STATE OF THE ART IN BEM APPLICATIONS TO FRACTURE MECHANICS OF ANISOTROPIC MEDIA

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ABSTRACT

This paper presents current state in the field of boundary integral equation method applied to static problems of three-dimensional anisotropic elastic bodies.

KEYWORDS

Boundary integral equation method, elastic anisotropy, crack problems.

INTRODUCTION

Fundamental Solutions. In the three-dimensional case fundamental solutions for the static problems can mostly easy be constructed by the use of Fourier transform applied to the operator of the equations of equilibrium yielding corresponding symbol. Fourier transform inversion of symbols, giving corresponding fundamental solutions, is succeeded only by one narrow subclass of elastic orthotropy, which includes isotropic and transversely isotropic materials, see Kroner (1953). For plane problems the situation is better, and analytical formulas are known for elastic materials with arbitrary anisotropy, these are mainly due to Kupradze and Basheleishvili (1954).

When three-dimensional problems with arbitrary anisotropy are concerned, only numerical methods can be used for reconstruction of the fundamental solutions from corresponding symbols. These methods can be divided into two groups. The first one is referred to disintegration of the Lebesgue's measure on hyperplanes (Radon transformation). Apparently first, this method was applied to static elasticity problems by Lifshitz and Rosenzveig (1949), and to dynamic problems by Hutorjanski (1985). Numerical experiments carried out by Wilson and Cruse (1978) revealed, that for arbitrary anisotropy additional approximations on spheres are needed to achieve appropriate accuracy in computing values of fundamental solutions under realistic computational time.

The second group is based on multipole decompositions of symbols, i.e. decompositions into the series of spherical harmonics. That, by the use of Bochner's inverting formulas, Bochner (1955), provides fundamental solutions either in the form of multipole series. For the fundamental solutions of statics this method was applied by Kuznetzov (1989). The latter method is used in the present article either for fundamental solutions and for singular or strongly-singular operators of the direct BEM formulations.

Periodic fundamental solutions are needed when perforated media or composite materials are regarded. The first attempt to construct such solutions was due to Hasimoto (1956) who applied it to investigation of the Stoke's flow through a periodic array of spheres. For general anisotropic elastic media periodic fundamental solutions were constructed by Kuznetzov (1991a).

Direct Boundary Element Methods. The first direct BEM formulations in the theory of elasticity is, to all appearances, due to Kupradze and Alexidze (1963). Their approach was based on the Somigliana identity written for the supplement Ω, where Ω is an open region under consideration. That approach leaded to the first-kind integral equations over unknown surface densities.

The "direct" approach which leads to the second-kind integral equations for the second boundary-value problem is due to Rizzo (1967). Analogous integral equations can in principle be obtained for other boundary-value problems of anisotropic elastic bodies, see Kuznetzov (1992).

Dislocations and Cracks in Anisotropic Media. Dislocations are of main interest for anisotropic crystals, mainly one narrow subclass of orthotropic crystals, which includes elastic transverse isotropy, was considered in the three-dimensional case, see Willis (1964), Tupholme (1974). In recent publications (Kuznetzov, 1991b, 1991e) were considered problems of interaction and arising of dislocations in general anisotropic media by the method of multipole decompositions.

Plane three-dimensional crack problems were considered by

Willis (1968) and Kuznetzov (1990) Their approach was based on the analysis of strongly-singular operators. Another method was proposed by Kunin et al. (1976), who developed Eshelby's method of transformation strain applied to ellipsoidal voids.

The first works on applications BIEM analysis to fracture mechanics were done by Cruse (1979). Two-dimensional problems with cracks of arbitrary loading were considered by Balanford et al. (1981), who considered also cracks on the interface between different media. This approach was developed by Goldstein and Perelmuter (1990) and implemented in a program package allowing to describe 3-d constructions with cracks under surface and body forces in 3-d and axially symmetrical bodies.

BASIC OPERATORS AND SYMBOLS

Initially anisotropic homogeneous elastic medium is regarded, with equations of equilibrium written in the form

$$\mathbf{A}(\partial_x)\mathbf{u} \equiv -\operatorname{div} \mathbf{C}\cdot\cdot(\nabla\mathbf{u}) \qquad (1)$$

where $\mathbf{u}$ is a displacement field, $\mathbf{C}$ is the fourth-order strongly elliptic elasticity tensor. The given anisotropic medium is assumed to be a hyperelastic one, so $\mathbf{C}$ is symmetric with respect to extreme pairs: $C^{ijmn} = C^{mnij}$.

