STRESS ANALYSIS OF INFINITE PLATE WITH TWO UNEQUAL ARBITRARILY ORIENTED CRACKS

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ABSTRACT

The elastostatic problem solved in this paper is of an isotropic homogeneous infinite plate, with two arbitrarily oriented cracks of different lengths subjected to uniform pressure and tangential shear at the crack surface. The problem is formulated in the complex plane using the Kolossoff-Muskhelishvili stress functions and further Schwarz alternating method is used to solve the problem of doubly connected region. Mode I and II stress intensity factors at all the four crack tips for various crack length ratios, crack angles and crack spacings are found. The fracture angles at the four crack tips are evaluated using strain energy density theory. The minimum strain energy density factor is also computed at all the crack tips and the crack initiation angle is found out.

KEYWORDS

Stress intensity factor, arbitrary oriented cracks, Schwarz alternating method,transformed stress function, corrected stress function, first approximation, second approximation.

INTRODUCTION

The boundary value problem in two dimensional elasticity can be reduced to the solutions of two complex stress functions given by (Muskhelishvili, 1953). By using this approach the problem of infinite plate with one hole of arbitrary counter can be solved by making use of a suitable mapping function which will map the region to a unit circle. This procedure is rather cumbersome for multiply connected regions. The Schwarz alternating method reduces a problem of a multiply connected region to a sequence of problems in simply connected regions (Sokolnikoff,1956). The problem considered is as shown in Fig.1. An infinite plate contains two cracks AB and CD of lengths 2a and 2g, makes angles β and $-\gamma$ with reference to x-axis, the center of two cracks are z_0 distance apart and subjected to uniform pressure P & Q as well as tangential shear T_1 and T_2 respectively.

APPROACH

The problem is solved for a single crack subjected to given loading condition. The stress functions valid near the first crack is first rotated, so that the axes of the stress function lies along and perpendicular to the second crack and then translated to the center of the second crack. These transformed stress functions give the boundary condition on the surface of second crack. To achieve true boundary condition on the second crack surface, a new problem is solved by applying negative of the boundary condition obtained from transformed stress function and actual boundary condition i.e. loading at the second crack surface. This gives the corrected stress functions. Superposition of transformed and corrected stress functions give the second approximation to the stress field valid near second crack.

Similarly starting from second crack, the second approximation stress functions valid near the first crack are found, the procedure is same as given by Ukadgaonker(1971).

FIRST APPROXIMATION

The Muskhelishvili stress functions for a crack C_1 of length '2a' in an infinite plate subjected to uniform normal pressure 'P' and uniform tangential shear 'T_1' can be obtained by mapping the crack in z-plane to unit circle in the ζ_1-plane using mapping function (Savin, 1951) as shown in Fig.2.

$$z = \omega(\zeta_1) = \frac{a}{2}\left(\zeta_1 + \frac{1}{\zeta_1}\right) \qquad (1)$$

The stress functions obtained are

$$\phi_1(\zeta_1) = -\frac{(P+iT_1)\,a}{2\,\zeta_1}$$

$$\psi_1(\zeta_1) = -\frac{(P+iT_1)\,a}{2\,\zeta_1}\left\{1 + \frac{(1+\zeta_1^2)}{(\zeta_1^2 - 1)}\right\} \qquad (2)$$

Similarly for crack C_2 as shown in Fig. 1. Stress functions are given by

$$\phi_2(\zeta_2) = -\frac{(Q+iT_2)\,g}{2\,\zeta_2}$$

$$\psi_2(\zeta_2) = -\frac{(Q+iT_2)\,g}{2\,\zeta_2}\left\{1 + \frac{(1+\zeta_2^2)}{(\zeta_2^2 - 1)}\right\} \qquad (3)$$

Where ζ_1-plane corresponds to the center of crack C_1 in the mapped plane and ζ_2-plane corresponds to the center of crack C_2 in the mapped plane.

The first approximation to the stress field can be obtained from the single crack solutions of the two cracks superposed on to each other. However this does not account the interaction effect of the other crack in the vicinity.

