\nabla\vec{\xi})\, dV. \quad (13)$$

The tearing modulus, defined as the derivative of G with respect of α, is very classical in stability analysis of one crack or a system of cracks [Nemat-Nassers et al. 1980]. This quantity depends on the rate displacement $\frac{\partial \mathbf{u}_\alpha}{\partial \alpha}$, solution of the following problem:

$$\left\{\begin{aligned} & \int_{\Omega_0} \mathbf{A}:\nabla\frac{\partial\mathbf{u}_\alpha}{\partial\alpha}:\nabla\mathbf{v}\, dV - \alpha\int_D \mathbf{A}:(\nabla\frac{\partial\mathbf{u}_\alpha}{\partial\alpha}.\nabla\vec{\xi}):\nabla\mathbf{v}\, dV - \alpha\int_D \mathbf{A}:(\nabla\frac{\partial\mathbf{u}_\alpha}{\partial\alpha}):(\nabla\mathbf{v}.\nabla\vec{\xi})\, dV \\ & +\alpha^2\int_D \mathbf{A}:(\nabla\frac{\partial\mathbf{u}_\alpha}{\partial\alpha}.\nabla\vec{\xi}):(\nabla\mathbf{v}.\nabla\vec{\xi})\, dV = \int_D \mathbf{A}:(\nabla\mathbf{u}_\alpha.\nabla\vec{\xi}):\nabla\mathbf{v}\, dV + \int_D \mathbf{A}:(\nabla\mathbf{u}_\alpha):(\nabla\mathbf{v}.\nabla\vec{\xi})\, dV \\ & - 2\alpha\int_D \mathbf{A}:(\nabla\mathbf{u}_\alpha.\nabla\vec{\xi}):(\nabla\mathbf{v}.\nabla\vec{\xi})\, dV, \qquad (14) \\ & \forall\ \mathbf{v}, \text{ admissible displacement field.} \end{aligned}\right.$$

From the numerical point of view, this formulation is convenient, because the rate displacement problem (14) is the same problem as the displacement problem (12) with another right hand side.

Deriving expression (13) and choosing $\mathbf{v} = \frac{\partial \mathbf{u}_\alpha}{\partial \alpha}$ in expression (14), we get the tearing modulus:

$$\left\{ \begin{array}{l} T(\alpha,\lambda) = \dfrac{\partial G(\alpha,\lambda)}{\partial \alpha} = -\displaystyle\int_D \mathbf{A}:(\nabla \mathbf{u}_\alpha . \nabla \vec{\xi}):(\nabla \mathbf{u}_\alpha . \nabla \vec{\xi})\, dV \\ + \displaystyle\int_{\Omega_0} \mathbf{A}:\nabla \frac{\partial \mathbf{u}_\alpha}{\partial \alpha} : \nabla \frac{\partial \mathbf{u}_\alpha}{\partial \alpha}\, dV - 2\alpha \int_D \mathbf{A}:(\nabla \frac{\partial \mathbf{u}_\alpha}{\partial \alpha} . \nabla \vec{\xi}):\nabla \frac{\partial \mathbf{u}_\alpha}{\partial \alpha}\, dV \\ + \alpha^2 \displaystyle\int_D \mathbf{A}:(\nabla \frac{\partial \mathbf{u}_\alpha}{\partial \alpha} . \nabla \vec{\xi}):(\frac{\partial \mathbf{u}_\alpha}{\partial \alpha} . \nabla \vec{\xi})\, dV. \end{array} \right. \quad (15)$$

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