Fourier transform

$$f^{\vee}(\boldsymbol{\xi}) = \int_{R^3} f(\mathbf{x}) \exp(-2\pi i\, \mathbf{x}\cdot\boldsymbol{\xi})\, dx$$

applied to the operator $\mathbf{A}$ gives its symbol:

$$\mathbf{A}^{\vee}(\boldsymbol{\xi}) = 4\pi^2\, \boldsymbol{\xi}\cdot\mathbf{C}\cdot\boldsymbol{\xi} \qquad (2)$$

Similarly the surface-traction symbol is defined

$$\mathbf{T}^{\vee}(\boldsymbol{\nu},\boldsymbol{\xi}) = 2\pi i\, \boldsymbol{\nu}\cdot\mathbf{C}\cdot\boldsymbol{\xi} \qquad (3)$$

Here $\boldsymbol{\nu}$ is the unit normal to a boundary surface $\partial\Omega$.

Using symbol $\mathbf{A}$ the symbol of fundamental solution of the equations (1) can be represented in the form

$$\mathbf{E}^{\vee}(\boldsymbol{\xi}) = \mathbf{A}^{\vee}_{0}(\boldsymbol{\xi}) \,/\, \det \mathbf{A}^{\vee}(\boldsymbol{\xi}), \tag{4}$$

where $\mathbf{A}^{\vee}_{0}$ is a matrix of algebraic complements of $\mathbf{A}^{\vee}(\boldsymbol{\xi})$. This expression shows, that $\mathbf{E}^{\vee}$ is strongly elliptic and positively homogeneous of order -2 with respect to $|\xi|$.

Let boundary $\partial\Omega$ be a compact two-dimensional submanifold in R^3 of the class $C^{m,\alpha}$, $m \geq 1$, $\alpha > 0$. An operator of boundary conditions on $\partial\Omega$ can be defined by the following formula

$$\mathbf{B}(\boldsymbol{\nu},\partial_x)\mathbf{u} \equiv (\mathbf{M}\cdot\mathbf{u} + \mathbf{N}\cdot\mathbf{T}(\boldsymbol{\nu},\partial_x)\mathbf{u})|_{\partial\Omega} = \mathbf{g} \tag{5}$$

where $\mathbf{M}$, $\mathbf{N}$ are square matrices. Operator $\mathbf{B}$ by single analytical expression allows to describe different types of boundary conditions, namely: $\mathbf{M} = \mathbf{I}$, $\mathbf{N} = 0$, corresponds to the first boundary-value problem; $\mathbf{M} = 0$, $\mathbf{N} = \mathbf{I}$ corresponds to the second one; $\mathbf{M} = \boldsymbol{\nu}\otimes\boldsymbol{\nu}$, $\mathbf{N} = \mathbf{I} - \boldsymbol{\nu}\otimes\boldsymbol{\nu}$ is for the third boundary-value problem; $\mathbf{M} = \mathbf{I} - \boldsymbol{\nu}\otimes\boldsymbol{\nu}$, $\mathbf{N} = \boldsymbol{\nu}\otimes\boldsymbol{\nu}$ is for the forth one. In an analogous way can be represented other boundary-value problems.

FUNDAMENTAL SOLUTIONS

<u>Nonperiodic Case.</u> Multipole decompositions applied to the symbol $\mathbf{E}^{\vee}$ yield

$$\mathbf{E}^{\vee}(\boldsymbol{\xi}) = \sum_{n=0,2,\ldots}^{\infty} \sum_{k=1}^{2n+1} \mathbf{E}_{nk}\, Y_k^n(\boldsymbol{\xi}') \,/\, |\xi|^2 \tag{6}$$

where Y_k^n are spherical harmonics, $\mathbf{E}_{nk}$ are matrix coefficients determined by the integration over unite sphere. Taking into account Bochner's inverting formula the expansion (6) can be inverted giving corresponding fundamental solution

$$\mathbf{E}(\mathbf{x}) = \sum_{n=0,2,\ldots}^{\infty} \gamma_n \sum_{k=1}^{2n+1} \mathbf{E}_{nk}\, Y_k^n(\mathbf{x}') \,/\, |x| \tag{7}$$

<u>Periodic Case.</u> Looking for periodic fundamental solution $\mathbf{E}_p$ in the form of trigonometric series we obtain

$$\mathbf{E}'_p(\mathbf{x}) = 1/V_Q \sum_{m^*\in\Lambda_0^*} \mathbf{E}^{\vee}(\mathbf{m}^*) \exp(-2\pi i\, \mathbf{m}^*\cdot\mathbf{x}) \tag{8}$$

Here Λ_0^* is the adjoined lattice without zero knot. It should be noted that $\mathbf{E}'_p$ is defined up to an additive constant.

