THE RESIDUAL STRESSES INFLUENCE UPON THE BODY WEDGED OUT BY THE RIGID INCLUSIONS

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ABSTRACT

The problems related to the indentation of a linear or penny-shaped cracks by a thin smooth rigid inclusions were considered (Barenblatt, Cherepanov, 1960; Maiti, 1980;Tsai, 1984; and anothers). This sort of problems are of interest to the modelling of fracture processed in composite elastic materials.

This work deals with the solution of the problem of the expanding of penny-shaped crack situated in the three dimension body in the field of residual stress by the rigid thin circular inclusion. The research was carried out in linearized theory by Guz' (1984) limits using the marks used in this work.

KEYWORDS

Residual or initial stress, crack, rigid inclusion, displacement, boundary condition, integral, equation, garmonical function.

BASIC FORMULAE

The penny-shaped crack of radius b is located in the plane $y_3=0$ inside the body stressed before (fig. 1).

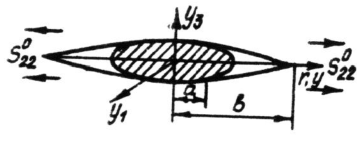

Fig.1. Indentation of a penny-shaped crack by a smooth disc inclusion.

Cartesian (y_1, y_2, y_3) and cylindrical (r, θ, y_3) coordinates in the initially deformed state are introduced. The initial stresses field is homogeneous, and satisfies following assumption $S^0_{33}=0$, $S^0_{11}=S^0_{22}\neq 0$. The coeficients of extending along the axes $\lambda_1=\lambda_2\neq\lambda_3$. The traction free plane surfaces of the crack are indented by a smooth, rigid oblate spheroidal inclusion. Because of the symmetricity of properties relating the plane $y_3=0$, mixed boundary conditions of the problem may be put as:

$$\sigma_{3r}=0 \ , \qquad r\geq 0 \tag{1}$$

$$u_3=\begin{cases} g(r) \ , & 0\leq r\leq a \\ 0 \ , & r>b, \ \sigma_{33}=0, a\leq r<b \end{cases} \tag{2}$$

Where $g(r)$ - is function of the shape of inclusion.

Solution of the static problems of the theory of elasticity for the bodies with an initial stress can be reduced to the finding of two functions φ_j $(j=1,2)$, which satisfy equation:

$$(\partial^2/\partial r^2 + \frac{1}{r}\frac{\partial}{\partial r} + n_j\frac{\partial^2}{\partial y_3^2})\ \varphi_j=0 \ , \ j=1,2.$$

For all this, according to the demands of uniqueness of solution of the linearized problem for equal and non-equal roots, different representations of solution with the help of function φ_j can have place.

There are expressions for displacements and stresses when $y_3=0$

$n_1=n_2$

$$u_3=\frac{m_1}{\sqrt{n_1}}\frac{\partial\varphi_1}{\partial z_1}+\frac{(m_2-1)}{\sqrt{n_1}}\frac{\partial\varphi_2}{\partial z_1} \ , \ \sigma_{33}=C_{44}\frac{\partial^2}{\partial z_1^2}[(1+m_1)l_1\varphi_1+ (1+m_2)l_2\varphi_2] \ , \ \sigma_{3r}=\frac{C_{44}}{\sqrt{n_1}}\frac{\partial^2}{\partial r\partial z_1}\Big[(1+m_1)\varphi_1+(1+m_2)\varphi_2\Big] \tag{3}$$

$n_1\neq n_2$

$$u_3=\frac{m_1}{\sqrt{n_1}}\frac{\partial\varphi_1}{\partial z_1}+\frac{m_2}{\sqrt{n_2}}\frac{\partial\varphi_2}{\partial z_2} \ , \ \sigma_{33}=C_{44}\Big[(1+m_1)l_1\frac{\partial^2\varphi_1}{\partial z_1^2} + (1+m_2)l_2\frac{\partial^2\varphi_2}{\partial z_2^2}\Big] \ , \ \sigma_{3r}=C_{44}\Big[\frac{1+m_1}{\sqrt{n_1}}\frac{\partial^2\varphi_1}{\partial r\partial z_1}+\frac{1+m_2}{\sqrt{n_2}}\frac{\partial^2\varphi_2}{\partial r\partial z_2}\Big]. \tag{4}$$

where $z_j=y_3/\sqrt{n_j}$, $j=1,2$.

The values of the quantities n_j, m_j, C_{44}, l_j are determined of the elastic properties of the material and by the field of the initial stresses (Guz', 1983).