SECOND APPROXIMATION

Starting from crack C_1, to account for the interaction effect of two cracks, the stress functions given (2) are transformed to the second crack position by rotating by an angle $\alpha = \beta + \gamma$, and translating to the second crack center O_2 through distance c_0 in the mapped plane given by

$$z_0 = \frac{a}{2}\left(c_0 + \frac{1}{c_0}\right) \qquad (4)$$

The transformed stress functions obtained are

$$\phi_{12}(\zeta_2) = -\frac{(P+iT_1)\,a\,e^{-2i\alpha}}{2\,(\zeta_2 + c_0)} \qquad (5)$$

$$\psi_{12}(\zeta_2) = -\frac{(P+iT_1)\,a}{2\,(\zeta_2+c_0)}\left\{1 + \frac{\left[1+(\zeta_2+c_0)^2\,e^{2i\alpha}\right]}{\left[(\zeta_2+c_0)^2\,e^{2i\alpha} - 1\right]}\right\} + \bar{c}_0\,\frac{(P+iT_1)\,a\,e^{-2i\alpha}}{2(\zeta_2+c_0)^2} \qquad (6)$$

These transformed stress functions give a boundary condition $f_{12}(t_2)$ on second crack as

$$f_{12}(t_2) = \phi_{12}(t_2) + \frac{\omega_2(t_2)}{\overline{\omega_2'(t_2)}}\,\overline{\phi_{12}'(t_2)} + \overline{\psi_{12}(t_2)} \qquad (7)$$

This boundary condition is not true boundary condition at the second crack boundary. In order to correct this a new problem of infinite plate with crack of length '2g', subjected to negative of the boundary condition (7) and true loading $(Q+iT_2)$ i.e. uniform pressure 'Q' and tangential shear T_2.

$$f_2(t_2) = -\frac{(Q+iT_2)\,g}{2\,\zeta_2}\left(t_2 + \frac{1}{t_2}\right) - f_{12}(t_2) \qquad (8)$$

The corrected stress function valid near the second crack are given by

$$\phi_{22}(\zeta_2) = -\frac{1}{2\pi i}\oint\frac{f_2(t_2)\,dt_2}{(t_2-\zeta_2)}$$

$$= -\frac{(Q+iT_2)\,g}{2\,\zeta_2} - \frac{(P-iT_1)\,a\,e^{2i\alpha}}{2\,(\bar{c}_0^2-1)^2}\left[\frac{2\bar{c}_0 + (3\bar{c}_0^2-1)\,\zeta_2}{(1+\bar{c}_0\zeta_2)^2}\right]$$

$$-\frac{(P-iT_1)\,a\,\bar{c}_0\,e^{2i\alpha}}{2\,(\bar{c}_0^2-1)^2}\left[\frac{2+\bar{c}_0\,(\bar{c}_0^2+1)\,\zeta_2}{(1+\bar{c}_0\zeta_2)^2}\right]$$

$$-\frac{(P-iT_1)\,a\,e^{-2i\alpha}}{(\bar{c}_0^2\,e^{-2i\alpha}-1)}\left\{\frac{\bar{c}_0 e^{-2i\alpha}(1+\bar{c}_0\zeta_2)+\zeta_2}{\left[(1+\bar{c}_0\zeta_2)^2 e^{-2i\alpha}-\zeta_2^2\right]}\right\}$$

$$+\frac{(P-iT_1)\,a\,c_0\,e^{2i\alpha}}{2\,\bar{c}_0^2}\left[\frac{1+2\bar{c}_0\zeta_2}{(1+\bar{c}_0\zeta_2)^2}\right] \qquad (9)$$

$$\psi_{22}(\zeta_2) = -\frac{1}{2\pi i}\oint \frac{\overline{f_2(t_2)}\,dt_2}{(t_2-\zeta_2)} - \frac{\zeta_2(1+\zeta_2^2)}{(\zeta_2^2-1)}\,\phi'_{22}(\zeta_2)$$

$$\psi_{22}(\zeta_2) = -\frac{(Q-iT_2)\,g}{2\,\zeta_2} - \frac{(P-iT_1)\,a\,e^{2i\alpha}}{\bar{c}_0\,(1+\bar{c}_0\zeta_2)}$$

$$-\frac{(P-iT_1)\,a\,e^{-2i\alpha}}{(c_0^2-1)^2}\left[\frac{(c_0^2+1)\,\zeta_2-2c_0}{(\zeta_2^2-1)}\right] - \frac{\zeta_2(1+\zeta_2^2)}{(\zeta_2^2-1)}\,\phi'_{22}(\zeta_2) \qquad (10)$$

The stress functions valid near the second crack can be obtained by superposing the transformed and corrected stress function as

$$\phi_2(\zeta_2) = \phi_{12}(\zeta_2) + \phi_{22}(\zeta_2)$$
$$\psi_2(\zeta_2) = \psi_{12}(\zeta_2) + \psi_{22}(\zeta_2) \qquad (11)$$

which are found to satisfy the boundary condition exactly on the second crack.

