STRAIN ENERGY RELEASE RATE AT A CRACK TIP; EVALUATIONS BY A MIXED HYBRID ELEMENT

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ABSTRACT

The brittle fracture mechanics uses, simultaneously with the toughness characteristics of materials, the strain energy release rate G which is an important feature, particularly in linear fracture mechanics. This work is specially focused on giving different ways to obtain numerically a value of G with the aid of one crack hybrid finite element. Evaluating G is performed in this paper with either the notion of stress intensity factors or the method of virtual crack extension, both used on the mixed hybrid element only. Comparison with other published results is presented for an isotropic material, and shows the applicability of the proposed hybrid approach.

KEYWORDS

Crack hybrid element, energy release rate, linear fracture mechanics.

INTRODUCTION

In brittle fracture analysis, the strain energy release rate G is characterized by the energy stored at the crack tip. Once associated with an experimental critical toughness of the material, it constitutes the primary parameter of crack initiation. If the stress field is correctly evaluated around the crack tip, then G can be locally given either "statically" in terms of the stress intensity factors or "kinematically" by using displacements calculated near the crack tip (Bui, 1978). These two techniques imply a refined mesh in the area of the crack if classical (displacement) finite elements are used. We propose a more convenient approach : only one super crack hybrid finite element is placed around the crack tip, and classical displacement finite elements are used everywhere else. This type of element was primary suggested by Tong, Pian & Lasry (1973). A refinement is no longer necessary and a good stress evaluation is still obtained. The stress intensity factors are directly available in the hybrid element and give a first evaluation of G. The virtual crack extension method of Hellen (1975) is the second technique used in this paper to evaluate the strain energy release rate. It is particularly efficient here because the variation of the potential energy concerns only the crack hybrid element. Two elementary stiffnesses need to be calculated : one in the initial state, and the second after the virtual extension. One application is proposed and compared with results issued from literature : a crack in mode I placed in an isotropic material.

GOVERNING EQUATIONS AND THE PIAN PRINCIPLE

Local equations : for simplicity, the analysis is restricted to plane linear elasticity and all the latin indices i, j, k and l vary from 1 to 2. A volume Ω made of N elements $\Omega^{(K)}$is considered :

$$\Omega = \prod_{K=1}^{N} \Omega^{(K)} \quad (1)$$

The equations to be solved are constituted by the small displacement hypothesis (2), the prescribed displacements (3) on the boundary Γ_u, the equilibrium (4) in each volume $\Omega^{(K)}$ and the static conditions (5) on the boundary Γ_σ. They are completed by an elastic behavior law (6) and continuity conditions (7) and (8) along interfaces Γ_{kl} between $\Omega^{(K)}$ and $\Omega^{(L)}$:

$$\varepsilon_{ij}(\vec{u}) = 1/2\ (u_{i,j} + u_{j,i}) \text{ on } \Omega^{(K)}, K = 1,N \quad (2)$$

$$u_i = \bar{u}_i \quad \text{on } \Gamma_u \quad (3)$$

$$\sigma_{ij,j} = 0 \quad \text{on } \Omega^{(K)} \quad (4)$$

$$\sigma_{ij}\, n_j = T_i^{\circ} \quad \text{on } \Gamma_\sigma \quad (5)$$

$$\varepsilon_{ij}^{(K)} = S_{ijkl}^{(K)}\, \sigma_{ij}^{(K)} \quad (6)$$

$$u_i^{(K)} = u_i^{(L)} \quad \text{on } \Gamma_{kl} \quad (7)$$

$$\sigma_{ij}^{(K)}\, n_j^{(K)} = \sigma_{ij}^{(L)}\, n_j^{(L)} \quad \text{on } \Gamma_{kl} \quad (8)$$

Variational approach : a primal approximation is used in the major part of the structure which is modelled with ordinary 4-node displacement finite elements. No further information will be given here on these elements because they are well known in literature. A mixed hybrid variational approach is applied in modelling the area near the crack tip. Pian's (1964) principle is chosen because it takes into account a dualized continuity of the stress vector at interfaces and it allows a kinematic compatibility with the already existing displacement elements.