The condition of the absence of tangencial stresses along all plane $y_3=0$ makes possible not to consider one of the function φ_j and to reduce the solution of the problem to the determination of the boundary conditions of the function f, which satisfies the equation

$$\Big[\frac{\partial^2}{\partial r^2}+\frac{\partial}{r\partial r}+\frac{\partial^2}{\partial y_3^2}\Big]f(r,y_3)=0 \ . \tag{5}$$

Displacements u_3 and stresses σ_{33} with $y_3=0$ with the help of this function can be determined out of the relationships

$$u_3=\frac{\omega}{(1+m_1)(1+m_2)}\frac{\partial f}{\partial y_3} \ , \ \sigma_{33}=C_{44}\lambda\frac{\partial^2 f}{\partial y_3^2} \ ,$$

$$\omega=\begin{cases}\dfrac{1+2m_1-m_2}{\sqrt{n_1}}, & n_1=n_2 \\ m_1-m_2 \ , & n_1\neq n_2\end{cases} \qquad \lambda=\begin{cases} l_1-l_2 \ , & n_1=n_2 \\ l_1\sqrt{n_1} - l_2\sqrt{n_2} \ , & n_1\neq n_2\end{cases} \tag{6}$$

Out of the boundary conditions (2), taking into consideration (6) we get such mixed problem of the theory of garmonical potential

$$\frac{\partial f}{\partial y_3}=\frac{(1+m_1)(1+m_2)}{\omega}g(r) \ , \ 0\leq r\leq a \ ,$$
$$\frac{\partial^2 f}{\partial y_3^2}=0 \ , \ a\leq r<b \ , \quad \frac{\partial f}{\partial y_3}=0 \ , \ r>b \ . \tag{7}$$

CONTACT STRESS AND STRESS INTENSITY FACTOR

This problem can be redused to the triple integral equation, which are easy to solve, for example, by the method given by Cooke (1963). At the same time, it is obvious that the received problem of garmonical potential coincides with the problem which appears in the classical theory of isotropic body elasticity, if $\frac{(1+m_1)(1+m_2)}{\omega}$ with be changed for $-\frac{\mu}{1-\nu}$.

Thus, it is possible to get the solution of the problem for the body with an initial stress if the solution of the corresponding problem of the linear theory of elastisity is known. In particular for the stresses in the plane $y_3=0$ it is possible to write such relationship

$$\sigma_{33}(r)=k\sigma^0_{33}(r) \quad . \tag{8}$$

Where σ^0_{33} - are the stresses of the corresponding problem of the linear-elastic isotropic body, k - coefficient, which determines the influence of the initial stresses:

$$k=\frac{1-\nu}{\mu}\;\frac{(1+m_1)(1+m_2)C_{44}\lambda}{\omega} \quad . \tag{9}$$

For the stress intensity factor K_1 in the point $r=b$ we can also write

$$K_1=kK^0_1 \quad , \tag{10}$$

where K^0_1 - stress intensity factor of the corresponding problem for the linear elastic isotropic body.

The range of value of the coefficient k is illustrated by an example of neo-hookean body and the theory of small initial deformations.

In this case we have

$$\begin{aligned} &n_1=\lambda_1^{-6},\; n_2=1\;,\; n_3=\lambda_1^{-6},\; \lambda_3=\lambda_1^{-2},\; C_{44}=\mu\lambda_1^{-4}, \\ &m_1=\lambda_1^{-6}\;,\; m_2=1\;,\; l_1=2\lambda_1^6(\lambda_1^6+1)^{-1},\; l_2=(\lambda_1^6+1)/2\;,\; \nu=0,5 \;. \end{aligned} \tag{11}$$

While putting expression (11) in the relationship (9) we shall get:

$$k=(\lambda_1^9+\lambda_1^6+3\lambda_1^3-1)/(2\lambda_1^4(\lambda_1^3+1)) \quad . \tag{12}$$

From (12) we get the value of k for different values of the coefficients of extending along the axes λ_1

λ_1	0,667	0,8	0,9	1	2	4	∞
k	0	0,7527	0,9282	1	2,0799	8,0058	∞

As we can see, the effect of initial stresses on stress intensity factor are rather substantial. When the initial

stresses tend to the values corresponding the surface unstability of half-space during the compression ($\lambda_1^9+\lambda_1^6+3\lambda_1^3-1=0$), the values of stress intensity factor tend to zero, and on the contrary, rise to high extend when $\lambda_1\to\infty$. The same qualitative behavior of stresses was considered by Guz' (1983) when he concidered the problem of wedging out the half-infinite crack by the rigid half-infinite wedge of finite thickness.