Similarly second approximation stress functions valid in the vicinity of the first crack are obtained starting from the stress functions valid for the second crack and transforming them to first crack position. The final form of the stress functions are given by

$$\phi_1(\zeta_1) = -\frac{(Q+iT_2)\,g\,e^{2i\alpha}}{2\,(\zeta_1-c_1)} - \frac{(P+iT_1)\,a}{2\,\zeta_1}$$

$$-\frac{(Q-iT_2)\,g\,e^{-2i\alpha}}{2\,(\bar{c}_1^2-1)^2}\left[\frac{2\bar{c}_1-(3\bar{c}_1^2-1)\,\zeta_1}{(1-\bar{c}_1\zeta_1)^2}\right]$$

$$-\frac{(Q-iT_2)\,g\,\bar{c}_1\,e^{-2i\alpha}}{2\,(\bar{c}_1^2-1)^2}\left[\frac{2-\bar{c}_1\,(\bar{c}_1^2+1)\,\zeta_1}{(1-\bar{c}_1\zeta_1)^2}\right]$$

$$-\frac{(Q-iT_2)\,g\,e^{2i\alpha}}{(\bar{c}_1^2\,e^{2i\alpha}-1)}\left\{\frac{\zeta_1-\bar{c}_1\,e^{-2i\alpha}(1-\bar{c}_1\zeta_1)}{\left[(1-\bar{c}_1\zeta_1)^2 e^{2i\alpha}-\zeta_1^2\right]}\right\}$$

$$-\frac{(Q-iT_2)\,a\,c_1\,e^{-2i\alpha}}{2\,\bar{c}_1^2}\left[\frac{1-2\bar{c}_1\zeta_1}{(1-\bar{c}_1\zeta_1)^2}\right] \qquad (12)$$

$$\psi_1(\zeta_1) = -\frac{(Q+iT_2)\,g}{2\,(\zeta_1-c_1)}\left\{1+\frac{\left[1+(\zeta_1-c_1)^2\,e^{-2i\alpha}\right]}{\left[(\zeta_1-c_1)^2\,e^{-2i\alpha}-1\right]}\right\}$$

$$-\bar{c}_1\frac{(Q+iT_2)\,g\,e^{2i\alpha}}{2(\zeta_1-c_1)^2} - \frac{(P-iT_1)\,a}{2\,\zeta_1} - \frac{(Q-iT_2)\,g\,e^{-2i\alpha}}{\bar{c}_1\,(1+\bar{c}_1\zeta_1)}$$

$$-\frac{(Q-iT_2)\,g\,e^{2i\alpha}}{(c_1^2-1)^2}\left[\frac{(c_1^2+1)\,\zeta_1+2c_1}{(\zeta_1^2-1)}\right] - \frac{\zeta_1(1+\zeta_1^2)}{(\zeta_1^2-1)}\,\phi'_{11}(\zeta_1) \qquad (13)$$

where

$$z_0 = \frac{g}{2}\left(c_1+\frac{1}{c_1}\right) \qquad (14)$$

which are found to satisfy the boundary condition at first crack exactly.