Pian's principle : the following admissible spaces Σ and U are defined :

$$\Sigma(\Omega) = \{ \tau / \tau = (\tau_{ij}),\ \tau_{ij} = \tau_{ji},\ \tau_{ij} \in L^2(\Omega^{(K)}),\ \tau_{ij,j} \in L^2(\Omega^{(K)}),\ \tau_{ij,j} = 0 / \Omega^{(K)},\ K = 1, N \} \quad (9)$$

$$U(\Omega) = \{ \vec{v} / \vec{v} = (v_i),\ \exists\ \vec{w}^{(K)} \in H^1(\Omega^{(K)}),\ \vec{w}^{(K)} = \vec{v} / \partial\Omega^{(K)},\ K = 1, N;\ \vec{v} = 0 / \Gamma_u \} \quad (10)$$

Then Pian's principle can be written as :

$$\mathcal{L}(\tau,\vec{v}) = \sum_{K=1}^{N} \Big[-\frac{1}{2} \int_{\Omega^{(K)}} S_{ijkl}\, \tau_{ij}\, \tau_{kl}\, d\Omega + \int_{\partial\Omega^{(K)}} \tau_{ij}\, n_j\, v_i\, d\Gamma - \int_{\partial\Omega^{(K)} \in \Gamma_\sigma} T_i^{\circ}\, v_i\, d\Gamma \Big] \quad (11)$$

where $T_i = \sigma_{ij}\, n_j$ (12)

In (11) and (12), n_j are the components of the outward normal to the surface considered for the evaluation of T_i (here, it is Γ_σ with T_i°).

Properties : the stationary conditions of this functional are :

$\exists\ (\sigma, \vec{u}) \in \Sigma \times U,\ \forall\ \delta\tau_{ij} \in \Sigma,\ \forall\ \delta v_i \in U,$

$$\sum_{K=1}^{N} \Big[\int_{\Omega^{(K)}} S_{ijkl}\, \sigma_{kl}\, \delta\tau_{ij}\, d\Omega - \int_{\partial\Omega^{(K)}} u_i\, \delta\tau_{ij}\, n_j\, d\Gamma \Big] = 0 \quad (13)$$

$$\sum_{K=1}^{N} \Big[\int_{\partial\Omega^{(K)}} \sigma_{ij}\, n_j\, \delta v_i\, d\Gamma \Big] = \sum_{K=1}^{N} \Big[\int_{\partial\Omega^{(K)} \in \Gamma_\sigma} T_i^{\circ}\, \delta v_i\, d\Gamma \Big] \quad (14)$$

First condition (13) appears as a behavior law written for the whole structure and relation (14) is equivalent to the virtual work principle. Using two bilinear forms and one scalar respectively given as

$$a(\sigma, \tau) = \sum_{K=1}^{N} \int_{\Omega^{(K)}} S_{ijkl}\, \sigma_{kl}\, \tau_{ij}\, d\Omega \quad (15)$$

$$b(\tau, \vec{v}) = - \sum_{K=1}^{N} \int_{\partial \Omega^{(K)}} \tau_{ij} \, n_j \, v_i \, d\Gamma \qquad (16)$$

$$L(\vec{v}) = - \sum_{K=1}^{N} \int_{\partial \Omega^{(K)} \in \Gamma_\sigma} T_i^{\circ} \, v_i \, d\Gamma \qquad (17)$$