Let us consider the rigid inclusions of particular form.

1.Suppose rigid inclusion is the disk of radius a and of stable thickness 2h. Contact with the surfase of the crack is assumed along the whole circle of disk for the inclusion of such form. Let us use the solution of the theory of elasticity of isotropic body. Which was received by Selvadurai (1984). Contact stresses in region $0\le r\le a$ according to the relationships (8), (9) and (13) of this work will be as those:

$$\sigma_{33}(r)=\frac{2h(1+m_1)(1+m_2)C_{44}\lambda}{\pi\omega(a^2-r^2)^{1/2}}\left\{\frac{r}{a}+\frac{r}{b\pi}+\frac{ar}{b^2\pi^2}+\right. \\ \left.+\frac{a^2r}{b^3\pi}\left[\frac{1}{\pi}+4\pi\left(\frac{r^2}{6a^2}-\frac{5}{96}\right)\right]+\Phi\left[\frac{a^4}{b^4}\right]\right\} . \tag{13}$$

Stresses outside crack will be expressed by the relationship

$$\sigma_{33}(r)=\frac{2h(1+m_1)(1+m_2)C_{44}\lambda}{\pi\omega(r^2-b^2)^{1/2}}\left\{\frac{ab}{2r^2}+\frac{a^2}{2\pi r^2}+\frac{a^3}{b^3}\left[\frac{b^2}{2\pi r^2}+\right.\right. \\ \left.\left.+\frac{3(2b^2-r^2)b^2}{16r^4}+\Phi\left[\frac{a^4}{b^4}\right]\right]\right\} . \tag{14}$$

Stress intensity factor K_1 in points $r=b$, which is determined as $K_1=\lim\limits_{r\to b}\sqrt{2\pi(r-b)}\,\sigma_{33}(r)$ can be put as

$$K_1=\frac{(1+m_1)(1+m_2)C_{44}\lambda h}{\sqrt{\pi b}\;\omega}\left\{\frac{a}{b}+\frac{a^2}{\pi b^2}+\frac{a^3}{b^3}\left[\frac{1}{\pi^2}+\frac{3}{8}\right]+\Phi\left[\frac{a^4}{b^4}\right]\right\} . \tag{15}$$

2.Suppose the surfase of rigid inclusion, which is in contact with the surfase of circular of crack, is described by the formula $g(r)=a-r^2/2R$.

Distribution of stresses near the crack with inclusion of such a form for the transversely isotropic materials was studied by Tsai (1984). For the determination of stress intensity factor $\mathbb{K}_1$ in points $r=b$ we get such relationship

$$\mathbb{K}_1 = \frac{8(1+m_1)(1+m_2)C_{44}\lambda a\sqrt{b}}{\pi\omega R\sqrt{\pi}}\,\mathbb{G}(a/b), \tag{16}$$

$$\mathbb{G}(a/b)=\frac{a}{b}\int_0^1 \frac{\mathbb{F}(\lambda, a/b)}{b^2/a^2-\lambda^2}\,d\lambda,\quad \mathbb{F}(\lambda, a/b)=(1-\lambda)^{1/2} f(a\lambda)\lambda(b^2/a^2-\lambda^2)^{1/2}$$

and the function $f(a\lambda)$ is determined from Fredgolm's equation

$$f(a\rho)=1-\int_0^1 f(a\lambda)\mathbb{M}(\rho,\lambda)\,d\lambda,$$

$$\mathbb{M}(\rho,\lambda)=\frac{4}{\pi^2}\int_0^\infty \frac{\lambda(1-\lambda)^{1/2}(l^2/a^2-\lambda)^{1/2}d\theta}{\left[\left(b^2/a^2-1\right)ch^2\theta+\left(1-\rho^2\right)\right]\left[\left(b^2/a^2-1\right)ch^2\theta+1-\lambda^2\right]}$$

The change of the value of coefficient of stresses in accordance with ratio a/b with different conditions of initial stresses is shown in fig.2 by an example of neo-hookean body.

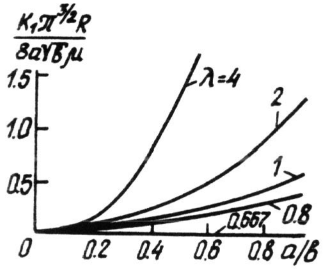

Fig.2. Normalized stress intensity factor.

The curves in fig.2 as well as analitical function (15) show the substantial influence of residual stresses (compressing as well as extending) on the limit equilibrium state of the body with a crack, with thin inclusion.

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