CALCULATION OF STRESS INTENSITY FACTORS

The mode I and mode II stress intensity factors for tip-C and tip-D can be found by using (9) as, (Sih et al., 1962)

$$K_{I_2} - iK_{II_2} = 2\sqrt{\frac{\pi}{g}}\,\phi_2'(\pm 1)$$

$$=2\sqrt{\frac{\pi}{g}}\Bigg\{\frac{(P+iT_1)\,a\,e^{-2i\alpha}}{2\,(c_0 \pm 1)^2} + \frac{(Q+iT_2)\,g}{2} - \frac{(P-iT_1)\,a\,e^{2i\alpha}\,(3\bar{c}_0^2-1)}{2\,(\bar{c}_0^2-1)^2\,(\bar{c}_0 \pm 1)^2}$$

$$+\frac{(P-iT_1)\,a\,\bar{c}_0\,e^{2i\alpha}\left[2\bar{c}_0 \pm (3\bar{c}_0^2-1)\right]}{(\bar{c}_0^2-1)^2\,(\bar{c}_0 \pm 1)^3} - \frac{(P-iT_1)\,a\,\bar{c}_0^2\,e^{2i\alpha}(1\pm\bar{c}_0^2)}{2\,(\bar{c}_0^2-1)^2\,(\bar{c}_0 \pm 1)^2}$$

$$+\frac{(P-iT_1)\,a\bar{c}_0^2\,e^{2i\alpha}\left[2 \pm\bar{c}_0\,(\bar{c}_0^2+1)\right]}{(\bar{c}_0^2-1)^2\,(\bar{c}_0 \pm 1)^3} - \frac{(P-iT_1)\,ae^{-2i\alpha}(\bar{c}_0^2e^{-2i\alpha}+1)}{(\bar{c}_0^2\,e^{-2i\alpha}-1)\left[(1\pm\bar{c}_0)^2e^{-2i\alpha}-1\right]}$$

$$-\frac{2\,(P-iT_1)\,a\,e^{-2i\alpha}\left[1-\bar{c}_0^2\,e^{-2i\alpha}\,(1\pm\bar{c}_0)^2\right]}{(\bar{c}_0^2\,e^{-2i\alpha}-1)\left[(1\pm\bar{c}_0)^2e^{-2i\alpha}-1\right]^2}$$

$$\mp\frac{(P-iT_1)\,a\,c_0\,e^{2i\alpha}}{(1\pm\bar{c}_0)^3}\Bigg\} \qquad (15)$$

Upper line signs are for tip-D and lower line signs for tip-C.

The mode I and mode II stress intensity factors for tip-A and tip-B can be found by using (12) and (13), as (Sih et al., 1962)

$$K_{I_1} - iK_{II_1} = 2\sqrt{\frac{\pi}{a}}\,\phi_1'(\pm 1)$$

$$=2\sqrt{\frac{\pi}{a}}\Bigg\{\frac{(Q+iT_2)\,g\,e^{2i\alpha}}{2\,(c_1 \mp 1)^2} + \frac{(P+iT_1)\,a}{2} - \frac{(Q-iT_2)\,g\,e^{-2i\alpha}\,(3\bar{c}_1^2-1)}{2\,(\bar{c}_1^2-1)^2\,(\bar{c}_1 \mp 1)^2}$$

$$+\frac{(Q-iT_2)\,g\,\bar{c}_1\,e^{-2i\alpha}\left[2\bar{c}_1 \mp (3\bar{c}_1^2-1)\right]}{(\bar{c}_1^2-1)^2\,(\bar{c}_1 \mp 1)^3} - \frac{(Q-iT_2)\,g\,\bar{c}_1^2e^{-2i\alpha}(1\mp\bar{c}_1^2)}{2\,(\bar{c}_1^2-1)^2\,(\bar{c}_1 \mp 1)^2}$$

$$+\frac{(Q-iT_2)g\bar{c}_1^2\,e^{-2i\alpha}\left[2 \mp\bar{c}_0\,(\bar{c}_0^2+1)\right]}{(\bar{c}_1^2-1)^2\,(\bar{c}_1 \mp 1)^3} - \frac{(Q-iT_2)\,g\,e^{2i\alpha}(\bar{c}_1^2e^{2i\alpha}+1)}{(\bar{c}_1^2\,e^{2i\alpha}-1)\left[(1\mp\bar{c}_1)^2e^{2i\alpha}-1\right]}$$

$$-\frac{2\,(Q-iT_2)\,g\,e^{2i\alpha}\left[1-\bar{c}_1^2\,e^{4i\alpha}\,(1\mp\bar{c}_1)^2\right]}{(\bar{c}_1^2\,e^{2i\alpha}-1)\left[(1\mp\bar{c}_1)^2e^{2i\alpha}-1\right]^2}$$

$$\pm\frac{(Q-iT_2)\,g\,c_1\,e^{-2i\alpha}}{(1\mp\bar{c}_1)^3}\Bigg\} \qquad (16)$$

The upper line signs are for tip-B and the lower line signs are for tip-A.