Pian's mixed hybrid functional (11) may now be transformed into (18) and relations (13) and (14) are then written as (19) and (20) :

$$\mathcal{L}(\tau, \vec{v}) = 1/2 \, a(\tau, \tau) + b(\tau, \vec{v}) - L(\vec{v}) \qquad (18)$$

$$a(\sigma, \tau) + b(\tau, \vec{u}) = 0 \qquad (19)$$

$$b(\sigma, \vec{v}) = L(\vec{v}) \qquad (20)$$

The solution $(\sigma, \vec{u})$ is a saddle point of Pian's functional; the works by Brezzi (1974) and Babuska (1973, 1974) show that the system (19,20) is equivalent to (21) :

$$\sup_{\tau \in \Sigma} \mathcal{L}(\sigma, \vec{v}) < \mathcal{L}(\sigma, \vec{u}) < \inf_{v \in U} \mathcal{L}(\tau, \vec{u}) \qquad (21)$$

CRACK HYBRID ELEMENT

Introduction : the static and kinematic properties of stress and displacement fields pointed out in (9) and (10) respectively are used. The stress field inside the super element must be in equilibrium and is chosen equal to the analytical solution for a crack in isotropic or orthotropic material. As the super hybrid element will be used here with linear 4-node rectangular displacement elements, the boundary displacements are chosen linear for compatibility reasons.

Stress fields : if the crack exists in an isotropic material, the complex function approach of Muskhelishvili (1975) is a convenient formulation for the analysis of the problem. The final solution is, for example, reported by Owen and Fawkes (1983, p 11). The case of an orthotropic material is analysed by Savin (1961), and the complete stress field can be found, for example, in the work by Courtade and Surry (1987).

Boundary displacement interpolation. The linear 9-node hybrid element is shown in figure 1. The length between two consecutive nodes p and p+1 is noted h, and s represents the coordinate used between these nodes. The two components u(s) and v(s) of the displacement vector along the boundary are linearly interpolated and written as :

$$\begin{Bmatrix} u(s) \\ v(s) \end{Bmatrix} = \begin{bmatrix} 1-\frac{s}{h} & 0 & \frac{s}{h} & 0 \\ 0 & 1-\frac{s}{h} & 0 & \frac{s}{h} \end{bmatrix} \begin{Bmatrix} u_p \\ v_p \\ u_{p+1} \\ v_{p+1} \end{Bmatrix} \qquad (22)$$

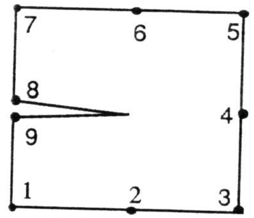

Fig. 1. The crack hybrid finite element used

Using relations (19) and (20) results in an equivalent stiffness matrix K_{hyb} for the crack hybrid element, needing the evaluation of compliance and boundary matrices implicitly defined in (15) and (16) respectively. A Gaussian quadrature is employed for integrating these two matrices and a specific procedure suggested by Owen and Fawkes (1983) is applied over the volume.

OBTAINING G

Stress intensity factors : for an elastic isotropic material in plane stress state, the strain energy release rate G in mode I is related to the stress intensity factors K_I and Young's modulus E :

$$G = K_I^2 / E \qquad (23)$$

The relation is slightly different for an elastic orthotropic material : with $S_{11} = 1/E_x$, $S_{22} = 1/E_y$, $S_{12} = -\nu_{xy}/E_x$, and $S_{33} = 1/G_{xy}$, it comes :

$$G = \left(\frac{S_{11} S_{22}}{2} \right)^{1/2} \left[\left(\frac{S_{22}}{S_{11}} \right)^{1/2} + \frac{2S_{12} + S_{33}}{2S_{11}} \right]^{1/2} \Pi \, k_I^2 \qquad (24)$$

Here, E_x and E_y are Young's moduli in x and y directions respectively, G_{xy} is the shear modulus and ν_{xy} is one Poisson's ratio of the plane xy; it has to be noted that the isotropic stress intensity factor K_I is different from the orthotropic corresponding value k_I :

$$K_I = k_I(\Pi)^{1/2} \quad (25)$$

Method of crack virtual extension : within the limits of linear elasticity given by equations (1) to (6), the paper by Hellen (1975) gives the strain energy release rate G in terms of variations of the potential energy Π and the length a of the crack :

$$G = -\delta\Pi / \delta a = -1/2 \; U_{nod}^t \; (\delta K / \delta a) \; U_{nod} \quad (26)$$