Following formula shows, that $\mathbf{E}'_p(\mathbf{x})$ possess the same local properties, as nonperiodic fundamental solution

$$\mathbf{E}'_p(\mathbf{x}) = \mathbf{E}(\mathbf{x}) + \mathbf{G}(\mathbf{x}), \qquad \mathbf{G} \in C^{\infty}(Q, R^3\otimes R^3) \tag{9}$$

<u>Weakly-Singular, Singular and Strongly-Singular Operators.</u> The multipole decomposition method can be applied for the construction of weakly-singular, singular and strongly-singular operators, arising when elastic potentials and their derivatives are confined on the supporting surfaces (Kuznetzov, 1992).

DISLOCATIONS AND CRACKS IN ANISOTROPIC BODIES

<u>Plane Cracks in Anisotropic Media.</u> Let a plane crack lie in the plane Π. The displacement field around it can be represented by the double-layer potential with the unknown density which is actually a jump of displacements on the crack boundaries. The kernel of a such operator is of the form

$$\mathbf{G}_0^{\sim}(\mathbf{x}',\boldsymbol{\xi}') = \lim_{x''\to\pm 0} \int \mathbf{T}^{\vee}(\boldsymbol{\nu}_{x'},\boldsymbol{\xi})\cdot\mathbf{E}^{\vee}(\boldsymbol{\xi})\cdot\mathbf{T}^{\vee t}(\boldsymbol{\nu}_{x'},\boldsymbol{\xi}) \exp(-2\pi i\xi'' x'')\, dx''$$

$$\boldsymbol{\xi}' = \boldsymbol{\xi} - (\boldsymbol{\xi}\cdot\boldsymbol{\nu}_{x'})\boldsymbol{\nu}_{x'}, \qquad \xi'' = \boldsymbol{\xi}\cdot\boldsymbol{\nu}_{x'}, \qquad \mathbf{x}' \in \partial\Omega \tag{10}$$

Here and so forth sign " ~ " refers to Fourier transform on variables belonging to fibers of cotangent bundle $T^*\partial\Omega$, and if no confusion can arise operator and its symbol or amplitude are denoted by the same letter.

Integral in (10) can be expressed in another form

$$\mathbf{G}_0^{\sim}(\boldsymbol{\xi}') = -\,\mathrm{sym}\,[\,(2\pi i)^3(\boldsymbol{\xi}'\cdot\mathbf{C}\cdot\boldsymbol{\xi}')\cdot H_{\xi''}\,\mathbf{E}^{\vee}(\boldsymbol{\xi}',0)\cdot(\boldsymbol{\xi}'\cdot\mathbf{C}\cdot\boldsymbol{\nu}) +$$

$$+ (2\pi i)^2(\xi'\cdot C\cdot\nu)\cdot E^\sim(\xi',0)\cdot(\xi'\cdot C\cdot\nu) +$$

$$+ (2\pi i)^2(\xi'\cdot C\cdot\xi')\cdot E^\sim(\xi',0)\cdot(\nu\cdot C\cdot\nu) +$$

$$+ (2\pi i)(\xi'\cdot C\cdot\nu)\cdot\partial_{x''}E^\sim(\xi',0)\cdot(\nu\cdot C\cdot\nu)] \qquad (11)$$

In (11) $H_{\xi''}E^\sim(\xi',0)$ is the zero value of Hilbert's transformation on ξ'':

$$H_{\xi''}E^\sim(\xi',0) = \text{V.P.}\int_{-\infty}^{\infty}\frac{E^\sim(\xi)}{2\pi i\ \xi''}\,d\xi'' \qquad (12)$$

Other symbols in (11) are defined similarly. Taking into account (11), we obtain

Proposition 1. Matrix symbol $G_0^\sim$ is a symbol of the matrix p.d.o. of the class $S^1(\partial\Omega,\ R^3\otimes R^3)$.

The symbol $G_0^\sim$ can also be represented in the form

$$G_0^\sim(\xi') = -(2\pi)^2\ |\xi|^2\ V^\sim(\xi') \qquad (13)$$

where $V^\sim \in S^{-1}$. Fourier transform of (13) gives

$$G_0 \equiv V \circ \Delta \qquad (14)$$

Decomposition (14) implies

Proposition 2. For any function $g \in H^s(\partial\Omega,R^3)$, $s \geq 1$ exclusion of the pole vicinity $\omega_{x'}$ in evaluation the strongly-singular integral $G_0(g(x'))$ gives an error of the order $O(\text{mes}(\omega_{x'}))$.

Scholium. The preceding proposition shows, that evaluation of the integrals with the kernel ${}_0G$ can be implemented by the existing computer programs on numerical integration provided the pole vicinity is excluded from the analysis.

An analogous approach can be applied to the analysis of the stress and displacement fields around isolated dislocations in anisotropic media.

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