FRACTURE ANGLE AND STRAIN ENERGY DENSITY FACTOR

The first postulate of strain energy density criterion (Sih, 1973) states that the crack propagation takes place in a radial direction along which strain energy density factor possesses a minimum stationary value, i.e. the crack will grow along an angle θ_0 at which,

$$\left.\frac{dS}{d\theta}\right|_{\theta=\theta_0} = 0 \quad \text{and} \quad \left.\frac{d^2S}{d\theta^2}\right|_{\theta=\theta_0} > 0 \qquad (17)$$

The strain energy density factor 'S' is given (Sih, 1973) as

$$16\,G\,S = a_{11}\,K_I^2 + 2\,a_{12}\,K_I\,K_{II} + a_{22}\,K_{II}^2 \qquad (18)$$

a_{11}, a_{12}, a_{22} are functions of θ angular coordinates and Poisson's ratio, where G is shear modulus.

The stress intensity factors obtained for various cases from Eqns. (15) and (16) and substituted in Eqns. (17) and (18) which gives the initial direction of crack growth for both the cracks.

RESULTS AND DISCUSSION

The stress intensity factors for various loading conditions and crack orientation are computed from Eqns.(15) and (16). Some of the results in the form of normalized stress intensity factors (SIF) are presented.

(a) Effect crack tip distance and geometry

Fig.3 shows that variation of mode I normalized SIF with crack tip distance d, for the case of equally pressurized cracks of various crack length combinations. The interaction effect of two cracks is vanished when d/a is more than 2.0 and the solution converges to single crack. It is observed that the crack initiation is occurred in the direction of crack tip itself, as the cracks are subjected to opening mode loading.

The variation of mode II normalized SIF with crack tip distance 'd' for the case of cracks are subjected to equal tangential shear is presented in Fig.4. It is observed that the mode I SIF is zero when the cracks are collinear with real axis, and mode II SIF have negative values. From this graph it is also observed that the interaction effect of two cracks vanished when d/a is more than 2.0. The crack initiation angle is -82° with respect to positive direction of crack tip. The above results are found in good agreement with results obtained by Ukadgaonker and Patil (1993).

(b) Effect of orientation of two equal cracks
The mode-I and mode-II normalized SIF variation is plotted in Fig. 5. by varying the angular orientation of one of the cracks, which are equally pressurized. It is found that when two crack tips are close to each other then the mode I SIF values are more and mode II SIF values have small magnitude. Fig. 5 shows small variation in SIF values with angular orientation of the cracks, for fixed center distance The variation of crack initiation angle for all crack tips, with respective positive direction of crack tip, with angular orientation of the crack is presented in Table-I.

CONCLUSIONS

Ukadgaonker and Naik (1991a,b) and Ukadgaonker and Korrane (1991) has solved the problem of infinite plate with two arbitrarily oriented cracks subjected to uniform loading at infinity, using Schwarz alternating method and has found that this method converges in the second approximation itself. From the present results also it can be concluded that the solution is converged in second approximation, because the stress functions (11), (12), and (13) satisfies boundary condition exactly at respective crack boundary and the solution emerges to single crack solution as crack tip distance is increased more than four time the crack length.

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Table-I Variation of crack initiation angle with angular orientation of crack.

$\beta = 10^0$

γ ↓	Tip-A	Tip-B	Tip-C	Tip-D
0	-179	-179	179	179
10	-178	-178	178	178
20	-177	-177	177	177
30	-176	-175	175	176
40	-176	-174	174	176
50	-175	-173	173	175
60	-175	-172	172	175
70	-174	-171	171	174
80	-174	-171	171	174
90	-174	-171	171	174

$\beta = 30^0$

γ ↓	Tip-A	Tip-B	Tip-C	Tip-D
0	-177	-177	177	177
10	-176	-175	175	176
20	-176	-174	174	176
30	-175	-173	173	175
40	-175	-172	172	175
50	-174	-171	171	174
60	-174	-171	171	174
70	-174	-171	171	174
80	-175	-171	171	175
90	-175	-171	171	175