U_{nod} are the nodal displacements of the part of the finite element mesh in the initial state, but which will be affected by the evolution of the crack length : usually, this part consists of a lot of classical elements because the mesh has to be refined around the crack tip. $\delta K / \delta a$ is the corresponding variation of the assembled stiffness matrices of the same area. As a macro hybrid element is utilized, a refinement is no longer necessary and the stiffness variation is evaluated only inside this macro element :

$$\delta K / \delta a = \left[K_{hyb}(a + \delta a) - K_{hyb}(a) \right] / \delta a \quad (27)$$

EXAMPLE

A central crack in an isotropic medium : to illustrate the ideas given above, the example proposed in the thesis by Petit (1990, p 47) is chosen: a symmetrical rectangular plate (half length 0.1 m, height 0.1 m, thickness 1 mm) with a central crack of initial length a = 0.05 m. Young's modulus E is equal to 200000 MPa and Poisson's ratio to 0.3. The first case of a coarse mesh (figure 2) is only reported here. The kinematic method uses nodal displacements calculated near the crack tip as suggested by Bui (1978); several authors such as Rice (1968) have reported path independant integral based on primal formulation : this integral is given here as J_{int} when calculated on the internal elements and J_{ext} on external elements near the crack tip. The method of virtual crack extension is used either with classical 8-node elements or with the hybrid element. All the results are shown in table 1. NG is the number of Gauss points per direction used for numerical integration inside each eighth of the hybrid element (Owen and Fawkes, 1983).

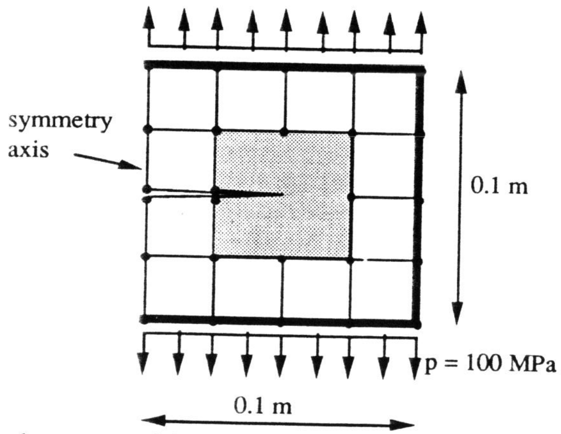

Fig. 2. Finite element model with one 9-node crack element

Table 1. Results for an isotropic material

used method	G and/or MPa.mm$^{-1/2}$	error %	K_I and/or N/mm	error %
kinematic method				- 9.8 %
Rice integral J_{int} (internal elements)	24.46	- 17.2 %		- 8.5 %
Rice integral J_{ext} (external elements)	25.55	- 13.5 %		- 6.1 %
virtual extension with displacement elements	25.56	- 13.5 %		- 6.7 %
stress intensity factors hybrid element NG = 2 one term in the series	30.11	+ 2 %	2454	1 %
virtual extension with hybrid element NG = 2 one term in the series	28.58	- 3.3 %	2391	- 1.6 %
reference	**29.54**		**2430**	

CONCLUSION

Two parameters can influence the modelling with one hybrid finite element : the number of complex terms included in the analytical solution chosen for the stress field, and the number of Gauss points for numerical integration. When the 9-node hybrid element is completely surrounded by other elements, good results are obtained with only one term in the analytical solution complex series and two Gauss points for numerical integration. The number of integration points has nearly no influence on the values of stress intensity factors but evaluating G by the virtual crack extension method is more sensitive to it. To avoid spurious modes when the nodal displacements of the hybrid element are not completely constrained by those of the environmental mesh, a higher number of terms in the complex series may be necessary.

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