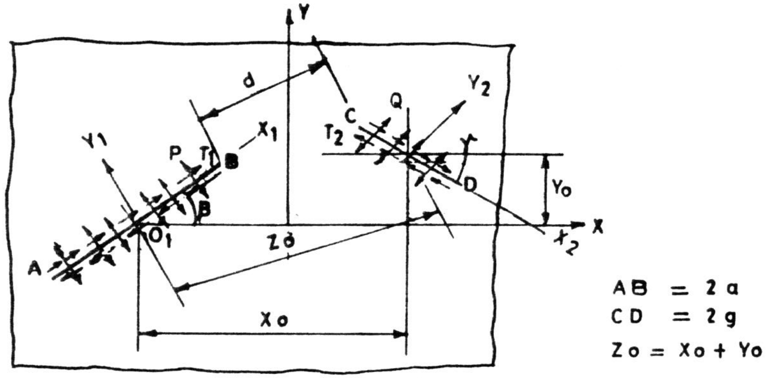
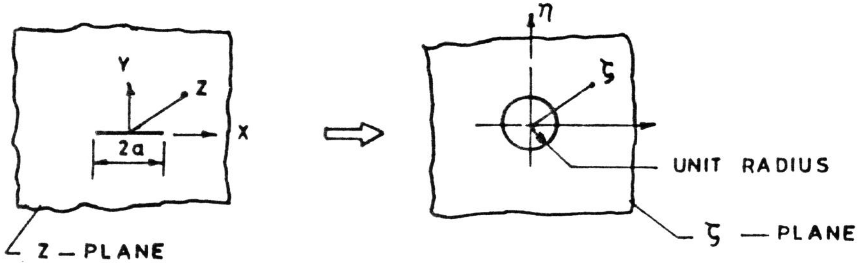


FIG. 1 STATEMENT OF PROBLEM



$$z = \frac{a}{2}\left(\zeta + \frac{1}{\zeta}\right)$$

FIG. 2 MAPPING OF OUTSIDE REGION OF UNIT CIRCLE.

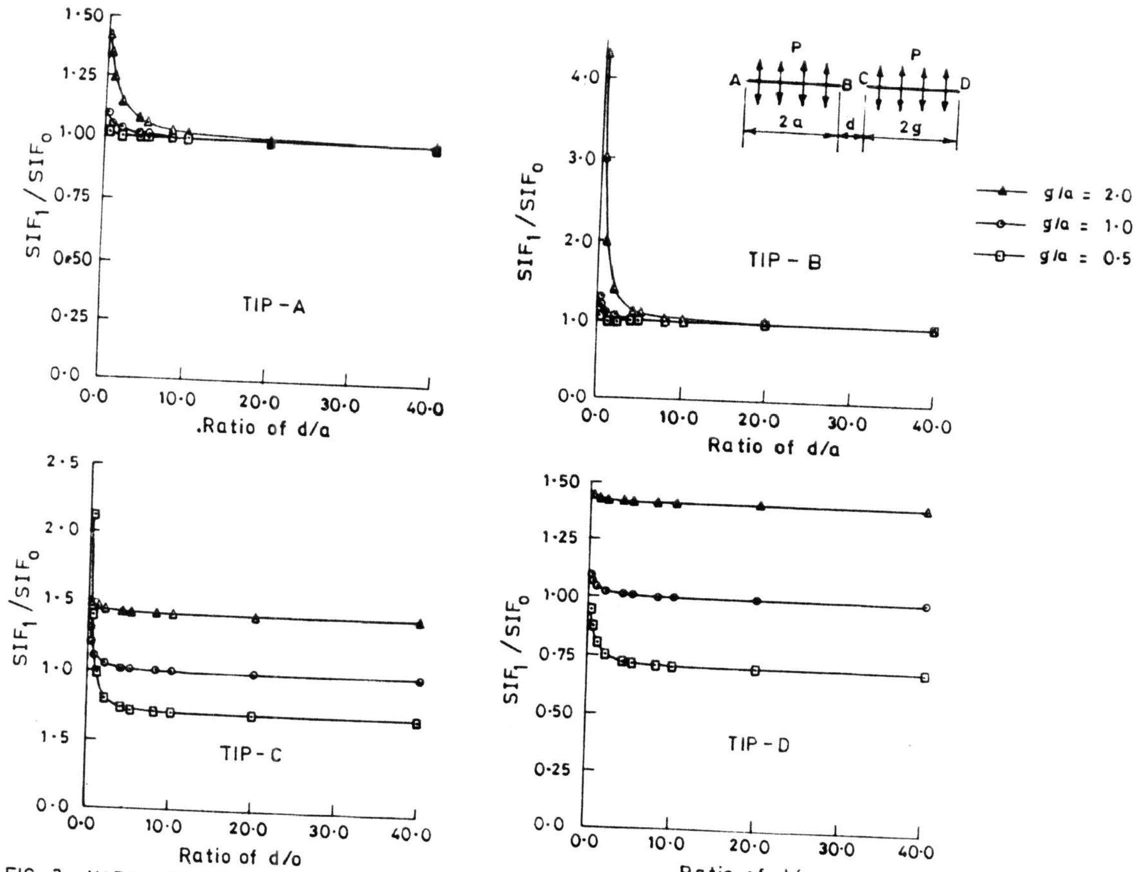


FIG. 3 VARIATION OF NORMALIZED SIF WITH CRACK TIP DISTANCE, CRACK LENGTH UNDER UNIFORM PRESSURE

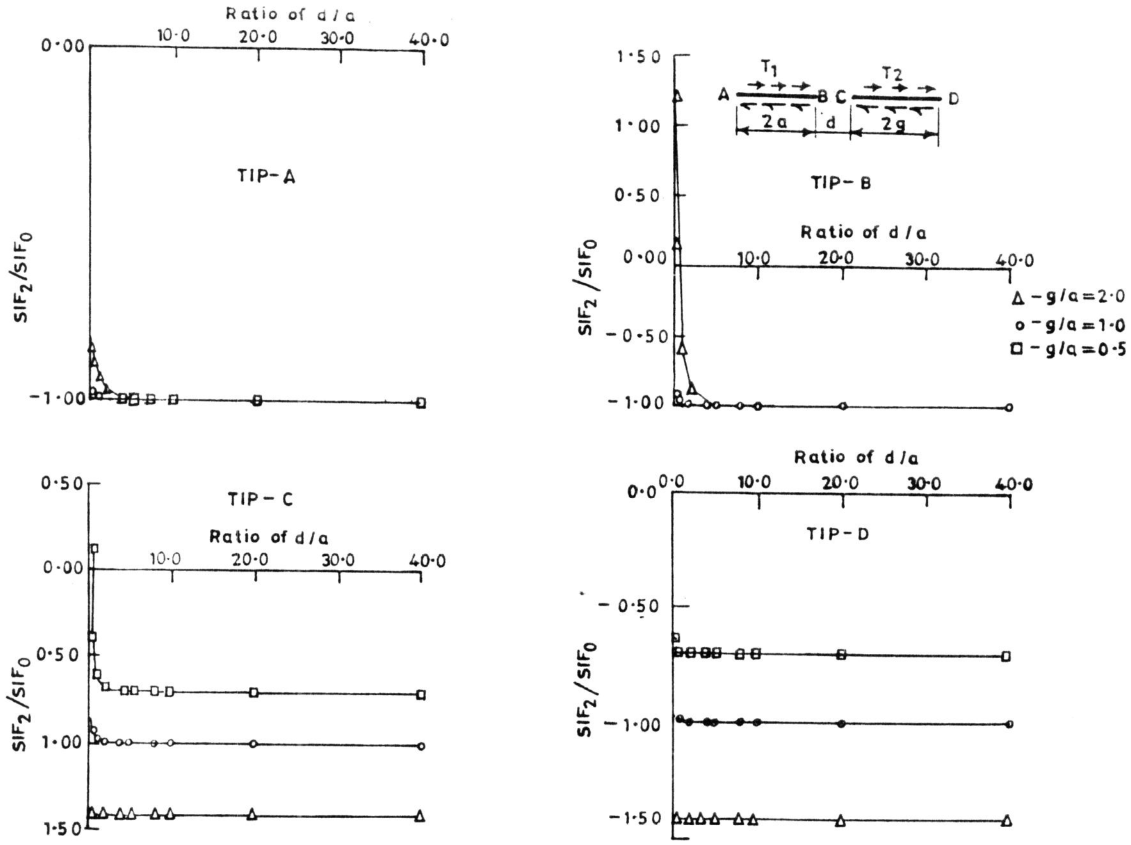


FIG. 4 VARIATION OF NORMALIZED SIF WITH CRACK TIP DISTANCE AND LENGTH OF CRACKS UNDER EQUAL SHEAR LOADING AT BOTH THE CRACKS

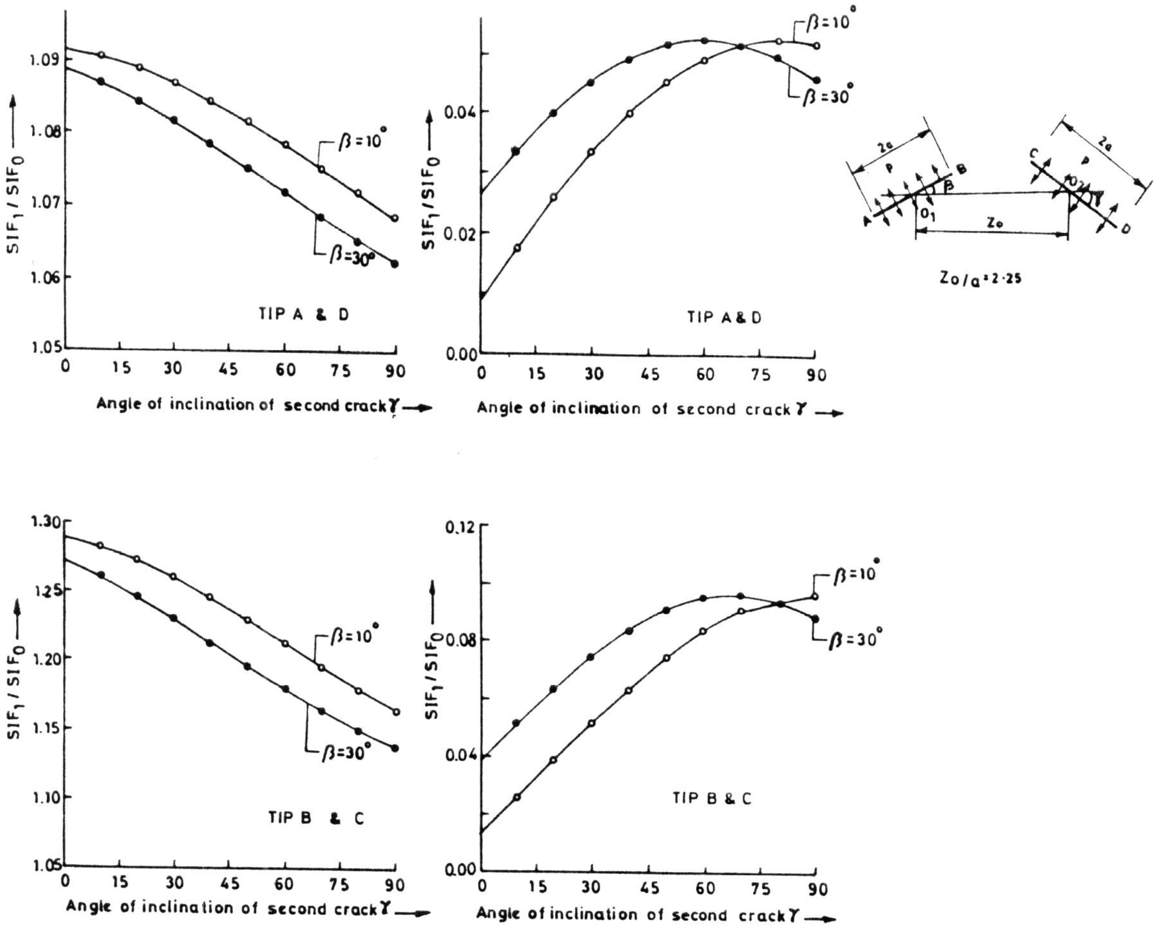


FIG. 5 - VARIATION OF NORMALIZED SIF WITH CRACK ANGLE